FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 3 (2020), 843-856
https://doi.org/10.22190/FUMI2003843H

EXISTENCE AND UNIQUENESS RESULTS FOR A COUPLED SYSTEM OF HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS

Mohamed Houas and Khellaf Ould Melha

Abstract. In this paper, we have studied existence and uniqueness of solutions for a coupled system of multi-point boundary value problems for Hadamard fractional differential equations. By applying principle contraction and Shaefer's fixed point theorem new existence results have been obtained.
Keywords: multi-point boundary value problems; Hadamard fractional differential equations; Shaefer's fixed point theorem.

1. Introduction

Differential equations of fractional order have proved to be very useful in the study of models of many phenomenons in various fields of science and engineering, such as: electrochemistry, physics, chemistry, viscoelasticity, control, image and signal processing. For more details, we refer the reader to [3, 5, 6, 7, 11, 12, 13, 14, 16, 18]. There has been a sign cant progress in the investigation of these equations in recent years, see [3, 8, 17, 18, 19]. More recently, a basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 3, 14, 16, 20, 22]. On the other hand, existence and uniqueness of solutions to boundary value problems for fractional differential equations has attracted the attention of many authors, see for example, [16, 17, 19] and the references therein. Moreover, the study of coupled systems of fractional order is also important in various problems of applied nature [2, 9, 10, 15, 24, 25]. Recently, many people have established the existence and uniqueness for solutions of some fractional systems, see [9, 10, 21, 23, 25] and the reference therein. In the last few decades, much attention has been focused on the study of the existence and uniqueness of solutions for boundary value problems of Riemann-Liouville type or Caputo type fractional

Received October 18, 2019; accepted February 20, 2020
2010 *Mathematics Subject Classification.* Primary 34A34; Secondary 34B10.

differential equations, see [21, 23, 24, 25]. There are few papers devoted to the research of the Hadamard fractional differential equations; see [2].

In this paper, we study the existence of solutions for a Hadamard coupled system of nonlinear fractional integro-differential equations given by:

$$\begin{cases} D^{\alpha}x\left(t\right)=f_{1}\left(t,y\left(t\right),D^{\delta}y\left(t\right)\right),1<\alpha\leq 2,t\in\left[1,T\right], \\ D^{\beta}y\left(t\right)=f_{2}\left(t,x\left(t\right),D^{\sigma}x\left(t\right)\right),1<\beta\leq 2,t\in\left[1,T\right], \\ x\left(1\right)=0,\ x\left(T\right)-\sum_{i=1}^{m}\lambda_{i}I^{p}x\left(\eta_{i}\right)=0, \\ y\left(1\right)=0,\ y\left(T\right)-\sum_{i=1}^{m}\mu_{i}I^{q}x\left(\xi_{i}\right)=0, \end{cases} \tag{1.1}$$

where $\sigma\leq\alpha-1,\delta\leq\beta-1$; $p,q>0$; $1<\eta_{i},\xi_{i}<T$ and $D^{\alpha},D^{\beta},D^{\delta}$ and D^{σ} are the Hadamard fractional derivatives, I^{p} and I^{q} are the Hadamard fractional integrals and f_{1},f_{2} are continuous functions on $[1,T]\times\mathbb{R}^{2}$.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4 an examples are treated illustrating our results.

2. Preliminaries

This section is devoted to the basic concepts of Hadamard type fractional calculus will be used throughout this paper [13].

Definition 2.1. The fractional derivative of $f:[1,\infty[\ \rightarrow\mathbb{R}$ in the sense of Hadamard is defined as:

$$D^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{f\left(s\right)}{s}ds,n-1<\alpha<n, \tag{2.1}$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the real number α and $log(t)=log_{e}(t)$.

Definition 2.2. The Hadamard fractional integral operator of order $\alpha>0$, for a continuous function f on $[1,\infty[$ is defined as:

$$I^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f\left(s\right)}{s}ds,\alpha>0, \tag{2.2}$$

where $\Gamma\left(\alpha\right):=\int_{0}^{\infty}e^{-u}u^{\alpha-1}du$.

Lemma 2.1. *Let $\alpha > 0$. Then*

$$I^{\alpha}D^{\alpha}x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{\alpha - i}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

We give also an auxiliary lemma to define the solutions for the problem (1.1).

