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THE LEVINSON-TYPE FORMULA FOR A CLASS OF STURM-LIOUVILLE EQUATION

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Abstract. The boundary value problem

$$-\psi'' + q(x)\psi = \lambda^2\psi, \quad 0 < x < \infty,$$

$$\psi'(0) - (\alpha_0 + \alpha_1\lambda)\psi(0) = 0$$

is considered, where λ is a spectral parameter, $q(x)$ is real-valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty$$

with $\alpha_0, \alpha_1 \geq 0$ ($\alpha_0, \alpha_1 \in \mathbb{R}$).

In this paper, for above-mentioned boundary value problem, the scattering data is considered and the characteristics properties (such as continuity of the scattering function $S(\lambda)$ and giving the Levinson-type formula) of this data are studied.
Keywords: Scattering data; scattering function; Gelfand-Levitan-Marchenko equation; Levinson-type formula.

1. Introduction

Consider the boundary value problem

$$-\psi'' + q(x)\psi = \lambda^2\psi, \quad 0 < x < \infty, \tag{1.1}$$

$$\psi'(0) - (\alpha_0 + \alpha_1\lambda)\psi(0) = 0, \tag{1.2}$$

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where $q(x)$ is real valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty \tag{1.3}$$

and α_0, α_1 are real numbers, also $\alpha_0, \alpha_1 \geq 0$.

Spectral analysis when the spectral parameter appearing linearly on the half line for the boundary value problem (1.1) was studied in [3, 4],(1.2). In the case $q(x) \equiv 0$, this boundary value problem is given by application to the heat transmission problem in [2]. In the wave theory of mathematical physics and geophysics, the applications of the problems can also be found [1, 5, 20, 21, 22, 23].

It is known [15, 16] that the function which can be unique represented in the from

$$e(x,\lambda) = e^{i\lambda x} + \int_0^\infty K(x,t)e^{i\lambda t}dt, \tag{1.4}$$

is a Jost solution of the equation (1.1) for any λ on closed upper half plane, where the kernel $K(x,t)$ satisfies the relation

$$|K(x,t)| \leq \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\}$$

with

$$\sigma(x) \equiv \int_x^\infty |q(t)|dt, \qquad \sigma_1(x) \equiv \int_x^\infty \sigma(t)dt$$

and

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt.$$

The function $e(x,-\lambda)$ satisfies the equation (1.1) for each $\lambda \in \mathbb{R}\setminus\{0\}$ and the functions $e(x,\lambda)$ and $e(x,-\lambda)$ form a fundamental set of solutions for the differential equation (1.1). Their Wronskian is as follows:

$$W\{e(x,\lambda), e(x,-\lambda)\} = e'(x,\lambda)e(x,-\lambda) - e(x,\lambda)e'(x,-\lambda) = 2i\lambda.$$

Let $\varpi(x,\lambda)$ denote the a special solution of the equation (1.1) that satisfies the initial conditions

$$\varpi(0,\lambda) = 1, \qquad \varpi'(0,\lambda) = \alpha_0 + \alpha_1\lambda.$$

The following lemma 1.1 and lemma 1.2 which have been proved in [9] should be given in order to achieve the aim of the manuscript:

Lemma 1.1. *The identity*

$$\frac{2i\lambda\varpi(x,\lambda)}{e'(0,\lambda)-(\alpha_0+\alpha_1\lambda)e(0,\lambda)} = e(x,-\lambda) - S(\lambda)e(x,\lambda)$$

holds for all real $\lambda \neq 0$ *where*

$$S(\lambda) = \frac{e'(0,-\lambda)-(\alpha_0+\alpha_1\lambda)e(0,-\lambda)}{e'(0,\lambda)-(\alpha_0+\alpha_1\lambda)e(0,\lambda)} \tag{1.5}$$

and

$$|S(\lambda)| = 1.$$

Here, the function $S(\lambda)$ is represented by the formula (1.5). This function is called the scattering function of the boundary value problem (1.1)-(1.3).

