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APPROXIMATION BY JAIN-SCHURER OPERATORS

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Abstract. In this paper we deal with Jain-Schurer operators. We give an estimate, related to the degree of approximation, via moduli of smoothness of the first and the second order. Also, we present a Voronovskaja-type result. Moreover, we show that the Jain-Schurer operators preserve the properties of a modulus of continuity. Finally, we study monotonicity of the sequence of the Jain-Schurer operators when the attached function is convex and non-decreasing.

Keywords: Jain-Schurer operators; monotonicity; moduli of smoothness; Voronovskaja-type result.

1. Introduction

In [19], Schurer constructed the following linear positive operators

$$S_{n,p}\left(f;x\right)=e^{-(n+p)x}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{\left(n+p\right)^{k}x^{k}}{k!}, \tag{1.1}$$

where $x\in[0,b],\ b<\infty,\ n\in\mathbb{N},\ p\geq 0$, and f is real valued and bounded function on $[0,\infty)$. The case $p=0$ gives the the well known Szász-Mirakjan operators. There are a number of generalizations of Szász-Mirakjan operators, here we cite only a few ([4], [6], [10], [11]) with references therein. Some works concerning Schurer's setting can be found in [3], [14], [20], [16] and [17]. Motivated by these statements, we extend the well known Jain operators in the Schurer's design. Recall that in [12], Jain constructed the following linear positive operators

$$P_{n}^{[\beta]}\left(f;x\right)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)w_{\beta}\left(k;nx\right),\quad x\in\left(0,\infty\right), \tag{1.2}$$

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and $P_n^{[\beta]}(f;0) = f(0)$, where $n \in \mathbb{N}$, $\beta \in [0,1)$, $f \in C[0,\infty)$, and for $0 < \alpha < \infty$, $w_\beta(k;\alpha)$ is given by

$$w_\beta(k;\alpha) := \frac{\alpha(\alpha+k\beta)^{k-1}}{k!}e^{-(\alpha+k\beta)}, \quad k \in \mathbb{N}\cup\{0\} \tag{1.3}$$

and it satisfies $\sum_{k=0}^{\infty} w_\beta(k;\alpha) = 1$. In the paper, the author studied convergence properties and the order of approximation by the sequence of these operators on any finite closed interval of $[0,\infty)$ by taking β as a sequence β_n such that $\beta_n \to 0$ as $n \to \infty$. For some interesting papers concerning Jain operators, we refer to [1], [2], [7], [9], [18], [23] and references therein. Obviously, the case $\beta = 0$ gives the well known Szász-Mirakjan operators [22].

In this work, for a fixed $p \in \mathbb{N}\cup\{0\}$, we consider the linear positive operators denoted by $S_{n,p}^{\beta}$, $n \in \mathbb{N}$, and defined as

$$S_{n,p}^{\beta}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) w_\beta(k;(n+p)x), \quad x \in (0,\infty) \tag{1.4}$$

and $S_{n,p}^{\beta}(f;0) = f(0)$, for $f \in C_B[0,\infty) := \{f \in C[0,\infty) : f \text{ is bounded}\}$, $\beta \in [0,1)$, and $w_\beta(k;(n+p)x)$ given by (1.3). We call $S_{n,p}^{\beta}$ as Jain-Schurer operators. Note that, each $S_{n,p}^{\beta}$ maps $C_B[0,\infty)$ into itself, and the case $p = 0$ covers the Jain operators: $S_{n,0}^{\beta} = P_n^{[\beta]}$, $n \in \mathbb{N}$. On the other hand, in the case $\beta = 0$, $S_{n,p}^{\beta}$ reduces to the Schurer extension of the Szász-Mirakjan operators given by (1.1). We obtain an estimate, which will be used next for the rate of convergence, with the help of the modulus of smoothness of a bounded and continuous function, and prove a Voronovskaja-type result. Moreover, we show that each Jain-Schurer operator preserves the properties of a general modulus of continuity. Finally, we investigate the monotonicity of the sequence of the Jain-Schurer operators $S_{n,p}^{\beta}(f)$, with respect to n, when the function f is convex and non-decreasing.