Lemma 2.2. *Let $g \in C([1,T], \mathbb{R})$, the solution of the boundary value problem*

$$\begin{cases} D^{\alpha}x(t) = g(t), 1 < \alpha \leq 2, t \in [1,T], \\ x(1) = 0, x(T) = \sum_{i=1}^{m} \lambda_i I^p x(\eta_i), \end{cases} \tag{2.4}$$

is given by:

$$\begin{array}{c} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ + \frac{(\log t)^{\alpha-1}}{\Pi} \left[\frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds \right. \\ \left. - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right], \end{array} \tag{2.5}$$

where

$$\Pi = \frac{1}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}}. \tag{2.6}$$

Proof. As argued in [13], for $c_i \in \mathbb{R}, i = 1, 2$, and by lemma 3, the general solution of equation of problem (2.4) is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{2.7}$$

Using the boundary conditions for (2.4), we find that $c_2 = 0$.

For c_1, we have

$$\begin{array}{c} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log T)^{\alpha-1} \\ = \frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}. \end{array} \tag{2.8}$$

which gives

$$\begin{array}{c} c_1 = \frac{\frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}} \\ - \frac{\frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}} \end{array} \tag{2.9}$$

Substituting the value of c_1 and c_2 in (2.7), we get (2.5). □

3. Main Results

Let us introduce the spaces $X = \{x : x \in C^1([1,T]), D^\sigma x \in C([1,T])\}$ and

$$Y = \{y : y \in C^1([1,T]), D^\delta y \in C([1,T])\}$$

endowed with the norm $\|x\|_X = \|x\| + \|D^\sigma x\|$; with

$$\|x\| = \sup_{t\in[1,T]} |x(t)|, \ \|D^\sigma x\| = \sup_{t\in[1,T]} |D^\sigma x(t)|,$$

and $\|y\|_Y = \|y\| + \left\|D^\delta y\right\|$; with

$$\|y\| = \sup_{t\in[1,T]} |y(t)|, \ \left\|D^\delta y\right\| = \sup_{t\in[1,T]} \left|D^\delta y(t)\right|.$$

Obviously, $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are a Banach spaces. The product space $\left(X \times Y, \|(x,y)\|_{X\times Y}\right)$ is also Banach space with norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$. Let us now introduce the quantities:

$$\begin{aligned}
N_1 &= \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^m \lambda_i(\log\eta_i)^{p+\alpha}}{\Gamma(p+\alpha+1)} + \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right),\\
N_2 &= \tfrac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)} + \tfrac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\tfrac{\sum_{i=1}^m \lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right),\\
N_3 &= \tfrac{(\log T)^\beta}{\Gamma(\beta+1)} + \tfrac{(\log T)^{\beta-1}}{|\Delta|}\left(\tfrac{\sum_{i=1}^m \mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)} + \tfrac{(\log T)^\beta}{\Gamma(\beta+1)}\right),\\
N_4 &= \tfrac{(\log T)^{\beta-\delta}}{\Gamma(\beta-\delta+1)} + \tfrac{\Gamma(\beta)(\log t)^{\beta-\delta-1}}{\Gamma(\beta-\delta)|\Delta|}\left(\tfrac{\sum_{i=1}^m \mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)} + \tfrac{(\log T)^\beta}{\Gamma(\beta+1)}\right),
\end{aligned}$$

which

$$\Lambda = \frac{1}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)}\sum_{i=1}^m \lambda_i(\log\eta_i)^{p+\alpha-1}},$$

and

$$\Delta = \frac{1}{(\log T)^{\beta-1} - \frac{\Gamma(\beta)}{\Gamma(q+\beta)}\sum_{i=1}^m \mu_i(\log\xi_i)^{q+\beta-1}}.$$

We list also the following hypotheses:

(H1) The functions $f_1, f_2 : [1,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H2) There exists a nonnegative continuous functions $a_i, b_i \in C([1,T]), i = 1,2$ such that for all $t \in [1,T]$ and $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| &\le a_1(t)|x_1-x_2| + b_1(t)|y_1-y_2|,\\
|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)| &\le a_2(t)|x_1-x_2| + b_2(t)|y_1-y_2|,
\end{aligned}$$

with

$$\begin{aligned}
\omega_1 = \sup\nolimits_{t\in[1,T]} a_1(t), \omega_2 = \sup\nolimits_{t\in[1,T]} b_1(t),\\
\varpi_1 = \sup\nolimits_{t\in[1,T]} a_2(t), \ \varpi_2 = \sup\nolimits_{t\in[1,T]} b_2(t).
\end{aligned}$$

