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ITERATIVE APPROXIMATIONS FOR GENERALIZED NONEXPANSIVE MAPPINGS USING K ITERATION PROCESS IN BANACH SPACES

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Abstract. Let H be a nonempty subset of a Banach space X. A mapping $T : H \to H$ is said to be generalized α-nonexpansive if there is a real number $\alpha \in [0, 1)$ such that for all $x, y \in H$, we have

$$\frac{1}{2}||x - Tx|| \leq ||x - y||$$

$$||Tx - Ty|| \leq \alpha||Tx - Ty|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||.$$

In this paper, we obtain some weak and strong convergence theorems for such mappings using K-iteration process in uniformly convex Banach space setting. Our results extend and improve many results in the literature.
Keywords: Banach space, nonexpansive mappings, iterative approximations.

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1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of all natural numbers and $\mathbb{R}$ stands for the set of all real numbers. A mapping T on a subset C of a Banach space X is called contraction if there is a real number $0 \leq \beta < 1$ such that

$$||Tx - Ty|| \leq ||x - y|| \; \forall x, y \in H. \tag{1.1}$$

If (1.1) is valid at $\beta = 1$, then T is called nonexpansive. A point $p \in H$ is called a fixed point of T if $p = Tp$. We shall denote the set of all fixed points of T by $F(T)$. We know that $F(T)$ is nonempty in the case when H is nonempty closed bounded convex and X is uniformly convex (see [1, 2, 3]). Notice that, $T : H \to H$ is called quasi-nonexpansive if for any $p \in F(T)$, we have

$$||Tx - Tp|| \leq ||x - p|| \; \forall x \in H.$$

In 2008, Suzuki [4] introduced a new class of nonexpansive mappings which is a condition on mappings called condition (C). The mapping $T : H \to H$ is said to satisfy condition (C) (sometimes called Suzuki-generalized nonexpansive) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y|| \; \forall x, y \in H.$$

In 2011, Aoyama and Kohsaka in [5] provided the notion of α-nonexpansive mappings which includes all nonexpansive maps. Suppose $\alpha \in [0, 1)$ be a fixed real, then, $T : H \to H$ is known as α-nonexpansive if the following holds for all $x, y \in H$,

$$||Tx - Ty||^2 \leq \alpha||Tx - Ty||^2 + \alpha||Ty - x||^2 + (1 - 2\alpha)||x - y||^2.$$

A natural question arises that how to extend futher the above mentioned class of mappings. To answer such a question, Pant and Shukla in [6] suggested a new class of mappings which they named as a class of α-nonexpansive mappings. This new class of mappings partially includes the class of α-nonexpansive mappings and fully includes the class of proved that the class Suzuki-generalized nonexpansive operators. Suppose $\alpha \in [0, 1)$ be a fixed real, then, $T : H \to H$ is known as generalized α-nonexpansive if the following holds for all $x, y \in H$,

$$\begin{aligned} & \frac{1}{2}||x - Tx|| \leq ||x - y|| \\ \Rightarrow \quad & ||Tx - Ty|| \leq \alpha||Tx - Ty|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||. \end{aligned}$$

There are also other wider classes of operators (see e.g., [21, 22] and others) which are more general than all of the above mappings; however, we restrict our study to the setting of generalized α-nonexpansive mappings.

One of the basic iterative schemes, the mostly used iterative scheme is due to Picard [7] as follows:

$$(1.2) \qquad \begin{cases} x_1 \in H, \\ x_{n+1} = Tx_n, n \in \mathbb{N}, \end{cases}$$

One of the drawbacks of the Picard iteration is that it does not converge to a fixed point of a nonexpansive operator in general. Hence we must use some other iterative methods to approximate fixed points of nonexpansive and generalized nonexpansive operators.

The iterative scheme of Mann [8] generates a sequence using the following formula:

$$(1.3) \qquad \begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n \in (0,1)$.

