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BEST PROXIMITY POINTS IN NON-ARCHIMEDEAN FUZZY METRIC SPACES *

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Abstract. In this article we establish best proximity point theorems for non-self proximal contractions in the setting of Non-Archimedean Fuzzy Metric Space which are more general than the notion of self-contractions. At the end of this paper, we support our theorems of proximity points by providing an example.

Keywords: Best proximity point, Non-Archimedean fuzzy metric space.

1. Introduction

In 1965 Zadeh([10]) introduced the concept of fuzzy set. Using the idea of fuzzy sets Kramosil and Michalek ([8]) introduced the concept of fuzzy metric space in the year 1975. Later on, George and Veermani([1]) modified the concept of fuzzy metric spaces and defined a Hausdorff topology on this fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and E-infinity theory. It has also been shown that every metric induces a fuzzy metric in Hausdroff topology.

In particular, Mihet([2]) proved a fuzzy Banach contraction result for complete Non-Archimedean fuzzy metric spaces ([2], Theorem 3.16); For more results on Non-Archimedean fuzzy metric space we suggest ([5], [11], [17], [18], [19], [20]).

On the other hand, Vetro and Salimi([6]) investigated the existence and uniqueness of the best proximity points in a Non-Archimedean fuzzy metric space, suggesting a way to obtain some points after the unavailability of fixed points, approximate points for non-self maps extend and fuzzify the existing results in metric spaces. For more results on best proximity point, we suggest ([3], [7], [4], [13] - [18], [22], [21]).

Best proximity point is one of the most interesting results in the extension of Fixed Point Theorem in metric space for non-self mapping such that A to B does not

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necessarily have a fixed point where A and B are nonempty closed subsets of a complete metric space (X, d), with $A \cap B = \phi$.
Let $T : A \to B$; then a point x in A is called a best proximity point of T, if $d(x, T(x)) = d(A, B) \neq 0$, whenever a non-self mapping T has no fixed point, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.
A best proximity point represents an optimal solution to the equation $T(x) = x$. Since best proximity point reduces to a fixed point, if the defined non-self mappings, the best proximity point theorems are natural generalizations of the fixed point theorems.
In this paper we establish best proximity point theorems for non-self proximal contractions in the setting of Non-Archimedean Fuzzy Metric Space which are more general than the notion of self-contractions. In this way we extend and fuzzify the existing results of Basha([12]) of metric spaces. Also, we provide one example in support of our theorems for best proximity point.
We recall the following definitions for our results.

Definition 1.1. ([10]) A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 1.2. ([9]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for every $a \in [0, 1]$;
4. $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.3. ([1]) A triplet $(X, M, *)$ is said to be a fuzzy metric space(in the sense of George and Veeramani), if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

(F_1) $M_{x,y}(t) > 0$,

(F_2) $M_{x,y}(t) = 1$ if and only if $x = y$,

(F_3) $M_{x,y}(t) = M_{y,x}(t)$,

(F_4) $M_{x,y}(\cdot) : (0, \infty) \to [0, 1]$ is continuous.

(F_5) $M_{x,y}(t) * M_{y,z}(s) \leq M_{x,z}(t + s)$,
If we replace F_5 by

(F_6) $M_{x,y}(t) * M_{y,z}(s) \leq M_{x,z} \max(t, s)$,

then the triplet $(X, M, *)$ is called non-Archimedean fuzzy metric space.
Note that, since (F_6) implies (F_4), each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 1.4. ([1]) Let $(X, M, *)$ be a fuzzy metric space (or non-Archimedean fuzzy metric space), then

(a) a sequence $\{x_n\}$ in X is said to be convergent to x if

$$lim_{n\to\infty} M_{x_n,x}(t) = 1,$$

for all $t > 0$;

(b) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$M_{x_n,x_m}(t) > 1 - \epsilon,$$

for all $t > 0$ and $n, m \geq n_0$;

(c) a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

2. Preliminaries

For a given two non-empty subsets A and B of a non-Archimedean fuzzy metric space $(X, M, *)$, the following notions are used through out this section:
$M(A, B, t) = \sup\{M(x, y, t) : x \in A \quad \text{and} \quad y \in B\}$
$A_0 = \{x \in A : M(x, y, t) = M(A, B, t) \quad \text{for some} \quad y \in B\}$
$B_0 = \{y \in B : M(x, y, t) = M(A, B, t) \quad \text{for some} \quad x \in A\}$
for all $t > 0$.

In 2011, Basha([12] introduced the concept of approximately compact, proximal contraction of first kind and proximal contraction of second kind in complete metric space.
Here, we are going to introduce these definitions in the setting of non-Archimedean fuzzy metric space.

Definition 2.1. A is said to be approximately compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $M(y, x_n, t) \to M(y, A, t)$ for some y in B and for all $t > 0$ has a convergent sub sequence.