(H3) There exists a nonnegative functions $l_1(t)$ and $l_2(t)$ such that

$$|f_1(t,x,y)| \le l_1(t), |f_2(t,x,y)| \le l_2(t) \text{ for each } t \in [1,T] \text{ and all } x,y \in \mathbb{R},$$

with

$$L_1 = \sup_{t\in[1,T]} l_1(t), L_2 = \sup_{t\in[1,T]} l_2(t).$$

Our first result is based on Banach contraction principle:

Theorem 3.1. *Suppose that the hypothesis (H2) holds.*
If

$$(N_1+N_2)(\omega_1+\omega_2)+(N_3+N_4)(\varpi_1+\varpi_2)<1, \tag{3.1}$$

then the boundary value problem (1.1) has a unique solution on $[1,T]$.

Proof. Consider the operator $\phi : X \times Y \to X \times Y$ defined by:

$$\phi(x,y)(t) := (\phi_1 y(t), \phi_2 x(t)), t \in [1,T], \tag{3.2}$$

where

$$\begin{aligned}\phi_1 y(t) := \quad & \tfrac{1}{\Gamma(\alpha)}\int_1^t \left(\log\tfrac{t}{s}\right)^{\alpha-1}\tfrac{f_1\left(s,y(s),D^\delta y(s)\right)}{s}ds \\ & +\tfrac{(\log t)^{\alpha-1}}{\Lambda}\left[\tfrac{\sum_{i=1}^m \lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{f_1\left(s,y(s),D^\delta y(s)\right)}{s}ds\right. \\ & \left.-\tfrac{1}{\Gamma(\beta)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\beta-1}\tfrac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right].\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\phi_2 x(t) := \tfrac{1}{\Gamma(\beta)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\beta-1}\tfrac{f_2(s,x(s),D^\sigma x(s))}{s}ds \\ +\tfrac{(\log t)^{\beta-1}}{\Delta}\left[\tfrac{\sum_{i=1}^m \mu_i}{\Gamma(q+\beta)}\int_1^{\xi_i}\left(\log\tfrac{\xi_i}{s}\right)^{\beta+q-1}\tfrac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right. \\ +\tfrac{(\log t)^{\beta-1}}{\Delta}\left[\tfrac{\sum_{i=1}^m \mu_i}{\Gamma(q+\beta)}\int_1^{\xi_i}\left(\log\tfrac{\xi_i}{s}\right)^{\beta+q-1}\tfrac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right.\end{aligned} \tag{3.4}$$

We shall prove that ϕ is contraction mapping.

Let $(x,y),(x_1,y_1) \in X \times Y$. Then, for each $t \in [1,T]$, we have:

$$\begin{aligned}|\phi_1 y(t)-\phi_1 y_1(t)| \quad & \le \tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds \\ & +\tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left[\tfrac{\sum_{i=1}^m \lambda_i}{\Gamma(p+\alpha)}\times\right. \\ & \int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds \\ & \left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.5}$$

Thanks to (H2),we obtain

$$
\begin{aligned}
&|\phi_1 y(t) - \phi_1 y_1(t)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y - D^\delta y_1\right\|}{s} ds \\
&+ \left|\frac{(\log T)^{\alpha-1}}{\Lambda}\right| \frac{\sum_{i=1}^m \lambda_i}{\Gamma(p+\alpha)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y - D^\delta y_1\right\|}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y - D^\delta y_1\right\|}{s} ds.
\end{aligned} \tag{3.6}
$$

Consequently,

$$
\begin{aligned}
&|\phi_1 y(t) - \phi_1 y_1(t)| \\
&\le \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{p+\alpha}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right](\omega_1+\omega_2) \\
&\times \left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right),
\end{aligned}
$$

which implies that

$$
\|\phi_1(y) - \phi_1(y_1)\| \le N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right),
$$