Now, denoting $e_j(t) = t^j$, $j \in \mathbb{N}\cup\{0\}$ and $\varphi_x^j(t) := (t-x)^j$, $j \in \mathbb{N}$, for the Jain operators $P_n^{[\beta]}$ we have (see, e.g., [11, Lemma 1])

Lemma 1.1. *For the operators $P_n^{[\beta]}$ given by (1.2), one has*

$$\begin{aligned}
P_n^{[\beta]}(e_0;x) &= 1,\\
P_n^{[\beta]}(e_1;x) &= \frac{x}{1-\beta},\\
P_n^{[\beta]}(e_2;x) &= \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3},\\
P_n^{[\beta]}(e_3;x) &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} + \frac{(1+2\beta)x}{n^2(1-\beta)^5},\\
P_n^{[\beta]}(e_4;x) &= \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} + \frac{(8\beta+7)x^2}{n^2(1-\beta)^6} + \frac{\left(6\beta^2+8\beta+1\right)x}{n^3(1-\beta)^7}.
\end{aligned}$$

Making use of Lemma 1.1, straightforward computation shows that moments and central moments of the Jain-Schurer operators are obtained as in the following lemmas, respectively:

Lemma 1.2. *For the operators* $S_{n,p}^{\beta}$ *given by (1.4), one has*

$$S_{n,p}^{\beta}\left(e_j;x\right)=P_n^{[\beta]}\left(e_j;\left(\frac{n+p}{n}\right)x\right),\quad j=0,1,\ldots.$$

Lemma 1.3. *For the operators* $S_{n,p}^{\beta}$ *given by (1.4), one has*

$$\begin{aligned}
S_{n,p}^{\beta}\left(\varphi_x^1;x\right) &= \left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta},\\
S_{n,p}^{\beta}\left(\varphi_x^2;x\right) &= \left(\beta+\frac{p}{n}\right)^2\frac{x^2}{\left(1-\beta\right)^2}+\left(1+\frac{p}{n}\right)\frac{x}{n\left(1-\beta\right)^3},\\
S_{n,p}^{\beta}\left(\varphi_x^4;x\right) &= \left(\beta+\frac{p}{n}\right)^4\frac{x^4}{\left(1-\beta\right)^4}+6\left(\beta+\frac{p}{n}\right)^2\left(1+\frac{p}{n}\right)\frac{x^3}{n\left(1-\beta\right)^5}\\
&\quad+\left(1+\frac{p}{n}\right)\frac{\left(4n\beta+3n+8p\beta+7p+8n\beta^2\right)}{n^3\left(1-\beta\right)^6}x^2+\left(1+\frac{p}{n}\right)\frac{\left(6\beta^2+8\beta+1\right)}{n^3\left(1-\beta\right)^7}x\,.
\end{aligned}$$

2. Modulus of smoothness K-Functional

In this part of the paper, we extend the result proved by Agratini for the Jain operators [2, Theorem 2] to the Jain-Schurer operators. To this aim, we recall the terminology that will be used in the results. As usual, let $C_B\left[0,\infty\right)$ denote the space of real valued, bounded and continuous functions defined on $[0,\infty)$ equipped with the norm given by

$$\left\|f\right\|=\sup_{x\in[0,\infty)}\left|f\left(x\right)\right|$$

for $f\in C_B\left[0,\infty\right)$. Also, let $UC_B\left[0,\infty\right)$ denote the space of all real valued bounded and uniformly continuous functions on $[0,\infty)$.

For a bounded, real valued function f on $[0,\infty)$ and $\delta>0$, the first modulus of smoothness, modulus of continuity, of f is defined by

$$\omega_1\left(f,\delta\right)=\sup_{|h|\leq\delta}\ \sup_{x,x+h\in[0,\infty)}\left|f\left(x+h\right)-f\left(x\right)\right|$$

and second modulus of smoothness of f is defined by

$$\omega_2\left(f,\delta\right)=\sup_{|h|\leq\delta}\ \sup_{x+2h\in[0,\infty)}\left|f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)\right|.$$

We have the following well known property of the modulus of smoothness (see, e.g., [3, p. 266, Lemma 5.1.1]).

