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FEKETE-SZEGÖ PROBLEM FOR CONCAVE UNIVALENT FUNCTIONS ASSOCIATED WITH FOX-WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. We investigate the Fekete-Szegö problem with real and complex parameter λ for the class $Co(\alpha)$ of concave univalent functions defined by Fox-Wright's generalized hypergeometric function.

Keywords: Concave, Univalent, Starlike functions, Fox-Wright's generalised hypergeometric function.

1. Introduction

Let S denote the class of all univalent analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

where $z \in \Delta = \{z : |z| < 1\}$. The problem of maximizing $|a_3 - \lambda a_2^2|$ is called Fekete-Szegö problem. The classical Fekete-Szegö inequality by means of Lowner's method for $f \in S$ is

$$|a_3 - \lambda a_2^2| \leq 1 + 2exp\left(\frac{-2\lambda}{1-\lambda}\right), \quad \lambda \in [0, 1) \tag{1.2}$$

When $\lambda \to 1$, we have $|a_3 - a_2^2| \leq 1$ The coefficient functional $\Lambda_\lambda(f) = a_3 - \lambda a_2^2$ plays an important role in function theory, namely $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f}\right)^2$$

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of locally univalent functions f in Δ. The problem of Fekete-Szegö has been solved for various subclasses of S, such as the class $S^*(\beta)$, $C(\beta)$, the subclass of close-to-convex function, close-to-convex functions of order α type β, etc. In [3] Bhowmik et al[4] solved the problem of Fekete-Szegö for the class of concave univalent functions given by (1.1).

In this paper we solve the Fekete-Szegö problem with real and complex parameter for the class of concave univalent function defined by Fox-Wright's generalized hypergeometric functions.

2. Preliminaries

The study of operators plays an important role in geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better (see [11, 12, 13, 14]) also see [21, 15, 22] and the references cited therein.

Let f be defined by (1.1). For $g \in S$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product of f and g is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta. \tag{2.1}$$

Now we briefly recall the definitions of the special functions and operators used in this paper. For complex parameters $\alpha_1, \ldots, \alpha_p$ $(\frac{\alpha_j}{A_j} \neq 0, -1, \ldots; j = 1, 2, \ldots p)$ and $\beta_1, \ldots, \beta_q$ $(\frac{\beta_j}{B_j} \neq 0, -1, \ldots; j = 1, 2, \ldots q)$ by Fox's H-function we mean the Wright's generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j, B_j > 0$, give (rather general and typical examples of $H-$functions, not reducible to $G-$functions):

$$\begin{aligned} {}_p\Psi_q\left(\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{matrix} : z \right) &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \ldots \Gamma(\alpha_p + nA_p)}{\Gamma(\beta_1 + nB_1) \ldots \Gamma(\beta_q + nB_q)} \frac{z^n}{n!} \\ &= H^{1,p}_{p,q+1}\left[-z \middle| \begin{matrix} (1-\alpha_1, A_1), \ldots, (1-\alpha_p, A_p) \\ (0,1), (1-\beta_1, B_1), \ldots, (1-\beta_q, B_q) \end{matrix} \right], \end{aligned} \tag{2.2}$$

with $1 + \sum\limits_{n=1}^{q} B_n - \sum\limits_{n=1}^{p} A_n \geq 0$, $(p, q \in \mathbb{N} = 1, 2, 3, ...)$ and for suitably bounded values of $|z|$.

Note that when $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, they turn into the generalized hypergeometric functions

$$
{}_p\Psi_q\left(\begin{array}{c}(\alpha_1,1),\ldots,(\alpha_p,1)\\(\beta_1,1),\ldots,(\beta_q,1)\end{array}:z\right)=\left[\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{i=1}^{q}\Gamma(\beta_j)}\right]{}_pF_q(\alpha_1,\ldots,\alpha_p:\beta_1,\ldots,\beta_q:z),
$$

(2.3)

$p \leq q+1;\ p,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}; z \in \Delta.$

Now we recall the linear operator due to Srivastava [21](see[15]) and Wright [22] in terms of the Hadamard product (or convolution) involving the generalized hypergeometric function. Let $l,m \in \mathbb{N}$ and suppose that the parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ are also positive real numbers, then corresponding to a function

$$
{}_l\Phi_m[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m};z]
$$

defined by

$$
{}_l\Phi_m[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m};z] = \Omega z {}_l\Psi_m[(\alpha_j,A_j)_{1,l}(\beta_j,B_j)_{1,m};z], \tag{2.4}
$$

where $\Omega = \left(\prod\limits_{j=0}^{l}\Gamma(\alpha_j)\right)^{-1}\left(\prod\limits_{j=0}^{m}\Gamma(\beta_j)\right)$. We consider a linear operator

$$
\mathcal{W}[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m}] : \mathcal{A}\to\mathcal{A}
$$

defined by the following Hadamard product (or convolution)

$$
\mathcal{W}[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m}](f)(z) \quad := \quad z\, {}_l\Phi_m[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m};z] * f(z).
$$