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\le \frac{1}{\Gamma(\alpha-\sigma)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\sigma-1} \frac{\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s} ds \\
&+ \frac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|} \times \\
&\left[\frac{\sum_{i=1}^m \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{\left|f_1\left(s,y(s),D^\delta y(s)\right) - f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s} ds\right. \\
&\left.+ \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{\left|f_1\left(s,y(s),D^\delta y(s)\right) - f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s} ds\right].
\end{aligned} \tag{3.7}
$$

By (H2), we have

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\le \frac{(\log T)^{\alpha-\sigma}(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)} \\
&+ \frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)(\omega_1+\omega_2) \\
&\times \left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).
\end{aligned} \tag{3.8}
$$

Hence,

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\le \left[\frac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)} + \frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right] \times \\
&(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).
\end{aligned} \tag{3.9}
$$

Therefore,

$$
|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \le N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.10}
$$

Consequently,

$$
\|D^\sigma \phi_1(y) - D^\sigma \phi_1(y_1)\| \le N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.11}
$$

By (**??**) and (3.11), we can write

$$\|\phi_1(y)-\phi_1(y_1)\|_X \le (N_1+N_2)(\omega_1+\omega_2)\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right). \tag{3.12}$$

With the same arguments as before, we have

$$\|\phi_2(x)-\phi_2(x_1)\|_Y \le (N_3+N_4)(\varpi_1+\varpi_2)\left(\|x-x_1\|+\|D^\sigma x-D^\sigma x_1\|\right). \tag{3.13}$$

And by,(3.12) and (3.13), we obtain

$$\begin{aligned}&\|\phi(x,y)-\phi(x_1,y_1)\|_{X\times Y}\\ &\le [(N_1+N_2)(\omega_1+\omega_2)+(N_3+N_4)(\varpi_1+\varpi_2)]\,\|(x-x_1,y-y_1)\|_{X\times Y}.\end{aligned} \tag{3.14}$$

Thanks to (3.1), we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution of the coupled system (1.1). □

The second main result is the following theorem:

Theorem 3.2. *Assume that the hypotheses (H1) and (H3) are satisfied.*
Then, the coupled system (1.1) has at least a solution on $[1,T]$.

Proof. We shall use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on $X\times Y$. It is to note that ϕ is continuous on $X\times Y$ in view of the continuity of f_1 and f_2 (hypothesis (H1)).

Now, We shall prove that ϕ maps bounded sets into bounded sets in $X\times Y$: Taking $r>0$, and $(x,y)\in B_r$, $B_r := \{(x,y)\in X\times Y; \|(x,y)\|_{X\times Y}\le r\}$, then for each $t\in[1,T]$, we have:

$$\begin{aligned}|\phi_1 y(t)| &\le \tfrac{1}{\Gamma(\alpha)}\int_1^t \left(\log\tfrac{t}{s}\right)^{\alpha-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\tfrac{(\log t)^{\alpha-1}}{|\Lambda|}\left[\tfrac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\tfrac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.15}$$

Thanks to (H3), we can write
(3.16)

$$\begin{aligned}|\phi_1 y(t)| &\le \tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}\tfrac{\sup_{t\in J} l_1(t)}{s}ds\\ &+\tfrac{(\log t)^{\alpha-1}}{|\Lambda|}\left[\tfrac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{\sup_{t\in J} l_1(t)}{s}ds\right.\\ &\left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\tfrac{\sup_{t\in J} l_1(t)}{s}ds\right].\\ &\le \sup_{t\in J} l_1(t)\left[\tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}+\tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned} &|\phi_1 y(t)| \\ &\le L_1\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\frac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right], t\in[1,T]. \end{aligned} \tag{3.17}$$

Hence, we have

$$\begin{aligned} &\|\phi_1(y)\| \\ &\le L_1\left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\frac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]=L_1N_1. \end{aligned} \tag{3.18}$$

On the other hand,

$$\begin{aligned} |D^{\sigma}\phi_1 y(t)| \le\ & \frac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\sigma-1}\frac{\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds \\ &+\frac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left[\frac{\sum_{i=1}^{m}\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds\right]. \end{aligned}$$

By (H3), we have,

$$\begin{aligned} &|D^{\sigma}\phi_1 y(t)| \\ &\le L_1\left[\frac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\frac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{3.19}$$

