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SOME REMARKS ON THE CLASSICAL PRIME SPECTRUM OF MODULES

Alireza Abbasi and Mohammad Hasan Naderi

Faculty of Science,Department of Mathematics, University of Qom,
Qom, Iran, P.O. Box 37161-46611

Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule P of M is called a classical prime submodule if $abm \in P$, for $a, b \in R$, and $m \in M$, implies that $am \in P$ or $bm \in P$. The classical prime spectrum of M, Cl.Spec(M), is defined to be the set of all classical prime submodules of M. We say M is classical primefule if $M = 0$, or the map ψ from Cl.Spec(M) to Spec$(R/\text{Ann}(M))$, defined by $\psi(P) = (P : M)/\text{Ann}(M)$ for all $P \in$ Cl.Spec(M), is surjective. In this paper, we study classical primeful modules as a generalization of primeful modules. Also, we investigate some properties of a topology that is defined on Cl.Spec(M), named the Zariski topology.
Keywords: Classical prime, Classical primeful, Classical top module

1. Introduction

Throughout the paper all rings are commutative with identity and all modules are unital. Let M be an R-module. If N is a submodule of M, then we write $N \leq M$. For any two submodules N and K of an R-module M, the residual of N by K is denoted by $(N : K) = \{r \in R : rK \subseteq N\}$. A proper submodule P of M is called a prime submodule if $am \in P$, for $a \in R$ and $m \in M$, implies that $m \in P$ or $a \in (P : M)$. Also, a proper submodule P of M is called a classical prime submodule if $abm \in P$, for $a, b \in R$ and $m \in M$, implies that $am \in P$ or $bm \in P$(see for example [5]). The set of prime(resp. classical prime) submodules of M is denoted by Spec(M)(resp. Cl.Spec(M)). The class of prime submodules of modules was introduced and studied in 1992 as a generalization of

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Corresponding Author: Mohammad hasan Naderi, Faculty of Science,Department of Mathematics, University of Qom, Qom, Iran, P.O. Box 37161-46611 | E-mail: mh.naderi@qom.ac.ir
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the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, classical prime, classical primary and classical quasi primary submodules, see [1, 8, 16, 4] and [7].

For a proper submodule N of an R-module M, the prime radical of N is $\sqrt[p]{N} = \cap\{P|P \in \mathcal{V}^*(N)\}$, where $\mathcal{V}^*(N) = \{P \in \text{Spec}(M) \mid N \subseteq P\}$. Also the classical prime radical of N is $\sqrt[Cl]{N} = \cap\{P|P \in \mathcal{V}(N)\}$, where $\mathcal{V}(N) = \{P \in \text{Cl.Spec}(M) \mid N \subseteq P\}$. If there are no such prime (resp. classical prime) submodules, $\sqrt[p]{N}$ (resp. $\sqrt[Cl]{N}$) is M. We say N is a radical (resp. classical radical) submodule, if $\sqrt[p]{N} = N$(resp. $\sqrt[Cl]{N} = N$).

The set of all maximal submodules of M is denoted by $\text{Max}(M)$. A Noetherian module M is called a semi-local (resp. a local) module if $\text{Max}(M)$ is a non-empty finite (resp. a singleton) set. A non-Noetherian commutative ring R is called a quasisemilocal (resp. a quasilocal) ring if R has only a finite number (resp. a singleton) of maximal ideals. An R-module M is called a multiplication (resp. weak multiplication) module if for every submodule (resp. prime submodule) of M, there exists an ideal I of R such that $N = IM$(see [14] and [2]). If N is a prime submodule of a multiplication R-module M, then $N_1 \cap N_2 \subseteq N$, where $N_1, N_2 \leq M$, implies that $N_1 \subseteq N$ or $N_2 \subseteq N$ (see for more detail [11] and [19]). An R-module M is called compatible if its classical prime submodules and its prime submodules coincide. All commutative rings and multiplicative modules are examples of compatible modules, (see for more detail [8]). A submodule N of M is said to be strongly irreducible if for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Strongly irreducible submodules have been characterized in [13].