In 1974, Ishikawa [9] extended the Mann scheme to the setting of two steps in the following manner:

$$(1.4) \qquad \begin{cases} x_1 \in H, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

Noor [11] first time used a three-step iterative scheme as follows:

$$(1.5) \qquad \begin{cases} x_1 \in H, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2007, Agarwal et al. [10] suggest a new two-step iterative scheme, which they named it S iteration. This scheme reads as follows:

$$(1.6) \qquad \begin{cases} x_1 \in H, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

In 2014, Abbas and Nazir [12] introduced the following new three step iteration process as follows:

$$(1.7) \qquad \begin{cases} x_1 \in H, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2014, Thakur et al. [13] used the following new iteration process:

$$(1.8)\qquad \begin{cases} x_1 \in H, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = T((1-\beta_n)x_n + \beta_n z_n), \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

Recently in 2018, Hussain et al. [14] have introduced the following new iteration process, called K iteration process:

$$(1.9)\qquad \begin{cases} x_1 \in H, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = T((1-\beta_n)T x_n + \beta_n T z_n), \\ x_{n+1} = T y_n, n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

They proved some weak and strong convergence results of K iteration process for the class of Suzuki generalized nonexpansive mappings. Also, they proved numerically that K iteration process is better than all of the above processes for the class of Suzuki generalized nonexpansive mappings. In this article, we extend their results to the general setting of generalized α-nonexpansive mappings. Our proofs use idea provided in [7, 23, 24].

To establish our results, we would like to provide some already known definitions and facts.

Definition 1.1. If $\{x_n\}$ denotes any bounded sequence of elements of a Banach space, namely, X and $\emptyset \neq H \subseteq X$ is convex and closed. Then for a fixed $x_0 \in X$, then

- (a_0) $r(x_0, \{x_n\}) = \limsup_{n\to+\infty} ||x_0 - x_n||$ is called asymptotic radius of $\{x_n\}$ on x_0;
- (a_1) $r(H, \{x_n\}) = \inf\{r(x_0, \{x_n\}) : x_0 \in H\}$ is called an asymptotic radius of $\{x_n\}$ with respect to P;
- (a_2) $A(H, \{x_n\}) = \{x_0 \in H : r(x_0, \{x_n\}) = r(H, \{x_n\})\}$ is called an asymptotic center of $\{x_n\}$ with respect to H.

Now we remark the following.

Remark 1.1. Among the other interesting properties, the set $A(H, \{x_n\})$ can have exactly one element, that is, it is singleton provided that X is uniformly convex (see e.g., [15, 16] and others).

Definition 1.2. [17, 18] Suppose X is a norm linear space. X is said to be

(i) uniformly convex if $\forall \varepsilon \in (0, 2]$, $\exists \delta > 0$: whenever $||x-y|| > \varepsilon$ and $(||x||, ||y|| \leq 1)$ imply $\frac{1}{2}||x+y|| \leq (1-\lambda)$;

(ii) enriched with the Opial's condition if and only if $\limsup_{n\to+\infty} ||x_n - x|| < \limsup_{n\to+\infty} ||x_n - y||$, $\forall y \in X - \{x\}$, where $\{x_n\}$ is any bounded sequence in X.

A condition (I) of an operator T is also needed.

Definition 1.3. [19] If a mapping $T : H \to H$ is such that there is a $C : [0, \infty) \to [0, \infty)$, where $C(0) = 0$ and $C(v) > 0$ for all $0 < v < \infty$ and also $||x - Tx|| \geq C(\mathrm{dist}(x, F(T)))$ for any chosen $x \in P$, then T is called mapping with condition (I).

The useful properties of generalized α-nonexpansive were established in [6]. The proposition is as follows.

Proposition 1.1. *Suppose $\emptyset \neq P$ be any subset of X and consider a selfmap $T : H \to H$.*

(v) T is generalized α-nonexpansive whenever it is Suzuki generalized nonexpansive.

(w) T is a quasi-nonexpansive whenever it is generalized α-nonexpansive.

(x) $F(T)$ is closed (in H) whenever T is generalized α-nonexpansive.