Consequently we obtain,

$$|D^{\sigma}\phi_1 y(t)|\le L_1N_2, t\in[1,T]. \tag{3.20}$$

Therefore,

$$\|D^{\sigma}\phi_1(y)\|\le L_1N_2. \tag{3.21}$$

Combining (3.18) and (3.21), yields

$$\|\phi_1(y)\|_X\le L_1(N_1+N_2). \tag{3.22}$$

Similarly, it can be shown that,

$$\|\phi_2(x)\|_Y\le L_2(N_3+N_4). \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\|\phi(x,y)\|_{X\times Y}\le L_1(N_1+N_2)+L_2(N_3+N_4). \tag{3.24}$$

Consequently

$$\|\phi(x,y)\|_{X\times Y}<\infty. \tag{3.25}$$

Next, we will prove that ϕ is equicontinuous on $[1,T]$: For $(x,y)\in B_r$, and $t_1,t_2\in [1,T]$, such that $t_1<t_2$. Thanks hypothesis (H3), we have:

$$\begin{aligned}|\phi_1 y(t_2)-\phi_1 y(t_1)| \leq & \frac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha)}\frac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{1}{s}ds \\ &-\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{1}{s}ds.\end{aligned}$$

Thus,

(3.26)

$$\begin{aligned}|\phi_1 y(t_2)-\phi_1 y(t_1)| \leq & \frac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ &+L_1\left|\frac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\right|\left(\frac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right).\end{aligned}$$

and using (H3),we obtain:

$$\begin{aligned}&|D^\sigma\phi_1 y(t_2)-D^\sigma\phi_1 y(t_1)| \\ &\leq \frac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-\sigma-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-\sigma-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-\sigma-1}\frac{1}{s}ds\right|\end{aligned}\tag{3.27}$$

Hence, by (3.26) and (3.27),we can write

$$\begin{aligned}\|\phi_1 y(t_2)-\phi_1 y(t_1)\|_X \leq & \frac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ &+L_1\left|\frac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\right|\left(\frac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right) \\ &+\frac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-\sigma-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-\sigma-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-\sigma-1}\frac{1}{s}ds\right| \\ &+\frac{L_1\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\left|\frac{(\log t_1)^{\alpha-\sigma-1}-(\log t_2)^{\alpha-\sigma-1}}{\Lambda}\right|\left(\frac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right).\end{aligned}\tag{3.28}$$

With the same arguments as before, we get

(3.29)

$$\begin{aligned}&\|\phi_1 x(t_2)-\phi_1 x(t_1)\|_Y \\ &\leq \frac{L_2}{\Gamma(\beta)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\beta-1}-\left(\log\frac{t_2}{s}\right)^{\beta-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_2}{\Gamma(\beta)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-1}\frac{1}{s}ds\right| \\ &+L_2\left|\frac{(\log t_1)^{\beta-1}-(\log t_2)^{\beta-1}}{\Delta}\right|\left(\frac{\sum_{i=1}^m\mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)}+\frac{(\log T)^\beta}{\Gamma(\beta+1)}\right) \\ &+\frac{L_2}{\Gamma(\beta-\delta)}\left|\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\beta-\delta-1}-\left(\log\frac{t_2}{s}\right)^{\beta-\delta-1}\right)\frac{1}{s}ds\right| \\ &+\frac{L_2}{\Gamma(\beta-\delta)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\beta-\delta-1}\frac{1}{s}ds\right| \\ &+\frac{L_2\Gamma(\beta)}{\Gamma(\beta-\delta)}\left|\frac{(\log t_1)^{\beta-\delta-1}-(\log t_2)^{\beta-\delta-1}}{\Delta}\right|\left(\frac{\sum_{i=1}^m\mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)}+\frac{(\log T)^\beta}{\Gamma(\beta+1)}\right).\end{aligned}$$

Thanks to (3.28) and (3.29), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|_{X\times Y}\to 0$ as $t_2\to t_1$ and by Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

Finally, we shall show that the set Ω defined by

$$\Omega=\{(x,y)\in X\times Y,(x,y)=\rho\phi(x,y),0<\rho<1\}, \tag{3.30}$$

is bounded:

Let $(x,y)\in\Omega$, then $(x,y)=\rho\phi(x,y)$, for some $0<\rho<1$. Thus, for each $t\in[1,T]$, we have:

$$x(t)=\rho\phi_1 y(t), y(t)=\rho\phi_2 x(t). \tag{3.31}$$

Then

$$\begin{aligned}\frac{1}{\rho}|x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\frac{\left|(\log T)^{\alpha-1}\right|}{|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.32}$$