Remark 2.1. *If* $f \in UC_B[0,\infty)$, *then* $\lim_{\delta\to 0^+}\omega_k(f,\delta)=0$ for $k=1,2$.

For convenience, we need the following Peetre's K-functional defined by

$$K(f,\delta)=\inf_{g\in C_B^2[0,\infty)}\{\|f-g\|+\delta\|g''\|\},$$

where $\delta>0$ and

$$C_B^2[0,\infty)=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}.$$

Note that the modulus of smoothness and the K-functional of an $f\in C_B[0,\infty)$ are related to each other as in the following sense: *There exist positive constants* C_1 *and* C_2 *such that*

$$C_1\omega_2(f,\delta)\le K\left(f,\delta^2\right)\le C_2\omega_2(f,\delta) \tag{2.1}$$

(see, e.g., [8, p. 177, Theorem 2.4]).

Below, we present a quantitative estimate to reach to the subsequent result concerning the rate of the approximation by $\left\{S_{n,p}^{\beta_n}(f;x)\right\}_{n\ge1}$.

Theorem 2.1. *Let* $p\in\mathbb{N}_0$ *be fixed,* $0\le\beta<1$ *and* $f\in C_B[0,\infty)$. *Then, for each* $x\in(0,\infty)$, *one has*

$$\left|S_{n,p}^{\beta}(f;x)-f(x)\right|\le\omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta}(x)\right), \tag{2.2}$$

where $C>0$ *is a positive constant and*

$$\delta_{n,p}^{\beta}(x):=\frac{1}{2}\sqrt{\left(\beta+\frac{p}{n}\right)^2\frac{x^2}{(1-\beta)^2}+\left(1+\frac{p}{n}\right)\frac{x}{2n(1-\beta)^3}}. \tag{2.3}$$

Proof. Consider an auxiliary operator

$$\overline{S}_{n,p}^{\beta}(f;x):=S_{n,p}^{\beta}(f;x)+f(x)-f\left(\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}\right) \tag{2.4}$$

for $f\in C_B[0,\infty)$, $n\in\mathbb{N}$. In this case, $\overline{S}_{n,p}^{\beta}$ are linear and positive and each operator preserves the linear functions. Now, let $g\in C_B^2[0,\infty)$. From Taylor's formula about an arbitrary fixed point x, one has

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-y)g''(y)\,dy \tag{2.5}$$

for $t\in[0,\infty)$. Application of the operators $\overline{S}_{n,p}^{\beta}$ on both sides of (2.5) gives that

$$\overline{S}_{n,p}^{\beta}(g;x)-g(x)=g'(x)\overline{S}_{n,p}^{\beta}(t-x;x)+\overline{S}_{n,p}^{\beta}\left(\int_x^t(t-y)g''(y)\,dy;x\right). \tag{2.6}$$

Taking (2.4) into account for $f(t)=\int\limits_x^t (t-y)\,g''(y)\,dy$, expression (2.6) reduces to

$$\overline{S}^{\beta}_{n,p}(g;x)-g(x)=S^{\beta}_{n,p}\left(\int\limits_x^t (t-y)\,g''(y)\,dy;x\right)-\int\limits_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}}\left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}-y\right]g''(y)\,dy.$$

Using the fact

$$\left|\int\limits_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}}\left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}-y\right]g''(y)\,dy\right|\leq\frac{1}{2}\left(S^{\beta}_{n,p}\left(\varphi^1_x;x\right)\right)^2\left\|g''\right\|,$$

by Lemma 1.3, we obtain

$$\begin{aligned}
&\left|\overline{S}^{\beta}_{n,p}(g;x)-g(x)\right|\\
\leq\ & S^{\beta}_{n,p}\left(\left|\int\limits_x^t (t-y)\,g''(y)\,dy\right|;x\right)+\left|\int\limits_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}}\left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}-y\right]\left|g''(y)\right|dy\right|\\
\leq\ & \frac{\left\|g''\right\|}{2}\left[S^{\beta}_{n,p}\left(\varphi^2_x;x\right)+\left(S^{\beta}_{n,p}\left(\varphi^1_x;x\right)\right)^2\right]\\
=\ & \frac{\left\|g''\right\|}{2}\left[2\left(\beta+\frac{p}{n}\right)^2\frac{x^2}{(1-\beta)^2}+\left(1+\frac{p}{n}\right)\frac{x}{n(1-\beta)^3}\right]. \qquad (2.7)
\end{aligned}$$