The function $S(\lambda)$ is meromorphic function on the upper half plane ($Im\lambda > 0$). The zeros of the function $\varphi(\lambda) \equiv e'(0,\lambda) - (\alpha_0 + \alpha_1\lambda)e(0,\lambda)$ are the poles of the function $S(\lambda)$.

Lemma 1.2. *The function* $\varphi(\lambda)$ *may have only a finite number of zeros* $\lambda_1, \lambda_2, ..., \lambda_n$ *on the half plane* $Im\lambda > 0$ *and all these zeros don't lie on the imaginary axis. The zeros* $\varphi(\lambda)$ *and* $\varphi_1(\lambda) \equiv e'(0,-\lambda) - (\alpha_0 + \alpha_1\lambda)e(0,-\lambda)$ *are complex conjugate each other and the number of these zeros is equal.*

The number m_k is referred to the multiplicity of the zeros λ_k, $(k = 1, 2, ..., n)$ of the equation $\varphi(\lambda) = 0$. These λ_k is called the singular values of the boundary value problem (1.1)-(1.3).

We denote

$$f_j(x) = i \operatorname*{Res}_{\lambda=\lambda_j} \frac{\varphi_2(\lambda)}{\varphi(\lambda)} e^{i\lambda x},$$

where $\varphi_2(\lambda) = \hat{e}\,'(0,\lambda) - (\alpha_0 + \alpha_1\lambda)\hat{e}(0,\lambda)$ and $\hat{e}(x,\lambda)$ is a solution of the equation (1.1) (see [18, p.299]). We shall call the polynomial

$$P_k(x) = e^{-i\lambda_k x} f_k(x), \qquad k = 1, 2, ..., n,$$

with degree of $m_k - 1$ the normalization poliynomial for boundary value problem (1.1)-(1.3).

Let

$$F_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_0 - S(\lambda)]d\lambda,$$

$$F(x) = \sum_{k=1}^{n} f_k(x) + F_s(x), \tag{1.6}$$

where $S_0 = \frac{a+i}{a-i}$.

The kernel function $K(x,t)$ of the special solution (1.4) satisfies the integral equation

$$F(x+y)+K(x,y)+\int_x^\infty K(x,t)F(t+y)dt=0, \qquad x<y<\infty \tag{1.7}$$

for each $x \geq 0$.

The equation (1.7) is called the main equation (also called Gelfand-Levitan-Marchenko equation) of the inverse boundary value problem (1.1)-(1.3). This main equation admits a uniquely solution $K(x,t)$ in the space $L_1(x,\infty)$ [9].

The set of values $\left\{S(\lambda),\lambda_k,P_k(x),(k=\overline{1,n})\right\}$ is referred to as the scattering data of the boundary value problem (1.1)-(1.3) (see [8]). The inverse scattering problem consists in uniquely recovering the coefficient $q(x)$ from the scattering data. Given the scattering data, we can use formula (1.6) to construct the function $F(x)$ and write out to main equation (1.7) for the unknown function $K(x,y)$. The main equation has a unique solution for every $x \geq 0$. Solving this equation, we find the kernel $K(x,y)$ of the solution (1.7) and hence potential $q(x)=-2\frac{dK(x,x)}{dx}$.

Note that the inverse problem of scattering theory on the half line for the boundary value problem (1.1)-(1.3) in the case $\alpha_1=0$ was completely solved in [6, 7, 15, 16]. Inverse problems in the half line with spectral parameter contained in the boundary conditions was investigated according to spectral function in [19], according to Weyl function in [21]-[23], and acording to scattering data [10]-[13]. In the case of non-selfadjoint, the similar problem was solved in [8]. The uniqueness of solution of inverse scattering problem for boundary value problem (1.1)-(1.3) is given in [9] by using the methods of [8] and [15]. Different from the classical case the zeros of Jost function not lie imaginary axis, lie complex plane and these zeros not simple. The boundary value problem (1.1)-(1.3) is not selfadjoint and for this reason, scattering data is differently defined. Therefore, the properties of the scattering data have to be investigated. The present work is devoted to give the properties of the scattering data of boundary problem (1.1)(1.3). Similar problem in the self-adjoint case was studied in [14, 17].