It is evident that every set is approximately compact with respect to itself. If A intersects B, then $A \cap B$ is contained in both A_0 and B_0. Further, it can be seen that if A is compact and B is approximately compact with respect to A, then the sets A_0 and B_0 are non-empty.

Definition 2.2. A mapping $T : A \to B$ is said to be a proximal contraction of first kind if there exists a non-negative number $k \in [0, 1)$ such that, for all u_1, u_2, x_1, x_2 in A,

$$\begin{cases} M(u_1, Tx_1, t) = M(A, B, t) \\ M(u_2, Tx_2, t) = M(A, B, t). \end{cases} \Rightarrow M(u_1, u_2, kt) \geq M(x_1, x_2, t)$$

for all $t > 0$.

It is easy to see that a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a non-self proximal contraction non-self-mapping is not necessarily a contraction.

Definition 2.3. A non-self mapping $T : A \to B$ is said to be a proximal contraction of second kind if there exists a non-negative real number $k \in [0, 1)$ and for all $t > 0$ such that for all u_1, u_2, x_1, x_2 in A,

$$M(Tu_1, Tu_2, t) \geq M(Tx_1, Tx_2, \frac{t}{k})$$

whenever x_1, x_2, u_1 and u_2 are elements in A satisfying the condition that

$$M(u_1, Tx_1, t) = M(A, B, t) \text{ and } M(u_2, Tx_2, t) = M(A, B, t).$$

The requirement for a self-mapping T to be a proximal contraction of second kind is that

$$M(T^2x_1, T^2x_2, t) \geq M(Tx_1, Tx_2, \frac{t}{k^2})$$

for all x_1 and x_2 in the domain of T.

It is remarkable that every contraction self-mapping is a proximal contraction of second kind but the converse is not true.
Consider R endowed with the Euclidean metric. Let the self mapping $T : [0, 1] \to [0, 1]$ be defined as

$$T(x) = \begin{cases} 0, & \text{if x is rational;} \\ 1, & \text{otherwise.} \end{cases}$$

Then, T is a proximal contraction of second kind but not a contraction. Also, a self-mapping that is a proximal contraction of second kind is not necessarily continuous.

Definition 2.4. Given $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve isometric distance with respect to g if

$$M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$$

for all x_1 and x_2 in A and $t > 0$.

3. Main Results

The following main result is a best proximity point theorem for non-self-mappings which are proximal contractions of the first kind as well as of the second kind.

Theorem 3.1. *Let $(X, M, *)$ be complete non-Archimedean fuzzy metric space. Let A and B be non-void closed subsets of $(X, M, *)$ such that A is approximatively compact with respect to B. Further suppose that A_0 and B_0 are non-void. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) T is a continuous proximal contraction of second kind.

(b) g is an isometry.

(c) $T(A_0)$ is contained in B_0.

(d) A_0 is contained in $g(A_0)$.

(e) T preserves isometric distance with respect to g.

Then, there exists an element x in A such that

$$M(gx, Tx, t) = M(A, B, t) \text{ for all } t > 0.$$

Further, if x^* is another element for which the preceding conclusion holds, then Tx and Tx^* are identical.

Proof. Let x_0 be a fixed element in A_0. Since $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, there exists an element x_1 in A_0 such that

$$M(gx_1, Tx_0, t) = M(A, B, t).$$

Again, since Tx_1 is an element of $T(A_0)$ which is contained in B_0, and A_0 is contained in $g(A_0)$, it follows that there is an element x_2 in A_0 such that

$$M(gx_2, Tx_1, t) = M(A, B, t).$$

Continuing in this way we will have x_n in A_0, and it is possible to find x_{n+1} in A_0 such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t).$$

for every positive integer n because of the fact that $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$. As T is a proximal contraction of second kind,

$$M(Tgx_{n+1}, Tgx_n, kt) \geq M(Tx_n, Tx_{n-1}, t).$$

Since T preserves isometric distance with respect to g,

$$M(Tx_{n+1}, Tx_n, kt) \geq M(Tx_n, Tx_{n-1}, t).$$

So, it follows that $\{Tx_n\}$ is a Cauchy sequence and hence it converges to some element y in B.
Further,

$$\begin{aligned} M(y, A, t) &\geq M(y, gx_n, t) \\ &\geq M(y, Tx_{n-1}, \tfrac{t}{2}) * M(Tx_{n-1}, gx_n, \tfrac{t}{2}) \\ &= M(y, Tx_{n-1}, \tfrac{t}{2}) * M(A, B, t) \\ &\geq M(y, Tx_{n-1}, \tfrac{t}{2}) * M(y, A, t) \end{aligned}$$