(y) For each two points $x, y \in H$, $||x - Ty|| \leq \left(\frac{3+\alpha}{1-\alpha}\right) ||x - Tx|| + ||x - y||$ holds, whenever T is generalized α-nonexpansive.

(z) If T is generalized α-nonexpansive, $\{x_n\}$ a sequence in X such that X has Opial's condition and $\{x_n\}$ is weakly convergent having weak limit a, then, $a \in F(T)$ whenever $\lim_{n\to+\infty} ||x_n - Tx_n|| = 0$.

Any uniformly convex Banach space enjoys the following interesting property.

Lemma 1.1. [20] *Suppose X denotes any uniformly convex Banach space. If $\limsup_{n\to+\infty} ||q_n|| \leq \eta$, $\limsup_{n\to+\infty} ||w_n|| \leq \eta$ and $\lim_{n\to+\infty} ||\theta_n q_n + (1-\theta_n) w_n|| = \eta$ for some $\eta \geq 0$ then, $\lim_{n\to\infty} ||w_n - q_n|| = 0$, where $\theta_n \in [i, j] \subset (0, 1)$ and $\{q_n\}$ and $\{w_n\}$ are sequences of elements of X.*

2. Main Results

From now on, X is uniformly convex Banach spaces.

Lemma 2.1. *If $\emptyset \neq H \subseteq X$ is closed convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then for each $p \in F(T)$, $\lim_{n\to+\infty} ||x_n - p||$ exists.*

Proof. If $p \in F(T)$, then using Proposition 1.1 (w),

$$\begin{aligned} ||z_n - p|| &= ||(1-\beta_n)x_n + \beta_n T x_n - p|| \\ &\leq (1-\beta_n)||x_n - p|| + \beta_n ||Tx_n - p|| \\ &\leq (1-\beta_n)||x_n - p|| + \beta_n ||x_n - p|| \\ &\leq ||x_n - p||, \end{aligned}$$

and

$$\begin{aligned} ||y_n - p|| &= ||T((1-\alpha_n)Tx_n + \alpha_n T z_n) - p|| \\ &\leq ||(1-\alpha_n)Tx_n + \alpha_n T z_n - p|| \\ &\leq (1-\alpha_n)||Tx_n - p|| + \alpha_n ||Tz_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||z_n - p||. \end{aligned}$$

They imply that

$$\begin{aligned} ||x_{n+1} - p|| &= ||Ty_n - p|| \\ &\leq ||y_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||z_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned}$$

Thus $\{||x_n - p||\}$ is bounded and nonincreasing, which implies that $\lim_{n\to+\infty} ||x_n - p||$ exists for each $p \in F(T)$. $\square$

Theorem 2.1. *If $\emptyset \neq H \subseteq X$ is closed convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then, $F(T) \neq \emptyset \iff \{x_n\}$ is bounded and the equation $\lim_{n\to+\infty} ||Tx_n - x_n|| = 0$ holds.*

Proof. If $F(T) \neq \emptyset$, then we may choose any $p \in F(T)$ and so using Lemma 2.1, we get $\lim_{n\to+\infty} ||x_n - p||$ exists and $\{x_n\}$ is bounded. Set the following

$$\lim_{n\to+\infty} ||x_n - p|| = \eta. \tag{2.1}$$

We have noted in Lemma 2.1, that

$$\limsup_{n\to+\infty} ||z_n - p|| \leq \limsup_{n\to+\infty} ||x_n - p|| = \eta. \tag{2.2}$$

By using Proposition 1.1 (w), we have

$$\limsup_{n\to+\infty} ||Tx_n - p|| \leq \limsup_{n\to+\infty} ||x_n - p|| = \eta. \tag{2.3}$$

Once more, looking into the proof of Lemma 2.1, the following is hold,

$$||x_{n+1} - p|| \leq (1-\alpha_n)||x_n - p|| + \alpha_n ||z_n - p||. \tag{2.4}$$