On the other hand, from (2.4) and Lemma 1.2, it can be easily obtained that

$$\left|\overline{S}^{\beta}_{n,p}(f;x)\right|\leq\left|S^{\beta}_{n,p}(f;x)\right|+2\left\|f\right\|\leq 3\left\|f\right\| \qquad (2.8)$$

for $f\in C_B[0,\infty)$. Thus, taking (2.4), (2.7) and (2.8) into account, for $f,g\in C_B[0,\infty)$ one has

$$\begin{aligned}
&\left|S^{\beta}_{n,p}(f;x)-f(x)\right|\\
\leq\ & \left|\overline{S}^{\beta}_{n,p}(f-g;x)-(f-g)(x)\right|+\left|\overline{S}^{\beta}_{n,p}(g;x)-g(x)\right|\\
&+\left|f(x)-f\left(\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}\right)\right|\\
\leq\ & \omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)\\
&+4\left\{\left\|f-g\right\|+\frac{1}{4}\left[\left(\beta+\frac{p}{n}\right)^2\frac{x^2}{(1-\beta)^2}+\left(1+\frac{p}{n}\right)\frac{x}{2n(1-\beta)^3}\right]\left\|g''\right\|\right\}.
\end{aligned}$$

Finally, taking infimum over all $g \in C_B^2[0,\infty)$ on the right hand-side of the last inequality and applying (2.1), we get

$$\begin{aligned}\left|S_{n,p}^{\beta}(f;x)-f(x)\right| &\leq \omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+K\left(f,\left(\delta_{n,p}^{\beta}(x)\right)^2\right)\\ &\leq \omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta}(x)\right),\end{aligned}$$

where $\delta_{n,p}^{\beta}(x)$ is given by (2.3). □

Note that the case $p=0$ in the above theorem reduces to Theorem 2 in [2].

Taking Remark 2.1 and (2.2) into account, we reach to the following conclusion:

Corollary 2.1. *i) If β is taken as a sequence β_n such that $0 \leq \beta_n < 1$ and $lim_{n\to\infty}\beta_n = 0$ and $f \in UC_B[0,\infty)$, then one gets $\lim_{n\to\infty} S_{n,p}^{\beta_n}(f;x) = f(x)$ on $[0,\infty)$ and the order of the approximation does not exceed to that of $\omega_1\left(f,\left(\beta_n+\frac{p}{n}\right)\frac{x}{1-\beta_n}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta_n}(x)\right)$.*

ii) If β is taken as a sequence β_n such that $0 \leq \beta_n < 1$ and $lim_{n\to\infty}\beta_n = 0$ and $f \in C_B[0,\infty)$, then $\left\{S_{n,p}^{\beta_n}(f)\right\}_{n\geq 1}$ converges uniformly to f on $[a,b]$, $0 \leq a < b < \infty$, by the well known Korovkin theorem.

3. A Voronovskaja-type result

In [9], Farcaş obtained the following Voronovskaja-type result for the Jain operator $P_n^{[\beta]}$ given by (1.2):

$$\lim_{n\to\infty} n\left\{P_n^{[\beta_n]}(f;x)-f(x)\right\}=\frac{x}{2}f''(x), \quad x>0,$$

for $f \in C_2[0,\infty)$, the space of all continuous functions having continuous second order derivative, where $0 \leq \beta_n < 1$ is a sequence such that $\lim_{n\to\infty}\beta_n = 0$.