Therefore, $M(y, gx_n, t) \to M(y, A, t)$. In view of the fact that A is approximatively compact with respect to $B, \{gx_n\}$ has a subsequence $\{gx_{n_k}\}$ converging to some z in A. Therefore, it can be concluded that

$$M(z, y, t) = \lim_{k\to\infty} M(gx_{n_k}, Tx_{n_k-1}, t) = M(A, B, t).$$

Eventually, z is a member of A_0. Since A_0 is contained in $g(A_0), z = gx$ for some x in A_0. As $g(x_{n_k}) \to g(x)$ and g is an isometry, $x_{n_k} \to x$. Since the mapping T is continuous, it follows that $Tx_{n_k} \to Tx$. Consequently, y and Tx are identical. Thus, it follows that

$$M(gx, Tx, t) = \lim_{k\to\infty} M(gx_{n_k}, Tx_{n_k-1}, t) = M(A, B, t).$$

Suppose that there is another element x^* such that

$$M(gx^*, Tx^*, t) = M(A, B, t).$$

Since T is a proximal contraction of second kind,

$$M(Tgx, Tgx^*, t) \geq M(Tx, Tx^*, \frac{t}{k})$$

As T preserves isometric distance with respect to g, we have

$$M(Tx, Tx^*, t) \geq M(Tx, Tx^*, \frac{t}{k})$$

which implies that $Tx = Tx^*$. This completes the proof of the theorem. □

If g is the identity mapping, then the preceding theorem yields the following corollary.

Corollary 3.1. *Let A and B be non-empty, closed subsets of a complete fuzzy metric space such that A is approximatively compact with respect to B. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \to B$ satisfy the following conditions.*

(a) T is a continuous proximal contraction of second kind.

(b) $T(A_0)$ is contained in B_0.

Then, there exists an element x in A such that $M(x, Tx, t) = M(A, B, t)$.
Moreover, if x^* is another best proximity point of T, then Tx and Tx^* are identical.

The following result provides another generalization of Banach's contraction principle to the case of non-self mappings.

Theorem 3.2. *Let X be a complete fuzzy metric space. Let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) T is a continuous proximal contraction of first kind.

(b) g is an isometry.

(c) $T(A_0)$ is contained in B_0.

(d) A_0 is contained in $g(A_0)$.

Then, there exists an element x in A such that

$$M(gx, Tx, t) = M(A, B, t) \text{ for all } t > 0.$$

Proof. Proceeding as in Theorem(3.1), there exists a sequence $\{x_n\}$ in A satisfying the following condition.

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t).$$

Since T is a proximal contraction of first kind, we have

$$M(gx_{n+1}, gx_n, kt) \geq M(x_n, x_{n-1}, t).$$

Since g is an isometry, it follows that

$$M(x_{n+1}, x_n, kt) \geq M(x_n, x_{n-1}, t).$$

Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some x in A. Since g and T are continuous, we have

$$M(gx, Tx, t) = \lim_{n\to\infty} M(gx_{n+1}, Tx_n, t) = M(A, B, t).$$

Suppose that there is another element x^* such that

$$M(gx^*, Tx^*, t) = M(A, B, t).$$

Since T is a proximal contraction of first kind and g is an isometry, we have

$$M(x, x^*, t) = M(gx, gx^*, t) \geq M(x, x^*, \frac{t}{k})$$

which implies that x and x^* are identical. This completes the proof of the theorem. □

If g is the identity mapping, then the preceding theorem yields the following best proximity point theorem.

Corollary 3.2. *Let X be a complete fuzzy metric space. Let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \to B$ satisfy the following conditions.*

(a) T is a continuous proximal contraction of first kind.

(b) $T(A_0)$ is contained in B_0.

Then, there exists an element x in A such that

$$M(x, Tx, t) = M(A, B, t) \text{ for } t > 0.$$

Example 3.1. Let $X = N$ and let $M : X \times X \times (0, +\infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by

$$M(x, y, t) = \begin{cases} \frac{x}{y}, \text{if } x \leq y, \\ \frac{y}{x}, \text{otherwise.} \end{cases}$$

for all $t > 0$.
Thus, $(X, M, *)$ is complete with $a * b = ab$ for all $a, b \in [0, 1]$.
Define the sets

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{6, 7, 8, 9, 10\},$$

so that $M(A, B, t) = \frac{5}{6}$. Clearly, A and B are nonempty closed subsets of $X; A_0(t) = \{5\}$ and $B_0(t) = \{6\}$.
Also define $T : A \to B$ by

$$T(x) = \begin{cases} 6, \text{if } x = 5, \\ x + 5, \text{otherwise.} \end{cases}$$

Notice that $T(A_0(t)) \subset B_0(t)$ and $g(x) = x$ so all the hypothesis of Theorem(3.1) holds true. Assume that $M(u, Tx, t) = M(A, B, t)$ for some $u, x \in A$, then $(u, x) = (5, 1)$ or $(u, x) = (5, 5)$. Next, putting $(u, x) = (5, 1)$ and $(v, y) = (5, 5)$, then $M(T^2u, T^2v, kt) = M(5, 5, kt) = 1 \geq M(u, v, t)$.
for all $t > 0$.
We conclude that all the hypotheses of Theorem(3.1) are satisfied, and so there exists a unique $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$ for all $t > 0$. Here, $x^* = 5$.

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