Let us give a brief description of the structure of our study. In Section 2, we prove the continuity of the scattering function on the whole axis. In Section 3, we derive the Levinson type formula.

2. The continuity of the scattering function

In this section, the continuity of the scattering function $S(\lambda)$ defined by (1.5) will be investigated.

Theorem 2.1. *The scattering function $S(\lambda)$ is continuous for all real points λ.*

Proof. It follows from lemma 1.1 that $\varphi(\lambda)\neq 0$ for all $\lambda\neq 0$. The continuity of the function $S(\lambda)$ can be obtained from hence.

When $\varphi(0) \neq 0$, the function $S(\lambda)$ is continuous for $\lambda = 0$ and $S(0) = 1$.

Let $\varphi(0) = 0$. Namely,

$$\begin{aligned}\varphi(0) &= e'(0,0) - \alpha_0 e(0,0) \\ &= -K(0,0) + \int_0^\infty K_x(0,t)dt - \alpha_0 \left[1 + \int_0^\infty K(0,t)dt\right] = 0. \end{aligned} \tag{2.1}$$

To complete proof, we shall investigate the continuity of the function $S(\lambda)$ in this case.

Now, putting $x = 0$ in the main equation (1.7), we obtain

$$K(0,y) + F(y) + \int_0^\infty K(0,t)F(t+y)dt = 0. \tag{2.2}$$

Substituting $x = 0$ after differentiating the main equation (1.7) with respect to x, we get

$$K_x(0,y) + F'(y) - K(0,0)F(y) + \int_0^\infty K_x(0,t)F(t+y)dt = 0. \tag{2.3}$$

After multiplying the equation (2.2) throughout by $-\alpha_0$ and adding to the equality (2.3), we have

$$K_x(0,y) - \alpha_0 K(0,y) - (\alpha_0 + K(0,0))F(y) + F'(y) + \int_0^\infty [K_x(0,t) - \alpha_0 K(0,t)]F(t+y)dt = 0. \tag{2.4}$$

Then, integrating the equality (2.4) with respect to y from z to ∞, we obtain

$$\begin{aligned}&\int_z^\infty [K_x(0,y) - \alpha_0 K(0,y)]dy - (\alpha_0 + K(0,0)) \int_z^\infty F(y)dy - F(z) \\ &\qquad\qquad + \int_0^\infty [K_x(0,t) - \alpha_0 K(0,t)] \int_{z+t}^\infty F(\xi)d\xi dt = 0.\end{aligned}$$

Put $K_1(z) = \int_z^\infty [K_x(0,y) - \alpha_0 K(0,y)]dy$. Then, from the last equality, the following relation is obtained:

$$K_1(z) - (\alpha_0 + K(0,0)) \int_z^\infty F(y)dy - F(z) - \int_0^\infty \left(\int_{z+t}^\infty F(\xi)d\xi \right) dK_1(t) = 0.$$

Using integration by parts and considering the following process

$$\int_0^\infty K'_x(x,t)|_{x=0}\int_{t+z}^\infty F(\xi)d\xi dt = \int_0^\infty K'_x(x,t)|_{x=0}\int_z^\infty F(y)dydt$$
$$-\int_0^\infty F(t+z)\int_t^\infty K'_x(x,\xi)|_{x=0}d\xi dt,$$

$$\int_z^\infty F'(y)dy - K(0,0)\int_z^\infty F(y)dy + \int_z^\infty K'_x(x,y)|_{x=0}dy$$
$$+\int_0^\infty K'_x(x,t)|_{x=0}\int_z^\infty F(y)dydt - \int_0^\infty F(t+z)\int_t^\infty K'_x(x,\xi)|_{x=0}d\xi dt = 0,$$

we get

$$K_1(z) - (\alpha_0 + K(0,0) + K_1(0))\int_z^\infty F(y)dy - F(z) - \int_0^\infty K_1(t)F(t+z)dt = 0.$$

Hence, when $\varphi(0) = 0$ (from (2.1)), $K_1(z)$ is the bounded solution of the equation

$$K_1(z) - \int_0^\infty K_1(t)F(t+z)dt = F(z), \qquad (0 \le z < \infty).$$

It is evident that the bounded solution of this equation is integrable on the half axis. It means that $K_1(z) \in L_1(0,\infty)$ (see [15], p. 211).