We observe that, for $f(z)$ of the form(1.1), we have

$$
\mathcal{W}[(\alpha_j,A_j)_{1,l};(\beta_j,B_j)_{1,m}]f(z) = z + \sum_{n=2}^{\infty}\sigma_n(\alpha_1)\, a_n z^n, \tag{2.5}
$$

where

$$
\sigma_n(\alpha_1) = \frac{\Omega\ \Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_l + A_l(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_m + B_m(n-1))}, \tag{2.6}
$$

for convenience, we write

$$
\mathcal{W}_m^l f(z) = \mathcal{W}[(\alpha_1,A_1),\ldots,(\alpha_l,A_l);(\beta_1,B_1),\ldots,(\beta_m,B_m)]f(z). \tag{2.7}
$$

Remark 2.1. Other interesting and useful special cases of the Fox-Wright generalized hypergeometric function ${}_l\Psi_m$ defined by (2.2) include (for example) the generalized Bessel function

$${}_l\Psi_m(-;(\nu+1,\mu);-z) \equiv \mathcal{J}_\mu^\nu.$$

For $\mu = 1$, corresponds essentially to the classical Bessel function $J^\nu(z)$, and the generalized Mittag-Leffler function ${}_l\Psi_m((1,1);(\mu,\lambda);z) \equiv E_\mu^\lambda$.

Remark 2.2. By setting $A_j = 1(j = 1,...,l)$ and $B_j = 1(j = 1,...,m)$ in (2.4), we are led immediately to the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\Omega_l F_m(z) \equiv \Omega_l F_m(\alpha_1,\dots\alpha_l;\beta_1,\dots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\dots(\alpha_l)_n}{(\beta_1)_n\dots(\beta_m)_n}\frac{z^n}{n!}, \tag{2.8}$$
$$(l \le m+1;\ l,m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}; z \in \mathcal{U})$$

where $\mathbb{N}$ denotes the set of all positive integers, $(\alpha)_n$ is the Pochhammer symbol.

By setting $A_j = 1(j = 1,...,l)$ and $B_j = 1(j = 1,...,m)$ the linear operator $\mathcal{W}_m^l$ contains the Dziok-Srivastava operator given by

$$\mathcal{H}_m^l f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{(n-1)}\dots(\alpha_l)_{(n-1)}}{(\beta_1)_{(n-1)}\dots(\beta_m)_{(n-1)}}\frac{z^n}{(n-1)!}$$

and as its various special cases contain linear operators like Hovlov operator, Carlson-Shaffer operator, Ruscheweyh derivative operator, generalized Bernardi-Libera-Livingston operator and fractional derivative operator as remarked below: It is of interest to note that the following are the special cases of the Dziok-Srivastava linear operator.

Remark 2.3. For $f \in \mathcal{A}, H_1^2(\alpha,1;\beta)f(z) = \mathcal{L}(\alpha,\beta)f(z)$ was considered by Carlson and Shaffer [6].

Remark 2.4. For $f \in \mathcal{A}, H_1^2(\delta+1,1;1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^\delta f(z)$ the symbol $\mathcal{D}^\delta f(z)$ was introduced by Ruscheweyh [20]

Remark 2.5. For $f \in \mathcal{A}, H_1^2(c+1,1;c+2)f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt = \mathcal{J}_c f(z)$ where $c > -1$. The operator $\mathcal{J}_c$ was introduced by Bernardi[5]. In particular, the operator $\mathcal{J}_1$ was studied earlier by Libera [18] and Livingston [19].

Remark 2.6. For $f \in \mathcal{A}, H_1^2(2,1;2-\lambda)f(z) = \Gamma(2-\lambda)z^\lambda \mathcal{D}_z^\lambda f(z) = \Omega^\lambda f(z), \quad \lambda \notin \mathbb{N}\setminus\{1\}$.