Accordingly, from (2.4), one has

$$||x_{n+1}-p||-||x_n-p|| \leq \frac{||x_{n+1}-p||-||x_n-p||}{\alpha_n} \leq ||z_n-p||-||x_n-p||.$$

Thus, we proved that $||x_{n+1}-p|| \leq ||z_n-p||$. Hence

$$\eta \leq \liminf_{n\to+\infty} ||z_n-p||. \tag{2.5}$$

If one combines (2.2) with (2.5), then

$$\eta = \lim_{n\to+\infty} ||z_n-p||. \tag{2.6}$$

Now using (2.6), we have

$$\eta = \lim_{n\to+\infty} ||z_n-p|| = \lim_{n\to+\infty} ||(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)||.$$

Hence,

$$\eta = \lim_{n\to+\infty} ||(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)||. \tag{2.7}$$

Now we can apply the Lemma 1.1 and hence we obtain the following required equation

$$\lim_{n\to\infty} ||Tx_n-x_n|| = 0.$$

Conversely, we need to prove that $F(T)$ is nonempty whenever $\{x_n\}$ is bounded with $\lim_{n\to+\infty} ||Tx_n-x_n|| = 0$. If $p \in A(H,\{x_n\})$. By Proposition 1.1 (y), we have

$$\begin{aligned}
r(Tp,\{x_n\}) &= \limsup_{n\to+\infty} ||x_n-Tp|| \\
&\leq \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{n\to+\infty} ||x_n-Tx_n|| + \limsup_{n\to+\infty} ||x_n-p|| \\
&= \limsup_{n\to+\infty} ||x_n-p|| \\
&= r(p,\{x_n\}).
\end{aligned}$$

Hence we have proved that $Tp \in A(H,\{x_n\}$. But in our case, the set $A(C,\{x_n\})$ has only one element. Thus, we must have $Tp = p$ and accordingly $F(T)$ is nonempty. $\square$

Now we are in the position to prove our weak convergence result.

Theorem 2.2. *If $\emptyset \neq H \subseteq X$ is closed convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then $\{x_n\}$ has a weak limit in $F(T)$ if X satisfies the Opial's condition.*

Proof. As we have establish in the Theorem 2.1 that $\{x_n\}$ is bounded and the equation $\lim_{n\to+\infty} ||Tx_n - x_n|| = 0$ holds. Furthermore, the space X is reflexive because it is uniformly convex. Consequently, one can suggests a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with the property that $\{x_{n_i}\}$ converges in the weak sense to a point $p_1 \in H$. By applying Proposition 1.1 (z), it seen that $p_1 \in F(T)$. If p_1 is a unique weak limit of $\{x_n\}$ then we are done. If not, then there is a point $p_2 \in H$ different form p_1 such that a subsequence $\{x_{n_j}\}$ of $\{x_n$ converges weakly to it. Again using Proposition 1.1 (z), $p_2 \in F(T)$. Now we use Lemma 2.1 and also the Opial condition of X to get

$$\begin{aligned}
\lim_{n\to+\infty} ||x_n - p_1|| &= \lim_{i\to+\infty} ||x_{n_i} - p_1|| \\
&< \lim_{i\to+\infty} ||x_{n_i} - p_2|| \\
&= \lim_{n\to+\infty} ||x_n - p_2|| \\
&= \lim_{j\to+\infty} ||x_{n_j} - p_2|| \\
&< \lim_{j\to+\infty} ||x_{n_j} - p_1|| \\
&= \lim_{n\to+\infty} ||x_n - p_1||.
\end{aligned}$$

Obviously, we found a contradiction. Hence we must have $a_1 = a_2$. Thus, we have reached to the conclusions. $\square$

Now we prove the following strong convergence result.