Note that a Voronovskaja-type result for a generalization of the Jain operators was obtained by Olgun et al. [18]. On the other hand, a Voronovskaja-type theorem as well as its a generalized form for Schurer setting of the Szász-Mirakjan operators were obtained by Sikkema in [20, p. 333].

In this part, we investigate a Voronovskaja-type result for the Jain-Schurer operators $S_{n,p}^{\beta}$, $n \in \mathbb{N}$.

Theorem 3.1. *Let $p \in \mathbb{N}_0$ be fixed and $0 \leq \beta_n < 1$ be a sequence such that $lim_{n\to\infty} n\beta_n = 0$. If f is bounded and continuous on $[0,\infty)$ and has the second order derivative at some $x \in (0,\infty)$, then one has*

$$\lim_{n\to\infty} n\left\{S_{n,p}^{\beta_n}(f;x)-f(x)\right\}=pxf'(x)+\frac{x}{2}f''(x).$$

Proof. From Taylor's formula, one has

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t-x)(t-x)^2, \tag{3.1}$$

at the fixed point $x \in [0,\infty)$, where $h(t-x)$ is bounded for all $t \in [0,\infty)$ and $\lim_{t\to x} h(t-x) = 0$. Application of the operators $S_{n,p}^{\beta}$ to (3.1) implies

$$\begin{aligned} n\left[S_{n,p}^{\beta_n}(f;x) - f(x)\right] &= f'(x)\, nS_{n,p}^{\beta_n}(t-x;x) + \frac{1}{2}f''(x)\, nS_{n,p}^{\beta_n}\left((t-x)^2;x\right) \\ &\quad + nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right). \end{aligned}$$

Using the facts $\lim_{n\to\infty} n\beta_n = 0$ and Lemma 1.3, it readily follows that

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}(t-x;x) = px$$

and

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left((t-x)^2;x\right) = x.$$

Hence, we have

$$\lim_{n\to\infty} n\left(S_{n,p}^{\beta_n}(f;x) - f(x)\right) = pxf'(x) + \frac{x}{2}f''(x) + \lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right).$$

It suffices to prove that $\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) = 0$. Indeed, defining $h(0) = 0$ and taking the fact $\lim_{t\to x} h(t-x) = 0$ into account, we get that h is continuous at x. Hence, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|h(t-x)| < \varepsilon$ for all t satisfying $|t-x| < \delta$. On the other hand, since $h(t-x)$ is bounded on $[0,\infty)$, there is an $M > 0$ such that $|h(t-x)| \leq M$ for all t. Therefore, we may write $|h(t-x)| \leq M\frac{(t-x)^2}{\delta^2}$ when $|t-x| \geq \delta$. So, these arguments enable one to write $|h(t-x)| \leq \varepsilon + M\frac{(t-x)^2}{\delta^2}$ for all t. The monotonicity and linearity of $S_{n,p}^{\beta_n}$ give that

$$\begin{aligned} S_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) &\leq \varepsilon S_{n,p}^{\beta_n}\left((t-x)^2;x\right) + \frac{M}{\delta^2}S_{n,p}^{\beta_n}\left((t-x)^4;x\right) \\ &= \varepsilon S_{n,p}^{\beta_n}\left(\varphi_x^2;x\right) + \frac{M}{\delta^2}S_{n,p}^{\beta_n}\left(\varphi_x^4;x\right). \end{aligned}$$

Making use of Lemma 1.3, with $\beta = \beta_n$,

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) = 0,$$

by the hypothesis on β_n, which completes the proof. $\square$

4. A Retaining Property

Recall that *A continuous and non-negative function* ω *defined on* $[0, \infty)$ *is called a modulus of continuity, if each of the following conditions is satisfied:*

i) $\omega(u+v) \leq \omega(u) + \omega(v)$ *for* $u, v, u+v \in [0, \infty)$*, i.e.,* ω *is semi-additive,*

ii) $\omega(u) \geq \omega(v)$ *for* $u \geq v > 0$*, i.e.,* ω *is non-decreasing,*

iii) $\lim_{u \to 0^+} \omega(u) = \omega(0) = 0$, ([15, p. 106]).