Let S_W^* denote the class of functions analytically defined as

$$S_W^* = \left\{ f \in S :\ \Re\left(\frac{z(\mathcal{W}_m^l f(z))'}{\mathcal{W}_m^l f(z)}\right) > 0, \quad z \in \Delta \right\}.$$

We now recall some facts about the class $Co(\alpha)$ of concave univalent functions. A function $f : \Delta \to \mathbb{C}$ is said to belong to the family $Co(\alpha)$ if f satisfies the following

conditions:
(i) f is analytic in Δ with the standard normalization $f(0) = f'(0)-1 = 0$. In addition it satisfies $f(1) = \infty$
(ii) f maps Δ conformally onto a set whose complement is convex with respect to $\mathbb{C}$.
(iii) the opening angle of $f(\Delta)$ at ∞ is less than or equal to $\pi\alpha$, $\quad \alpha \in (1, 2]$.
This class has been extensively studied in the recent years. A detailed discussion about concave univalent functions can be found in [1] and [2] and the references therein. The analytic characterization for functions in $Co(\alpha), \alpha \in (1, 2]$ is that $f \in Co(\alpha)$ if and only if $\Re(P_f(z)) > 0$ in Δ, where

$$P_f(z) = \frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{(1+z)}{(1-z)} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

Let S^* denote the class of starlike functions. In [3], the authors used this characterization and proved the following theorem:

Theorem 2.1. *Let $\alpha \in (1, 2]$. A function $f \in Co(\alpha)$ if and only if there exists a $\phi \in S^*$ such that $f(z) = \Lambda_\phi$, where $\Lambda_\phi = \int_0^z \frac{1}{(1-t)^{\alpha+1}}\left(\frac{t}{\phi(t)}\right)^{(\alpha-1)/2} dt$.*

The lemma stated below is also noteworthy.

Lemma 2.1. *Let $\phi(z) = z + b_2z^2 + b_3z^3 + \cdots \in S^*$. Then $|b_3 - \lambda b_2^2| \le max\{1, |3 - 4\lambda|\}$, which is sharp for the Koebe function k if $|\lambda - 3/4| \ge 1/4$ and for $(k(z)^2)^{\frac{1}{2}} = \frac{z}{1-z^2}$ if $|\lambda - 3/4| \le 1/4$.*

Recall from Theorem 2.1 that $f \in Co(\alpha)$ if and only if there exists a function $\phi \in S_W$, $\quad \phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ such that

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}}\left(\frac{z}{W[\alpha_1]\phi(z)}\right)^{\frac{\alpha-1}{2}} \tag{2.9}$$

where f has the form (1.1). Comparing the coefficients of z and z^2 on both sides of (2.9), we get

$$\begin{aligned} a_2 &= \frac{\alpha+1}{2} - \frac{\alpha-1}{4}\sigma_2(\alpha_1)\phi_2 \\ a_3 &= \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\alpha^2-1}{6}\sigma_2(\alpha_1)\phi_2 - \frac{\alpha-1}{6}\sigma_3(\alpha_1)\phi_3 + \frac{\alpha^2-1}{24}(\sigma_2(\alpha_1))^2\phi_2^2. \end{aligned}$$

Computation of $a_3 - \lambda a_2^2$ yields

$$\begin{aligned} a_3 - \lambda a_2^2 = & \quad \tfrac{(\alpha+1)^2}{4}\left[\tfrac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] + \left[\tfrac{\alpha^2-1}{4}\right]\left(\lambda - \tfrac{2}{3}\right)\sigma_2(\alpha_1)\phi_2 \\ & - \quad \tfrac{\alpha-1}{6}\sigma_3(\alpha_1)\left[\phi_3 - \left(\tfrac{2(\alpha+1)-3\lambda(\alpha-1)}{8\sigma_3(\alpha_1)}\right)(\sigma_2(\alpha_1))^2\phi_2^2\right]. \end{aligned} \tag{2.10}$$

We investigate the maximum value of the function $|a_3 - \lambda a_2^2|$ by considering several cases depending on the range of λ.

Case 1: Let $\lambda \in \left(-\infty, \frac{2(\alpha+1)((\sigma_2(\alpha_1))^2-8\sigma_3(\alpha_1))}{3(\sigma_2(\alpha_1))^2(\alpha-1)}\right)$. Note that this assumption is equivalent to

$$\frac{2(\alpha+1)(\sigma_2(\alpha_1))^2 - 3\lambda(\alpha-1)(\sigma_2(\alpha_1))^2}{8\sigma_3(\alpha_1)} \geq 1$$

and the first term in the last expression is non-negative. Using Lemma for the last term in (12) and since $|\phi_2| \leq 2/\sigma_2(\alpha_1)$ we have

$$\begin{aligned} |a_3 - \lambda a_2^2| \quad &\leq \quad \frac{\alpha^2+5}{6} + \frac{(\alpha^2-1)\sigma_2(\alpha_1)(\sigma_2(\alpha_1)+2\sigma_3(\alpha_1))}{6\sigma_3(\alpha_1)} \\ &- \quad \lambda\left[\frac{(\alpha+1)^2\sigma_3(\alpha_1) + 2\sigma_3(\alpha_1)\sigma_2(\alpha_1)(\alpha^2-1) + (\sigma_2(\alpha_1))^2(\alpha-1)^2}{4\sigma_3(\alpha_1)}\right]. \end{aligned}$$