Theorem 2.3. *If $\emptyset \neq H \subseteq X$ is compact convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then $\{x_n\}$ has a strong limit in $F(T)$.*

Proof. As we have proved in the Theorem 2.1 that $\lim_{n\to+\infty} ||x_n - Tx_n|| = 0$. By assumption of the compactness and convexity of H, hence we have the existence of a strongly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ having a strong limit q for some $q \in H$. By Proposition 1.1 (y), we have

$$||x_{n_j} - Tq|| \leq \left(\frac{3+\alpha}{1-\alpha}\right) ||x_{n_j} - Tx_{n_j}|| + ||x_{n_j} - q||.$$

We may put $j \to +\infty$ and hence obtain $Tq = q$. By Lemma 2.1, $\lim_{n\to+\infty} ||x_n - q||$ exists, that is, q is the strong limit of $\{x_n\}$. $\square$

In [6], Pant and Shukla gave an interesting example of generalized α-nonexpansive operators which fails to satisfy the condition (C) of Suzuki.

Example 2.1. ([6, Example 6.1]) Let $H = [-1, 1]$ be endowed with $|.|$ and define $T : C \to C$ by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [-1,0), \\ -x, & \text{if } [0,1] \setminus \{\frac{1}{2}\}, \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Theorem 2.4. *If $\emptyset \neq P \subseteq X$ is closed convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then $\{x_n\}$ has a strong limit in $F(T)$ if the equation $\liminf_{n\to+\infty} dist(x_n, F(T)) = 0$ holds.*

Proof. For any point p of $F(T)$, we have proved in Lemma 2.1 that $\lim_{n\to+\infty} ||x_n - p||$. It follows that $\lim_{n\to\infty} dist(x_n, F(T))$ exists. Accordingly, we can write

$$\lim_{n\to+\infty} dist(x_n, F(T)) = 0. \tag{2.8}$$

We want to construct a Cauchy sequence in $F(T)$. Looking into (2.8), one can construct a subsequence $\{z_r\}$ in $F(T)$ and $\{x_{n_r}\}$ of $\{x_n\}$ with the property $||x_{n_r} - z_r|| \leq \frac{1}{2^r}$ $\forall r \in \mathbb{N}$. On the other hand, $\{x_r\}$ is nonicreasing, we have

$$||x_{n_{r+1}} - z_r|| \leq ||x_{n_r} - z_r|| \leq \frac{1}{2^r}.$$

Consequently, we have

$$\begin{aligned} ||z_{r+1} - z_r|| &\leq ||z_{r+1} - x_{n_{r+1}}|| + ||x_{n_{r+1}} - z_r|| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^r} \\ &\leq \frac{1}{2^{r-1}} \to 0, \text{ if } r \to +\infty. \end{aligned}$$

Finally, we have constructed a Cauchy sequence $\{z_r\}$ as above in $F(T)$ and hence this sequence converges to some p. But according to Proposition 1.1 (x), $F(T)$ is closed in H, we must have $p \in F(T)$. Lemma 2.1 suggests that $\lim_{n\to+\infty} ||x_n - p||$ exists. Thus, p must be the strong limit for $\{x_n\}$. $\square$

Theorem 2.5. *If $\emptyset \neq H \subseteq X$ is closed convex and consider a selfmap $T : H \to H$ with $F(T) \neq \emptyset$. If T is generalized α-nonexpansive and $\{x_n\}$ denotes the K iterates (1.9). Then $\{x_n\}$ has a strong limit in $F(T)$ if T admits a condition (I).*

Proof. We have proved in the Theorem 2.1, that, $\lim_{n\to+\infty} ||x_n - Tx_n|| = 0$. It follows that

$$\liminf_{n\to+\infty} ||x_n - Tx_n|| = 0.$$

By using the condition (I), one can easily deducts the following equation

$$\liminf_{n\to+\infty} \mathrm{dist}(x_n, F(T)) = 0.$$

It seen that all the condition of Theorem 2.4 are proved and so $\{x_n\}$ has a strong limit in $F(T)$. $\square$

3. Numerical Example

In this section, using Example 2.1 of a generalized α-nonexpansive mapping, we compare the convergence of K iteration process with the other leading iterations. Suppose for $n \geq 1$ and we choose $\alpha_n = 0.70$, $\beta_n = 0.65$ are sequences in $(0, 1)$ and $x_1 = -0.9$. Table 4.1 and Figure 4.1 show the efficiency of K iteration process in the class of generalized α-nonexpansive mappings.