In [13], Li proved that each Bernstein polynomial preserves the properties of modulus of continuity on $[0, 1]$. Motivated by this result, in this section we will show that each Jain-Schurer operator has this preservation property as well. In the proof, we need the following Jensen formula

$$(4.1) \quad (u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u (u+k\beta)^{k-1} v [v+(m-k)\beta]^{m-k-1},$$

where u, v, and $\beta \in \mathbb{R}$ (see, e.g., [3, p. 326]).

Theorem 4.1. *Let* $p \in \mathbb{N}_0$ *be fixed and* $0 \leq \beta < 1$*. If* ω *is a bounded modulus of continuity on* $[0, \infty)$*, then for each* $n \in \mathbb{N}$*,* $S_{n,p}^{\beta}(\omega; x)$ *is also a modulus of continuity.*

Proof. Let $x, y \in [0, \infty)$ *and* $x \leq y$*. From the definition of* S_n^{β}*, we have*

$$(4.2) \qquad S_{n,p}^{\beta}(\omega; y) = \sum_{j=0}^{\infty} \omega\left(\frac{j}{n}\right) \frac{(n+p)\, y[(n+p)\, y + j\beta]^{j-1}}{j!} e^{-[(n+p)y+j\beta]}.$$

Taking $u = (n+p)\, x$*,* $v = (n+p)\, y - (n+p)\, x$ *and* $m = j$ *in (4.1), we obtain*

$$\begin{aligned} & (n+p)\, y\, ((n+p)\, y + j\beta)^{j-1} \\ = & \sum_{k=0}^{j} \binom{j}{k} (n+p)\, x \left[(n+p)\, x + k\beta\right]^{k-1} \\ & \times (n+p)(y-x) \left[(n+p)(y-x) + (j-k)\,\beta\right]^{j-k-1}. \end{aligned}$$

Substituting this expression into (4.2) we get

$$\begin{aligned}
S_{n,p}^{\beta}(\omega;y) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\omega\left(\frac{j}{n}\right)\binom{j}{k}\frac{1}{j!}(n+p)x\left[(n+p)x+k\beta\right]^{k-1} \\
&\quad\times(n+p)(y-x)\left[(n+p)(y-x)+(j-k)\beta\right]^{j-k-1}e^{-[(n+p)y+j\beta]} \\
&= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\omega\left(\frac{j}{n}\right)\frac{1}{k!(j-k)!}(n+p)x\left[(n+p)x+k\beta\right]^{k-1} \\
&\quad\times(n+p)(y-x)\left[(n+p)(y-x)+(j-k)\beta\right]^{j-k-1}e^{-[(n+p)y+j\beta]} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{k+l}{n}\right)\frac{1}{k!l!}(n+p)x\left[(n+p)x+k\beta\right]^{k-1} \\
&\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+k\beta+l\beta]}. \qquad (4.3)
\end{aligned}$$

On the other hand, from (1.3), we have

$$e^{(n+p)(y-x)}=\sum_{l=0}^{\infty}\frac{(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}}{l!}e^{-l\beta}.$$

Therefore, $S_{n,p}^{\beta}(\omega;x)$ *may be written as*

$$\begin{aligned}
S_{n,p}^{\beta}(\omega;x) &= \sum_{k=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{(n+p)x[(n+p)x+k\beta]^{k-1}}{k!}e^{-[(n+p)x+k\beta]} \\
&= \sum_{k=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{(n+p)x[(n+p)x+k\beta]^{k-1}}{k!}e^{-[(n+p)y+k\beta]}e^{(n+p)(y-x)} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{1}{k!l!}(n+p)x\left[(n+p)x+k\beta\right]^{k-1} \\
&\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+k\beta+l\beta]}. \qquad (4.4)
\end{aligned}$$

Subtracting (4.4) from (4.3), we obtain

$$\begin{aligned}
&S_{n,p}^{\beta}(\omega;y)-S_{n,p}^{\beta}(\omega;x) \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\omega\left(\frac{k+l}{n}\right)-\omega\left(\frac{k}{n}\right)\right\}\frac{1}{k!l!}(n+p)x\left[(n+p)x+k\beta\right]^{k-1} \\
&\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+(k+l)\beta]}. \qquad (4.5)
\end{aligned}$$