Returning to the representation $\varphi(\lambda)$, we have

$$\varphi(\lambda) = i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt - (\alpha_0 + \alpha_1\lambda)\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right]$$
$$= i\lambda K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt - \alpha_0\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right]$$
$$-\alpha_1\lambda\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right],$$

where

$$
\begin{aligned}
& -K(0,0)+\int_0^\infty K_x(0,t)e^{i\lambda t}dt-\alpha_0-\alpha_0\int_0^\infty K(0,t)e^{i\lambda t}dt= \\
&=-K(0,0)-\int_0^\infty e^{i\lambda t}d\left(\int_t^\infty K(0,y)dy\right)-\alpha_0+\alpha_0\int_0^\infty e^{i\lambda t}d\left(\int_t^\infty K(0,y)dy\right) \\
&=-K(0,0)+\int_x^\infty K_x(0,y)dy-\alpha_0\int_0^\infty K(0,y)dy+i\lambda\int_0^\infty e^{i\lambda t}\int_t^\infty K_x(0,y)dydt \\
& -i\alpha_0\lambda\int_0^\infty e^{i\lambda t}\int_0^\infty K(0,y)dydt \\
&=i\lambda\int_0^\infty\int_t^\infty (K_x(0,y)-\alpha_0K(0,y))dye^{i\lambda t}dt \\
&=i\lambda\int_0^\infty K_1(t)e^{i\lambda t}dt.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\varphi(\lambda)&=i\lambda\left[1+\int_0^\infty K_1(t)e^{i\lambda t}dt-i\alpha_1\left(1+\int_0^\infty K(0,t)e^{i\lambda t}dt\right)\right] \\
&=i\lambda\widetilde{K}(\lambda).
\end{aligned}
\tag{2.5}
$$

where

$$\widetilde{K}(\lambda)=1-i\alpha_1+\int_0^\infty K_1(t)e^{i\lambda t}dt-i\alpha_1\int_0^\infty K(0,t)e^{i\lambda t}dt.$$

Similarly, we get

$$\varphi_1(\lambda)=-i\lambda\widetilde{K_1}(\lambda), \tag{2.6}$$

where

$$\widetilde{K_1}(\lambda)=1+i\alpha_1+\int_0^\infty K_1(t)e^{-i\lambda t}dt-i\alpha_1\int_0^\infty K(0,t)e^{-i\lambda t}dt.$$

Consequently, from the equality (1.5)

$$S(\lambda)=-\frac{\widetilde{K_1}(\lambda)}{\widetilde{K}(\lambda)}.$$

Taking into account lemma 1.1 (see [9]) and by using the formulas (2.5) and (2.6), we can write

$$2\varpi(x,\lambda) = \widetilde{K}(\lambda)[e(x,-\lambda) - S(\lambda)e(x,\lambda)].$$

It can be seen that $\widetilde{K}(\lambda) \neq 0$, otherwise it would be $\varphi(x,0) = 0$. But, this can not be true since $\varphi(0,0) = 1$. So, $S(\lambda)$ is continuous at $\lambda = 0$ and by condition (2.1) it holds $S(\lambda) = -\frac{\widetilde{K_1}(0)}{\widetilde{K}(0)}$.

This completes the proof the theorem. □

3. The Levinson-Type formula

We give the formula that expresses the relation between the increment of the argument of the scattering function $S(\lambda)$ and the singular number λ_k of boundary value problem (1.1)-(1.3).

Theorem 3.1. *The following formlua is valid:*

$$-\frac{1-S(0)}{2} - \frac{1}{2\pi}\arg S(\lambda)|_{-\infty}^{\infty} + 1 = 2[m_1 + m_2 + ... + m_n], \tag{3.1}$$

where m_k $(k = 1,2,...,n)$ *is the multiplicity of the singular number* λ_j $(j = 1,2,...,n)$.