4. Conclusions

As we have already concluded, the class of operators defined by Suzuki [4] is properly contained in the class of generalized α-nonexpansive operators. Using the K iteration, Hussain et al. in [14] proved basic strong and weak convergence theorems for the class of Suzuki operators. Here, we improved their results to the general framework of generalized α-nonexpansive operators. By performing a numerical experiment, it has been shown that the K iteration is still more effective than the other iterative schemes in the setting of generalized α-nonexpansive operators.

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CONFLICT OF INTEREST

There are no conflicts of interest to this work.

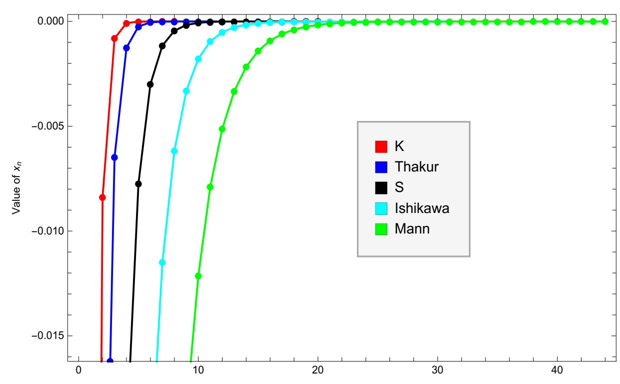

FIG. 4.1: Graphical comparison between K and other schemes using Example 2.1.

Table 4.1: Comparative experiment results of K, Thakur, S, Ishikawa and Mann iteration process for T of Example 2.1.

n	K	Thakur	S	Ishikawa	Mann
1	-0.9	-0.9	-0.9	-0.9	-0.9
2	-0.08690625	-0.17381250	-0.34762500	-0.58500000	-0.58500000
3	-0.00839188	-0.03356753	-0.13427015	-0.38025000	-0.58500000
4	-0.00081034	-0.00648273	-0.05186184	-0.24716250	-0.58500000
5	-0.00007824	-0.00125197	-0.02003163	-0.16065562	-0.58500000
6	-0.00000755	-0.00024178	-0.00773722	-0.10442615	-0.58500000
7	-0.00000072	-0.00004669	-0.00298850	-0.06787700	-0.58500000
8	-0.00000007	-0.00000901	-0.00115430	-0.04412005	-0.58500000
9	-0.00000000	-0.00000174	-0.00044585	-0.04412005	-0.58500000
10	-0.00000000	-0.00000033	-0.00017221	-0.02867803	-0.58500000
11	-0.00000000	-0.00000006	-0.00006651	-0.01864072	-0.58500000
12	-0.00000000	-0.00000001	-0.00002569	-0.01211646	-0.58500000
13	-0.00000000	-0.00000000	-0.00000992	-0.00787570	-0.58500000
14	-0.00000000	-0.00000000	-0.00000383	-0.00511920	-0.58500000
15	-0.00000000	-0.00000000	-0.00000148	-0.00332748	-0.58500000
16	-0.00000000	-0.00000000	-0.00000057	-0.00216286	-0.58500000
17	-0.00000000	-0.00000000	-0.00000022	-0.00140586	-0.58500000
18	-0.00000000	-0.00000000	-0.00000008	-0.00091381	-0.58500000
19	-0.00000000	-0.00000000	-0.00000003	-0.00059397	-0.58500000
20	-0.00000000	-0.00000000	-0.00000001	-0.00038608	-0.58500000
21	-0.00000000	-0.00000000	-0.00000000	-0.00025095	-0.58500000
22	-0.00000000	-0.00000000	-0.00000000	-0.00016312	-0.58500000
23	-0.00000000	-0.00000000	-0.00000000	-0.00010602	-0.58500000

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