Case 2: Let $\lambda \geq \frac{2(\alpha+2)}{3(\alpha+1)}$. This implies that the first term in (12) is non-positive. Also, we have $\lambda \geq 2/3$. Therefore

$$\left(\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{8}\right)\frac{(\sigma_2(\alpha_1))^2}{\sigma_3(\alpha_1)} \leq \frac{(\sigma_2(\alpha_1))^2}{2\sigma_3(\alpha_1)}.$$

By lemma (2.1),

$$\left|\phi_3 - \left(\frac{2(\alpha+1) - 3\lambda(\alpha-1)}{8}\right)\phi_2^2\frac{(\sigma_2\alpha_1)^2}{\sigma_3\alpha_1}\right| \leq 3 - \frac{2}{\sigma_3(\alpha_1)}.$$

Using this inequality and the fact that $|\phi_2| \leq 2/\sigma_2(\alpha_1)$, in (12), and simplifying the above inequality, we get

$$\begin{aligned} |a_3 - \lambda a_2^2| \quad &\leq \frac{\alpha-1}{2} - \left(\frac{(\alpha+1)(\alpha+2)}{6} + (\alpha^2-1)\sigma_3(\alpha_1) - \frac{\alpha-1}{2\sigma_3(\alpha_1)}\right) \\ &+\lambda\left(\frac{(\alpha+1)^2}{4} + \frac{(\alpha^2-1)\sigma_3(\alpha_1)}{2} - \frac{(\alpha+1)(\alpha+2)}{6}\right). \end{aligned}$$

Case 3: To get the complete solution to Fekete-Szegö problem, consider the case

$$\lambda \in \left(\frac{2(\alpha+1)((\sigma_2(\alpha_1))^2 - 8\sigma_3(\alpha_1))}{3(\sigma_2(\alpha_1))^2(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)}\right).$$

We now deal with this case by using (11) and (12) together with the representation formula for $\phi \in S_W^*$, namely

$$\frac{z(W[\alpha_1]\phi(z))'}{W[\alpha_1]\phi(z)} = \frac{1+zw(z)}{1-zw(z)}, \tag{2.11}$$

where $w : \Delta \to \bar{\Delta}$ is a function analytic in Δ with Taylor's series $w(z) = \sum_{n=0}^{\infty} c_n z^n$. Substituting the respective Taylor's series on both sides of (2.11), and comparing the coefficients of z and z^2, we get

$$\phi_2 = \frac{2c_0}{\sigma_2(\alpha_1)}, \quad \phi_3 = \frac{c_1 + 3c_0^2}{\sigma_3(\alpha_1)}. \tag{2.12}$$

Substituting these in (2.13) we get

$$a_3 - \lambda a_2^2 = \frac{(\alpha+1)^2}{4}\left(\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right) + \frac{(\alpha^2-1)}{2}\left(\lambda - \frac{2}{3}\right)c_0$$
$$+ \left(\frac{-(\alpha-1)}{2} + \frac{(\alpha^2-1)}{6} - \frac{\lambda(\alpha^2-1)}{4}\right)c_0^2 - \left(\frac{\alpha-1}{6}\right)c_1$$
$$= A + Bc_0 + Cc_0^2 + Dc_1,$$

where

$$A = \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\lambda(\alpha+1)^2}{4}; \quad B = (\alpha^2-1)(\frac{\lambda}{2} - \frac{1}{3})$$
$$C = \frac{-(\alpha-1)(4-2\alpha+3\lambda(\alpha-1))}{12} \quad \text{and} \quad D = \frac{-(\alpha-1)}{6}.$$

Using the fact that $|c_1| \le 1 - |c_0|^2$, we have

$$|a_3 - \lambda a_2^2| \le |A + Bc_0 + Cc_0^2| + |D|(1 - |c_0|^2). \tag{2.13}$$

Let $c_0 = re^{i\theta}$. We now find the maximum value of (2.13). For this, we first find the maximum of $|A + Bc_0 + Cc_0^2|^2$, where we fix r and vary θ. Substituting for c_0, we get $|A + Bc_0 + Cc_0^2|^2 = f(r,\theta)$ where

$$f(r,\theta) = (A - Cr^2)^2 + B^2r^2 + (2ABr + 2BCr^3)cos\theta + 4ACr^2cos^2\theta. \tag{2.14}$$