Thanks to (H3), we can write

$$\begin{aligned}\frac{1}{\rho}|x(t)| &\leq \frac{L_1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\\ &+\frac{L_1(\log T)^{\alpha-1}}{|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{1}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{1}{s}ds\right].\end{aligned} \tag{3.33}$$

Therefore,

$$|x(t)|\leq\rho L_1\left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\frac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right]. \tag{3.34}$$

Hence,

$$|x(t)|\leq\rho L_1N_1. \tag{3.35}$$

On the other hand,

$$\begin{aligned}&\frac{1}{\rho}|D^\sigma x(t)|\\ &\leq\frac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\sigma-1}\frac{\left|f\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\frac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.36}$$

By (H3), we have

$$\begin{aligned} &\frac{1}{\rho}\left|D^{\sigma}x\left(t\right)\right| \\ &\leq L_1\left[\frac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\frac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{3.37}$$

Therefore,

$$\begin{aligned} &\left|D^{\sigma}x\left(t\right)\right| \\ &\leq \rho L_1\left[\frac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\frac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{3.38}$$

Thus,

$$\left\|D^{\sigma}\left(x\right)\right\| \leq \rho L_1 N_2. \tag{3.39}$$

From (3.35) and (3.39) we get

$$\|x\|_X \leq \rho L_1\left(N_1+N_2\right). \tag{3.40}$$

Analogously, we can obtain

$$\|y\|_Y \leq \rho L_2\left(N_3+N_4\right). \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\|(x,y)\|_{X\times Y} \leq \rho\left[L_1\left(N_1+N_2\right)+L_2\left(N_3+N_4\right)\right]. \tag{3.42}$$

Hence,

$$\|\phi\left(x,y\right)\|_{X\times Y} < \infty. \tag{3.43}$$

This shows that the set Ω is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that ϕ has at least a fixed point, which is a solution of coupled system (1.1). □

4. Examples

Example 4.1. Let us consider the Hadamard coupled system:

$$\begin{cases} D^{\frac{5}{3}}x\left(t\right)=\frac{1}{8(t+2)^2}\left(\frac{|y(t)|}{1+|y(t)|}+\frac{t\left|D^{\frac{1}{2}}y(t)\right|}{2\pi\left(1+\left|D^{\frac{1}{2}}y(t)\right|\right)}\right)+\cos\left(1+t+t^2\right), t\in[1,e]\,, \\ D^{\frac{5}{3}}y\left(t\right)=\frac{1}{20\pi+t^2}\left(\sin\left|x\left(t\right)\right|+\frac{t+t^2}{\pi}\sin\left|D^{\frac{1}{3}}x\left(t\right)\right|\right)+\ln\left(2+t^2\right), t\in[1,e]\,, \\ x\left(1\right)=0, x\left(e\right)=2I^{\frac{3}{2}}x(\frac{5}{4})+I^{\frac{3}{2}}x(\frac{4}{3})+\frac{7}{4}I^{\frac{3}{2}}x(\frac{7}{5}), \\ y\left(1\right)=0, y\left(e\right)=\frac{3}{2}I^{\frac{4}{3}}x(\frac{7}{6})+\frac{6}{5}I^{\frac{4}{3}}x(\frac{5}{4})+\frac{7}{6}I^{\frac{4}{3}}x(\frac{3}{2}). \end{cases} \tag{4.1}$$

For this example, we have for $t\in[1,e]$

$$\begin{aligned} f_1(t,x,y) &= \frac{1}{8(t+2)^2}\left(\frac{|x|}{1+|x|}+\frac{t|y|}{2\pi(1+|y|)}\right)+\cos\left(1+t+t^2\right),\ x,y\in\mathbb{R}, \\ f_2(t,x,y) &= \frac{1}{20\pi+t^2}\left(\sin|x|+\frac{t+t^2}{\pi}\sin|y|\right)+\ln\left(2+t^2\right),\ x,y\in\mathbb{R}. \end{aligned}$$