Using the semi-additivity property of ω*, we get*

$$\begin{aligned}
& S_{n,p}^{\beta}(\omega;y)-S_{n,p}^{\beta}(\omega;x) \\
\leq\ & \sum_{k=0}^{\infty}\frac{(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1}}{k!}e^{-k\beta} \\
& \times\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{(n+p)\,(y-x)\left[(n+p)\,(y-x)+l\beta\right]^{l-1}}{l!}e^{-[(n+p)y+l\beta]} \\
=\ & e^{(n+p)x}\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{(n+p)\,(y-x)\left[(n+p)\,(y-x)+l\beta\right]^{l-1}}{l!}e^{-[(n+p)y+l\beta]} \\
=\ & \sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)\frac{(n+p)\,(y-x)\left[(n+p)\,(y-x)+l\beta\right]^{l-1}}{l!}e^{-[(n+p)(y-x)+l\beta]} \\
=\ & S_{n,p}^{\beta}(\omega;y-x),
\end{aligned}$$

which shows the semi-additivity of $S_{n,p}^{\beta}$*. From (4.5) it readily follows that* $S_{n,p}^{\beta}(\omega;y)\geq S_{n,p}^{\beta}(\omega;x)$ *for* $y\geq x$*, i.e.,* $S_{n,p}^{\beta}$ *is non-decreasing. Moreover, since the series is uniformly convergent, it follows that* $\lim_{x\to 0^+}S_{n,p}^{\beta}(\omega;x)=S_{n,p}^{\beta}(\omega;0)=\omega(0)=0$*. This comples the proof.* □

5. Monotonicity of the sequence of the Jain-Schurer operators

In [5], Cheney and Sharma proved that the sequence of Szász-Mirakjan operators $P_n^{[0]}(f)$ is non-increasing in n, when f is convex. The purpose of this section is to observe the monotonicity of the sequence of the Jain-Schurer operators when the attached function is convex and non-decreasing and $p\neq 0$. In the case $p=0$, we obtain monotonicity of the sequence of Jain operators in n when f is convex. For the proof, we further need the following Abel-Jensen formula

$$(u+v+m\beta)^m=\sum_{k=0}^{m}\binom{m}{k}(u+k\beta)^k\,v\left[v+(m-k)\,\beta\right]^{m-k-1} \tag{5.1}$$

for non-negative real number β, where $u,v\in\mathbb{R}$ and $m\geq 1$ (see, e.g., [21]). Reasoning as in [5], we present the following result:

Theorem 5.1. *Let* f *be a non-decreasing and convex function on* $[0,\infty)$*. Then, for all* n*,* $S_{n,p}^{\beta}(f)$ *is non-increasing in* n *when* $p\neq 0$*. For the case* $p=0$*, the same result holds when* f *is only convex on* $[0,\infty)$*.*

Proof. From (1.3), with $\alpha=x$, it is obvious that

$$e^x=\sum_{k=0}^{\infty}\frac{x\,(x+k\beta)^{k-1}}{k!}e^{-k\beta}. \tag{5.2}$$

Since $S_{n,p}^{\beta}(f;0)=f(0)$, we study only for $x>0$. Taking the definition of $S_{n,p}^{\beta}$ and (5.2) into consideration, one has

$$
\begin{aligned}
& S_{n,p}^{\beta}(f;x)-S_{n+1,p}^{\beta}(f;x) \\
= \; & e^{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]} \\
& -\sum_{k=0}^{\infty} f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]} \\
= \; & \sum_{l=0}^{\infty}\frac{x\,(x+l\beta)^{l-1}}{l!}e^{-l\beta}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]} \\
& -\sum_{k=0}^{\infty} f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}.
\end{aligned}
$$