To get the upper bounds of $|a_3 - \lambda a_2^2|$, we have to find the largest value of $f(r,\theta)$ where $r \in (0.1]$. Letting $cos\theta = x$, in (2.14), it becomes

$$h(x) = (A - Cr^2)^2 + B^2r^2 + (2ABr + 2BCr^3)x + 4ACr^2x^2, \quad x \in [-1,1]. \tag{2.15}$$

To do this we consider several subclasses of

$$\Big(\frac{2(\alpha+1)(\sigma_2(\alpha_1))^2 - 8\sigma_3(\alpha_1)}{3(\sigma_2(\alpha_1))^2(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)}\Big).$$

Case 3A: Let

$$\lambda \in \left(\frac{2(\alpha+1)((\sigma_2(\alpha_1))^2 - 8\sigma_3(\alpha_1))}{3(\sigma_2(\alpha_1))^2(\alpha-1)}, \frac{2(\alpha-2)}{3(\alpha-1)}\right).$$

In this range, $C > 0, B < 0$ and $A + Cr^2 > 0$, for $r \in (0,1]$. Therefore $h(x)$ attains its maximum value for any $r \in (0,1]$, at $x = -1$. Let

$$g(r) = A - Br + Cr^2 + \frac{\alpha-1}{6}(1 - r^2).$$

We now find the maximum value of $g(r)$.

$$g'(r) = -B + 2Cr - \frac{\alpha-1}{3}r.$$

since $g'(0) = -B > 0$ and

$$g'(1) = -B + 2C - \frac{\alpha - 1}{3} = \frac{\alpha - 1}{6(-6\lambda + 4(\alpha - 1))} > 0,$$

the maximum value is attained at the boundary that is at $r = 1$. Therefore

$$|a_3 - \lambda a_2^2| \leq g(r) \leq g(1) = \frac{2\alpha^2 + 1}{3} - \lambda\alpha^2. \tag{2.16}$$

Case 3B: Let

$$\lambda = \frac{2(\alpha - 2)}{3(\alpha - 1)}.$$

Clearly, $C = 0$ and hence $h(x) = A^2 + B^2r^2 + 2Brx$ is a linear function of x which has maximum value at $x = -1$. Proceeding as in the previous case, we get the same maximum value for $|a_3 - \lambda a_2^2|$.

Case 3C: Suppose

$$\lambda \in \left(\frac{2(\alpha - 2)}{3(\alpha - 1)}, \frac{2(\alpha - 1)}{3\alpha}\right).$$

In this interval h is monotonic decreasing for $x \in [-1, 1]$. $h(x)$ has its maximum at $x(r) = \frac{-B}{4}(\frac{1}{Cr} + \frac{r}{A})$. It is sufficient to show that $x(r)$ is monotonic increasing and $x(r) < -1$.The assertion that $x(r)$ is monotonic increasing is clear. To prove that $x(1) < -1$ is equivalent to showing that $j(\lambda) = \alpha^2(3\lambda - 2)^2 - 4 + 3\lambda > 0$. This is easily verified. Hence we also get here the same upper bound for $|a_3 - \lambda a_2^2|$ as in the case of $3A$ and $3B$. From the cases $3A, 3B, 3C$ we conclude that

$$|a_3 - \lambda a_2^2| \leq \frac{2\alpha^2 + 1}{3} - \lambda\alpha^2, \lambda \in \left(\frac{2(\alpha + 1)(\sigma_2(\alpha_1))^2 - 8\sigma_3(\alpha_1)}{3(\sigma_2(\alpha_1))^2(\alpha - 1)}, \frac{2(\alpha - 2)}{3\alpha}\right). \tag{2.17}$$

Case 3D: Let

$$\lambda \in \left[\frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3}\right).$$

The roots of $j(\lambda) = 0$ are

$$\lambda_1 = \frac{4\alpha^2 - 1 - \sqrt{8\alpha^2 + 1}}{6\alpha^2}, \quad \text{and} \quad \lambda_2 = \frac{4\alpha^2 - 1 + \sqrt{8\alpha^2 + 1}}{6\alpha^2}.$$

Note that $\lambda_2 > \lambda_1$. For $\lambda \in [\frac{2\alpha_1 - 1)}{3\alpha}, \lambda_1)$, h has maximum value at $x = -1$ and g has maximum value at

$$r_m = \frac{-B}{-2C + \frac{\alpha - 1}{3}} \in (0, 1].$$

Hence the maximum of Fekete-Szegö functional is

$$g(r_m) = A - Br_m + Cr_m^2 + \frac{\alpha - 1}{6}(1 - r_m^2)$$
$$= \frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)}. \tag{2.18}$$