Taking $x, y, x_1, y_1 \in \mathbb{R}, t \in [1, e]$, then:

$$\begin{aligned} |f_1(t,x,y) - f_1(t,x_1,y_1)| &\leq \tfrac{1}{8(t+2)^2}|x-x_1| + \tfrac{t}{16\pi(t+2)^2}|y-y_1|, \\ |f_2(t,x,y) - f_2(t,x_1,y_1)| &\leq \tfrac{1}{20\pi+t^2}|x-x_1| + \tfrac{t+t^2}{\pi(20\pi+t^2)}|y-y_1|. \end{aligned}$$

So, we can take

$$a_1(t) = \frac{1}{8(t+2)^2}, b_1(t) = \frac{t}{16\pi(t+2)^2},$$

and

$$a_2(t) = \tfrac{1}{20\pi+t^2}, b_2(t) = \tfrac{t+t^2}{\pi(20\pi+t^2)}.$$

It follows then that

$$\begin{gathered} \omega_1 = \sup_{t\in[1,e]} a_1(t) = \tfrac{1}{72}, \omega_2 = \sup_{t\in[1,e]} b_1(t) = \tfrac{1}{144\pi}, \\ \varpi_1 = \sup_{t\in[1,e]} a_2(t) = \tfrac{1}{20\pi+1}, \varpi_2 = \sup_{t\in[1,e]} b_2(t) = \tfrac{e+e^2}{\pi(20\pi+e)}, \\ N_1 = 1,3234, N_2 = 1,5028, N_3 = 1,3153, N_4 = 1,4974. \end{gathered}$$

and $\Delta = 1.0246, \Lambda = 1.020$,

$$(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) = 0,305\,4 < 1.$$

Hence by Theorem 5, then the system (4.1) has a unique solution on $[1, e]$.

Example 4.2. Consider the following coupled system:

$$\text{(4.2)} \qquad \begin{cases} D^{\frac{7}{4}}x(t) = \frac{\sin\left(|y(t)| + \left|D^{\frac{3}{4}}y(t)\right|\right)}{t^2+5t+2}, t \in [1,e], \\ D^{\frac{6}{5}}y(t) = \frac{\cos\left(|x(t)| + \left|D^{\frac{1}{5}}x(t)\right|\right)}{t^2+t+20\pi}, t \in [1,e], \\ x(1) = 0, x(e) = \frac{6}{5}I^{\frac{5}{3}}\left(\frac{5}{4}\right) + \frac{7}{4}I^{\frac{5}{3}}\left(\frac{4}{3}\right) + \frac{7}{4}I^{\frac{5}{3}}\left(\frac{7}{5}\right), \\ y(1) = 0, y(e) = \frac{3}{2}I^{\frac{7}{5}}\left(\frac{7}{6}\right) + \frac{6}{5}I^{\frac{7}{5}}\left(\frac{5}{4}\right) + \frac{7}{6}I^{\frac{7}{5}}\left(\frac{3}{2}\right). \end{cases}$$

Then, we have

$$\begin{aligned} f_1(t,x,y) &= \frac{\sin\left(|y(t)| + \left|D^{\frac{3}{4}}y(t)\right|\right)}{t^2+5t+2}, x, y \in \mathbb{R}, \\ f_2(t,x,y) &= \frac{\cos\left(|x(t)| + \left|D^{\frac{1}{5}}x(t)\right|\right)}{t^2+t+20\pi}, x, y \in \mathbb{R}. \end{aligned}$$

Let $x, y \in \mathbb{R}$ and $t \in [1, e]$. Then

$$|f_1(t,x,y)| \leq \tfrac{1}{t^2+5t+2}, |f_2(t,x,y)| \leq \tfrac{1}{t^2+t+20\pi}.$$

So we take

$$l_1(t) = \frac{1}{t^2+5t+2}, l_2(t) = \frac{1}{t^2+t+20\pi}.$$

Then

$$L_1 = 0,1250, L_2 = 0,0154.$$

Thanks to Theorem 6, the system (4.2) has at least one solution on $[1, e]$.