By simple calculations, one can write

$$
\begin{aligned}
(5.3) \quad & S_{n,p}^{\beta}(f;x)-S_{n+1,p}^{\beta}(f;x) \\
= \; & \sum_{l=0}^{\infty}\frac{x\,(x+l\beta)^{l-1}}{l!}e^{-l\beta}\sum_{k=l}^{\infty} f\left(\frac{k-l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}e^{-[(n+1+p)x+(k-l)\beta]} \\
& -\sum_{k=0}^{\infty} f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]} \\
= \; & \sum_{k=0}^{\infty} e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k} f\left(\frac{k-l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}\frac{x\,(x+l\beta)^{l-1}}{l!}\right. \\
& \left.-f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}\right\} \\
= \; & \sum_{k=0}^{\infty} e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k} f\left(\frac{l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}}{l!}\frac{x\,[x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}\right. \\
& \left.-f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}\right\} \\
= \; & \sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\times \\
& \left\{\sum_{l=0}^{k}\binom{k}{l}\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}f\left(\frac{l}{n}\right)-f\left(\frac{k}{n+1}\right)\right\}.
\end{aligned}
$$

Now, it only remains to show that the curly bracket in the last formula must be non-negative. For this, we denote

$$\alpha_l:=\binom{k}{l}\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}>0,$$

and

$$x_l=\frac{l}{n}$$

for $l=0,1,...,k$. Now, replacing u with $(n+p)\,x$, v with x, m with k and k with l in (4.1) we evidently get
(5.4)

$$\sum_{l=0}^{k}\alpha_l=\frac{1}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}=1.$$

On the other hand, it follows that

$$\begin{aligned}\sum_{l=0}^{k}\alpha_l x_l &= \frac{1}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\times\\ &\quad \sum_{l=0}^{k}\binom{k}{l}\frac{l}{n}\,(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}\\ &= \frac{k\,(n+p)\,x}{n\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\times\\ &\quad \sum_{l=0}^{k-1}\binom{k-1}{l}[(n+p)\,x+\beta+l\beta]^{l}x\,[x+(k-l-1)\,\beta]^{k-l-2}. \qquad (5.5)\end{aligned}$$

Making use of the Abel-Jensen formula given by (5.1) for $u=(n+p)\,x+\beta,\ v=x,\ k=l,\ m=k-1$, (5.5) reduces to

$$\begin{aligned}\sum_{l=0}^{k}\alpha_l x_l &= \frac{k\,(n+p)\,x}{n\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}[(n+1+p)\,x+k\beta]^{k-1}\\ &= \frac{k\,(n+p)}{n\,(n+1+p)}. \qquad (5.6)\end{aligned}$$

Taking into account (5.4), (5.6) and the convexity of f, (5.3) reduces to

$$\begin{aligned}(5.7)\,&S_{n,p}^{\beta}\,(f;x)-S_{n+1,p}^{\beta}\,(f;x)\\ &=\sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k}\alpha_l f\left(\frac{l}{n}\right)-f\left(\frac{k}{n+1}\right)\right\}\\ &\geq\sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\left\{f\left(\frac{k\,(n+p)}{n\,(n+1+p)}\right)-f\left(\frac{k}{n+1}\right)\right\}.\end{aligned}$$

It is obvious that when $p=0$, (5.7) gives the non-negativity of $S_{n,0}^{\beta}\,(f;x)-S_{n+1,0}^{\beta}\,(f;x)$ under the convexity of f, which means that the sequence of Jain operators is non-increasing in n under the convexity of the function. On the other hand, for $p\in\mathbb{N}$ it follows that

$$\frac{k\,(n+p)}{n\,(n+1+p)}=\frac{k}{n+1}\,\frac{n+1}{n}\,\frac{n+p}{n+1+p}=\frac{k}{n+1}\,\frac{1+\frac{p}{n}}{\left(1+\frac{p}{n+1}\right)}.$$

Hence, one has

$$\frac{k\,(n+p)}{n\,(n+1+p)} \geq \frac{k}{n+1}$$

by the fact that $\frac{1+\frac{p}{n}}{1+\frac{p}{n+1}} \geq 1$. Then, the result follows directly from the non-decreasingness of f. $\square$

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