For

$$\lambda \in [\lambda_1, \frac{2}{3}), r_0 = \frac{B}{2c\left(1 + \sqrt{1 - \frac{B^2}{4AC}}\right)} \in (0, 1]$$

is the unique solution of $x(r) = -1$ in the interval (0,1]. Clearly $r_m < r_0$ for $\lambda < 2/3$. Further,the function

$$k(r) = \sqrt{h(x(r))} + \frac{\alpha - 1}{6}(1 - r^2) = (A - Cr^2)\sqrt{1 - \frac{B^2}{4AC}} + \frac{\alpha - 1}{6}(1 - r^2)$$

is monotonic decreasing for $r > r_0$. Hence the maximum value of $|a_3 - \lambda a_2^2|$ is $g(r_m)$.

Case 3E: Let $\lambda = \frac{2}{3}$. Then, $B = 0, C = \frac{-(\alpha-1)}{6}$. Thus $|a_3 - \lambda a_2^2| = \frac{\alpha}{3}$.

From 3D and 3E, we conclude that

$$|a_3 - \lambda a_2^2| \leq \frac{\alpha(10 - 9\lambda) - (3\lambda - 2)}{9(2 - \lambda) + 3\alpha(3\lambda - 2)}. \tag{2.19}$$

Case 3F: Let $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$. Here $B > 0$. The function $x(r)$ is monotonic decreasing

$$r_1 = \frac{B}{-2c(1 + \sqrt{1 - \frac{B^2}{4AC}}} \in (0, 1]$$

is the unique solution of $x(r) = 1$ lying in (0,1].For $r < r_1$, $h(x) \leq h(1)$, consider the function

$$l(r) = A + Br + Cr^2 + \frac{(\alpha - 1)}{6}(1 - r^2).$$

The maximum value of this function is obtained at

$$r_n = \frac{B}{-2C + \frac{\alpha - 1}{3}}, r_n > r_1.$$

Since $k(r)$ is monotonic increasing, the maximum value of the Fekete-Szegö functional is

$$k(1) = (A - C)\sqrt{1 - \frac{B^2}{4AC}} = \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}}$$

which is attained for $c_0 = e^{i\theta_0}$, where $cos\theta_0 = \frac{-B(A+C)}{4AC}$. In this case, we have

$$|a_3 - \lambda a_2^2| \leq \alpha(1 - \lambda)\sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - \alpha^2(3\lambda - 2)^2}}, \ \lambda \in \left(\frac{2}{3}, \lambda_2\right]. \tag{2.20}$$

Case 3G: Let

$$\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right].$$

Since $x(1) < -1$ for these λ, the number

$$r_2 = \frac{B}{-2C(1-\sqrt{1-\frac{B^2}{4AC}})}$$

satisfies $x(r_2) = -1$ and $r_2 \in (0,1)$. For $r \leq r_2$, we can make similar considerations as in the preceding case.The maximum is attained at $x = 1$ or $x = -1$. For the values of λ under consideration, $A + C < 0$ and $A + Cr^2 < 0$ and therefore the maximum of (12) is attained at $x = -1$, i.e. for $c_0 = -r$. Hence for $r \in (r_2, 1]$, the maximum function is

$$n(r) = -A + Br - Cr^2 + \frac{\alpha-1}{6}(1-r^2). \tag{2.21}$$

Since $-C > \frac{\alpha-1}{6}, B > 0$, we get $n(r) \leq n(1)$ in this interval. Hence

$$|a_3 - \lambda a_2^2| \leq n(1) = -A + B - C = \lambda\alpha^2 - \frac{2\alpha^2+1}{3} \tag{2.22}$$

whenever

$$\lambda \in (\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}].$$

Equations (2.17),(2.18),(2.19),(2.20) and Case 3G give the following theorem:

Theorem 2.2. *For $\alpha \in (1,2]$, let $f \in Co(\alpha)$ have the expansion (1.1). Then:*

$$|a_3-\lambda a_2^2| \leq \begin{cases} \frac{\alpha^2+5}{6} + \frac{(\alpha^2-1)\sigma_2(\alpha_1)[\sigma_2(\alpha_1)+2\sigma_3(\alpha_1)]}{6\sigma_3(\alpha_1)} \\ -\lambda[\frac{(\alpha+1)^2\sigma_3(\alpha_1)+2\sigma_3(\alpha_1)\sigma_2(\alpha_1)(\alpha^2-1)+\sigma_2^2(\alpha_1)(\alpha-1)^2}{4\sigma_3(\alpha_1)}], & \lambda \in [-\infty, \frac{2(\alpha+1)\sigma_2^2(\alpha_1)-8\sigma_3(\alpha_1)}{3(\alpha-1)\sigma_2^2(\alpha_1)}] \\ \frac{2\alpha^2+1}{3} - \lambda\alpha^2, & \lambda \in [\frac{2(\alpha+1)\sigma_2^2(\alpha_1)-8\sigma_3(\alpha_1)}{3(\alpha-1)\sigma_2^2(\alpha_1)}, \frac{2(\alpha-1)}{3\alpha}] \\ \frac{\alpha(10-9\lambda)-(3\lambda-2)}{9(2-\lambda)+3\alpha(3\lambda-2)}, & \lambda \in [\frac{2(\alpha-1)}{3\alpha}, \frac{2}{3}] \\ \alpha(1-\lambda)\sqrt{\frac{12(1-\lambda)}{(4-3\lambda)^2-\alpha^2(3\lambda-2)^2}}, & \lambda \in [\frac{2}{3}, \lambda_2] \\ \lambda\alpha^2 - \frac{(2\alpha^2+1)}{3}, & \lambda \in [\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}] \\ \frac{\alpha-1}{2} - \left(\frac{(\alpha+1)(\alpha+2)}{6} + (\alpha^2-1)\sigma_3(\alpha_1) - \frac{\alpha-1}{2\sigma_3(\alpha_1)}\right) \\ +\lambda\left(\frac{(\alpha+1)^2}{4} + \frac{(\alpha^2-1)\sigma_3(\alpha_1)}{2} - \frac{(\alpha+1)(\alpha+2)}{6}\right), & \lambda \in [\frac{2(\alpha+2)}{3(\alpha+1)}, \infty) \end{cases}$$

where $\lambda_2 = \frac{4\alpha^2-1+\sqrt{8\alpha^2+1}}{6\alpha^2}$. The inequalities are sharp.

When λ is complex, we have the following:

Theorem 2.3. *Let $f \in Co(\alpha), \alpha \in [1,2]$ have the expansion (1.1). If λ is a complex number, then*

$$|a_3 - \lambda a_2^2| \leq \max\left\{\{1, \frac{1}{12}(\alpha+1)\, v(\alpha,\lambda)\}\right\},$$

where

$$v(\alpha,\lambda) = |(2-3\lambda)(\alpha+1)+2| + 2(\alpha-1)|3\lambda-2| + \frac{\alpha-1}{\alpha+1}|4-[2(\alpha+1)-3\lambda(\alpha-1)]|.$$

Proof. Substituting (11) in (9) we have

$$\begin{aligned} 12(a_3-\lambda a_2^2) &= (\alpha+1)[(2-3\lambda)(\alpha+1)+2] + 2(\alpha^2-1)(3\lambda-2)c_0 \\ &\quad -(\alpha-1)[6-[2(\alpha+1)-3\lambda(\alpha-1)]]c_0^2 + 2(1-\alpha)c_1. \end{aligned}$$

For λ complex,

$$\begin{aligned} 12|a_3-\lambda a_2^2| &\leq (\alpha+1)|(2-3\lambda)(\alpha+1)+2| + 2(\alpha^2-1)|3\lambda-2||c_0| \\ &\quad -(\alpha-1)|6-[2(\alpha+1)-3\lambda(\alpha-1)]||c_0|^2 + 2(1-\alpha)|c_1|. \end{aligned}$$

Using the fact that $|c_0| \leq 1$ and $|c_1| \leq 1-|c_0|^2$,

$$\begin{aligned} 12|a_3-\lambda a_2^2| &\leq (\alpha+1)[(2-3\lambda)(\alpha+1)+2] + 2(\alpha^2-1)(3\lambda-2)|c_0| \\ &\quad -(\alpha-1)[6-[2(\alpha+1)-3\lambda(\alpha-1)]]|c_0|^2 + 2(1-\alpha)(1-|c_0|^2). \end{aligned}$$

Thus, $12|a_3-\lambda a_2^2| \leq (\alpha+1)v(\alpha,\lambda)$ for $Re\, v(\alpha,\lambda) > 0$

where

$$v(\alpha,\lambda) = |(2-3\lambda)(\alpha+1)+2| + 2(\alpha-1)|3\lambda-2| + \frac{\alpha-1}{\alpha+1}|4-[2(\alpha+1)-3\lambda(\alpha-1)]|.$$