Proof. For sufficiently little $\varepsilon > 0$ and sufficiently large $R > 0$, let

$$\Gamma_{R,\varepsilon} = C_R^+ \cup C_\varepsilon^- \cup [-R,-\varepsilon] \cup [\varepsilon, R],$$

where C_R^+ and C_ε^- are circles with centers in origin and corresponding radius of R and ε, respectively (Fig. 1). Orientation on the C_R^+ is positive and on the C_ε^- negative.

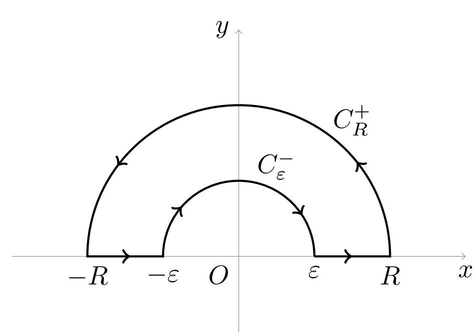

Figure 1: The Graph of $\Gamma_{R,\varepsilon}$.

Let us apply argument principle to $\varphi(\lambda)$ function. This function is regular on the upper half plane and continuous on the closed half plane $Im\lambda \geq 0$. When moving from $-\infty$ to ∞ on the whole real axis and passing origin from top along with half circle with radius ε, the change in the argument of $\varphi(\lambda)$ is equal to number of its pole points times 2π:

$$\arg\varphi(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} + \arg\varphi(\lambda)|_{\Gamma_\varepsilon} + \arg\varphi(\lambda)|_{\Gamma_R} = 2\pi[m_1 + m_2 + ... + m_n] \tag{3.2}$$

or

$$\frac{1}{2\pi i}\left(\int\limits_{C_R^+} + \int\limits_{C_\varepsilon^-} + \int\limits_{\varepsilon}^{R} + \int\limits_{-R}^{-\varepsilon}\right) d\ln\varphi(\lambda) = m_1 + m_2 + ... + m_n.$$

From (1.5), the scattering function $S(\lambda)$ has the form

$$S(\lambda) = \frac{\varphi_1(\lambda)}{\varphi(\lambda)}$$

for real λ. It is clear from here that $\arg S(\lambda) = -2\arg\varphi(\lambda)$.

Using the last equality, we have

$$\arg\varphi(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} = -\frac{1}{2}\arg S(\lambda). \tag{3.3}$$

Considering (3.3) in the equality (3.2), we obtain

$$-\frac{1}{2}\arg S(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} + \arg\varphi(\lambda)|_{C_\varepsilon^-} + \arg\varphi(\lambda)|_{C_R^+} = 2\pi[m_1 + m_2 + ... + m_n]. \tag{3.4}$$

According to theorem 2.1, the function $S(\lambda)$ is continuous on the whole real axis. Hence,

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\left\{-\frac{1}{2}\arg S(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]}\right\} = -\frac{1}{2}\arg S(\lambda)\Big|_{-\infty}^{\infty}, \tag{3.5}$$

$$\lim_{\varepsilon\to 0}\arg\varphi(\lambda)|_{C_\varepsilon^-} = \begin{cases} 0, & \text{if } \varphi(0) \neq 0, \\ -\pi, & \text{if } \varphi(0) = 0, \end{cases} = -\frac{\pi(1 - S(0))}{2} \tag{3.6}$$

and

$$\lim_{R\to\infty}\arg\varphi(\lambda) = \pi \tag{3.7}$$

from lemma 1.1.

Taking into account the equalities (3.5), (3.6) and (3.7) in the equality (3.4), we have

$$-\frac{1}{2}\arg S(\lambda)\Big|_{-\infty}^{\infty}+\pi+\begin{cases}0, & \text{if } \varphi(0)\neq 0,\\ -\pi, & \text{if } \varphi(0)=0,\end{cases}=2\pi[m_1+m_2+...+m_n]$$

From this last equality, the formula (3.1) is obtained, which proves the theorem. □

The note that, this formula is called the Levinson-type formula for the boundary value problem (1.1)-(1.3).

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