REFERENCES

1. M. R. Ali, A.R. Hadhoud and H.M. Srivastava: *Solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method*. Advances in Difference Equations. Article ID 115, (2019), 1-14.
2. B. Ahmad, S.K. Ntouyas: *A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations*. Fract. Calc. Appl. Anal. 17(2), (2014), 348-360.
3. A. Anber, S. Belarbi and Z. Dahmani: *New existence and uniqueness results for fractional differential equations*. Analele Stiintifice ale Universitatii Ovidius Constanta Seria Matematica. Vol. 21(3), (2013), 33-41.
4. C.Z. Bai and J.X. Fang: *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations*. Applied Mathematics and Computation. 150(3), (2004) 611-621.
5. D. Delbosco, L. Rodino: *Existence and uniqueness for a nonlinear fractional differential equation*. J. Math. Anal. Appl. 204, 3(4), (1996), 429-440.
6. K. Diethelm, N.J. Ford: *Analysis of fractional differential equations*. J. Math. Anal. Appl. 265(2), (2002), 229-248 .
7. K. Diethelm, G. Walz: *Numerical solution of fraction order differential equations by extrapolation*. Numer. Algorithms. 16(3), (1998), 231-253,
8. M. Houas, Z. Dahmani, M. Benbachir: *New results for a boundary value problem for differential equations of arbitrary order*. International Journal of Modern Mathematical Sciences. 7(2), (2013), 195-211.
9. M. Houas, Z. Dahmani: *New results for a coupled system of fractional differential equations*. Facta Universitatis (NIS) Ser. Math. Inform. 28(2),(2013), 133-150.
10. M. Houas, Z. Dahmani: *New results for a system of two fractional differential equations involving n Caputo derivatives*. Kragujevac J. Math. 38 (2014), 283–301.
11. A.M.A. El-Sayed: *Nonlinear functional differential equations of arbitrary orders*. Nonlinear Anal. 33(2), (1998), 181-186.
12. V. Gafiychuk, B. Datsko, and V. Meleshko: *Mathematical modeling of time fractional reaction-diffusion systems*. Journal of Computational and Applied Mathematics. 220(1-2), (2008), 215-225.
13. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo: *Theory and applications of fractional differential equations*. North-Holland Mathematics Studies, 204, Elsevier Science B.V. Amsterdam, 2006.

14. V. Lakshmikantham, A.S. Vatsala: *Basic theory of fractional differential equations*. Nonlinear Anal. 69(8), (2008), 2677-2682.

15. J. Liang,Z. Liu, X. Wang: *Solvability for a couple system of nonlinear fractional differential equations in a Banach space*. Fractional Calculus and Applied Analysis. 16(1), (2013), 51-63.

16. F. Mainardi, Fractional calculus: *Some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics*. Springer, Vienna. 1997.

17. S.K. Ntouyas: *Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions*. Journal of Fractional Calculus and Applications. 3(9),(2012), 1-14.

18. I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary, L. Dorcak: *Analogue realizations of fractional-order controllers. Fractional order calculus and its applications*. Nonlinear Dynam. 29(4),(2002), 281-296.

19. M.U. Rehman ,R.A Khan and N.A Asif :*Three point boundary value problems for nonlinear fractional differential equations*. Acta Mathematica *Scientia* 2011,31B(4), 1337-1346.

20. H. M. Srivastava, A.M. A. El-Sayed and F. M. Gaafar: *A class of nonlinear boundary value problems for an arbitrary fractional-order differential equation with the Riemann-Stieltjes functional integral and infinite-point boundary conditions*. Symmetry. 10(10), (2018), 1-13.

21. X. Su: *Boundary value problem for a coupled system of nonlinear fractional differential equations*. Applied Mathematics Letters. 22(1),(2009, 64-69 .

22. I. Tejado, B. M. Vinagre, O.E. Traver, J. Prieto-Arranz and C. Nuevo-Gallardo: Back to basics: Meaning of the parameters of fractional order PID controllers. Mathematics.7(2019), Article ID 533, 1-10.

23. J. Wang, H. Xiang, Z. Liu: *Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations*. International Journal of Differential Equations. Article ID 186928, 12 pages, 2010.

24. W. Yang: *Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions*. Comput. Math. Appl. 63,(2012), 288-297.

25. Y. Zhang, Z. Bai, T. Feng: *Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance*. Comput. Math. Appl. 61,(2011) 1032-1047.

Mohamed HOUAS
Laboratory FIMA, UDBKM, University of Khemis Miliana, Algeria
houasmed@yahoo.fr

Khellaf OULD MELHA
Department of Mathematics
University of Chlef, Algeria
k.ouldmelha@univ-chlef.dz