□

Remark 2.7. If $l = 2$ and $m = 1$ with $\alpha_1 = \mu+1(\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu+2$,where J_μ is a Bernardi operator [5] defined by

$$J_\mu f(z) := \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt \equiv H_1^2(\mu+1,1;\mu+2)f(z).$$

Note that the operator J_1 was studied earlier by Libera and Livingston. Various other interesting corollaries and consequences of our main results (which are asserted in above Theorems) can be derived similarly. Further, by setting $A_j = 1(j = 1,...,l)$ and $B_j = 1(j = 1,...,m)$, and specific choices of parameters l, m, α_1, β_1 the various results presented in this paper would provide interesting extensions and generalizations of S_W^*. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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REFERENCES

1. F.G.Avkhadiev,and K.J. Wirths,*Concave schlict functions with bounded opening angle at infinity,* Lobachevskii J. Math. **17**, (2005), 3-10.
2. F. G. Avkhadiev,Ch. Pommerenke and K.J.Wirths, *Sharp inequalities for the coefficients of concave schlict functions,* Comment. Math.Hele. **81**, (2006), 801-807.
3. S. Bhowmik, S. Ponnusamy,and K.J.Wirths, *Characterization and the pre-Schwarzian norm estimate for concave univalent functions,* Monatschfte f *ü*r Mathematik, online first DOI 10.007\ s00605-009-0146-7, (2009).
4. S. Bhowmik, S. Ponnusamy,and K.J.Wirths, *On the Fekete-Szegö problem for concave univalent functions,* Journal of Math.Analysis and appl. **373**, (2011), 432-448.
5. S. D. Bernardi, *Convex and starlike univalent functions,* Trans. Amer. Math. Soc.,**135** (1969), 429–446.
6. B. C. Carlson and S. B. Shaffer, it Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. **15** (1984), 737–745.
7. J.H. Choi,Y.C. Kim and T. Sugawa,*A general approach to Fekete-Szegȯ problem,* J.Math.Soc.Japan **59**, No. (3), (2007), 707-727.
8. J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function,* Intergral Transform Spec. Funct.,**14** (2003), 7–18.
9. P.L. Duren, *Univalent functions,* Springer Verlag, New York. Inc. (1983).
10. M. Fekete Szegö, G, Eine Bemerkung *ü ber ungerade schlict Funktionen,* J.London Math Soc. **8,** (1933), 85-89
11. V.N. Mishra, H.H. Khan, K. Khatri, L.N. Mishra, Hypergeometric Representation for Baskakov-Durrmeyer-Stancu Type Operators, Bulletin of Mathematical Analysis and Applications, Volume 5 Issue 3 (2013), Pages 18-26.
12. V.N. Mishra, K. Khatri, L.N. Mishra,*On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators,* Journal of Ultra Scientist of Physical Sciences, Vol. 24, No. (3)A, 2012, pp. 567-577.
13. V.N. Mishra, P. Sharma *Direct estimates for Durrmeyer-Baskakov-Stancu type operators using Hypergeometric representation,* Journal of Fractional Calculus and Applications, Vol. 6 (2) July 2015, pp. 1-10.
14. V.N. Mishra, *Some Problems on Approximations of Functions in Banach Spaces,* Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee - 247 667, Uttarakhand, India.
15. G. Murugusundaramoorthy, T.Rosy, *Subclasses of analytic functions associated with Fox-Wright's generalised hypergeometric functions based on Hilbert space operator,* Stud. Univ. Babes-Bolyai Math.,**56**, No. (3), (2011), 61-72.
16. W. Koepf,*On the Fekete-Szegö problem for close-to-convex functions,* Proc. Amer. Math. Soc. **101**, (1987), 420-433.
17. A. Pfluger, *The Fekete-Szegö inequality for complex parameter,* Complex Variables Theory Appl. **7**, (1986), no. 1-3, 149-160.
18. R. J. Libera,*Some classes of regular univalent functions* Proc. Amer. Math. Soc., **16** (1965), 755–758.
19. A. E. Livingston, *On the radius of univalence of certain analytic functions,* Proc. Amer. Math. Soc., **17** (1966), 352–357.

20. S. Ruscheweyh, *New criteria for univalent functions,* Proc. Amer. Math. Soc. **49** (1975), 109–115.

21. H. M. Srivastava, *Some Fox's-Wright generalized hypergeometric functions and associated families of convolution operators,* Applicable Analysis and Discrete Mathematics, **1** (2007), 56-71.

22. E.M.Wright, it The asymptotic expansion of the generalized hypergeometric function, Proc. London. Math. Soc.,**46** (1946), 389–408.

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