Let M be an R-module. For any subset E of M, we consider classical varieties denoted by $\mathcal{V}(E)$. We define $\mathcal{V}(E) = \{P \in \text{Cl.Spec}(M) : E \subseteq P\}$. Then

(a) If N is a submodule generated by E, then $\mathcal{V}(E) = \mathcal{V}(N)$.

(b) $\mathcal{V}(0_M) = \text{Cl.Spec}(M)$ and $\mathcal{V}(M) = \emptyset$.

(c) $\bigcap_{i\in I} \mathcal{V}(N_i) = \mathcal{V}(\sum_{i\in I} N_i)$, where $N_i \leq M$

(d) $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L)$, where $N, L \leq M$.

Now, we assume that $\mathcal{C}(M)$ denotes the collection of all subsets $\mathcal{V}(N)$ of $\text{Cl.Spec}(M)$. Then, $\mathcal{C}(M)$ contains the empty set and $\text{Cl.Spec}(M)$, and also $\mathcal{C}(M)$ are closed under arbitrary intersections. However, in general, $\mathcal{C}(M)$ is not closed under finite union. An R-module M is called a classical top module if $\mathcal{C}(M)$ is closed under finite unions, i.e., for every submodules N and L of M, there exists a submodule K of M such that $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(K)$, for in this case, $\mathcal{C}(M)$ satisfies the axioms for the closed subsets of a tological space, then in this case, $\mathcal{C}(M)$ induce a topology on $\text{Cl.Spec}(M)$. We call the induced topology the classical quasi-Zariski topology(see [9]).

In this paper, we introduce the notion of classical primeful modules and also we investigate some properties of classical quasi-Zariski topology of $\text{Cl.Spec}(M)$. In Section 2, we introduce the notion of classical primeful modules as a generalization of primefule modules. In particular, in Proposition 2.3, it is proved that if M is a classical primeful R-module, then $\text{Supp}(M) = \text{V}(\text{Ann}(M))$. Then we get some properties of classical top modules. In Section 3, we get some properties of classical quasi-Zariski topology of Cl.Spec(M) and also we get some properties of classical top modules.

2. Classical primeful module

The notion of primeful modules was introduced by Chin P. Lu in [18] as follows:

Definition 2.1. An R-module M is primeful if either $M = (0)$, or $M \neq (0)$ and the map $\phi : \text{Spec}(M) \longrightarrow \text{Spec}(R/\text{Ann}(M))$, defined by $\phi(P) = (P : M)/\text{Ann}(M)$ for all $P \in \text{Spec}(M)$, is surjective.

Now, we extend the notion of primeful modules to classical primeful modules.

Definition 2.2. Suppose $\text{Cl.Spec}(M) \neq \emptyset$, then the map ψ from Cl.Spec(M) to Spec$(R/\text{Ann}(M))$ defined by $\psi(P) = (P : M)/\text{Ann}(M)$ for all $P \in \text{Cl.Spec}(M)$, will be called the natural map of Cl.Spec(M).

An R-module M is classical primeful if either

(i) $M = (0)$, or

(ii) $M \neq (0)$ and the map $\psi : \text{Cl.Spec}(M) \longrightarrow \text{Spec}(R/\text{Ann}(M))$ from above is surjective.

Lemma 2.1. *Let M be a classical top R-module. Then the natural map $\psi : \text{Cl.Spec}(M) \longrightarrow \text{Spec}(R/\text{Ann}(M))$ is injective.*

Proof. Let $P, Q \in \text{Cl.Spec}(M)$. If $\psi(P) = \psi(Q)$, then

$$(P : M)/\text{Ann}(M) = (Q : M)/\text{Ann}(M).$$

So $(P : M) = (Q : M)$ and then $P = Q$. □

Theorem 2.1. *Let M be a classical top R-module. Then, If R satisfies ACC on prime ideals, then M satisfies ACC on classical prime submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq \dots$ be an ascending chain of classical prime submodules of M. This induces the following chain of prime ideals, $\psi(N_1) \subseteq \psi(N_2) \subseteq \dots$, where ψ is the natural map

$$\psi : \text{Cl.Spec}(M) \longrightarrow \text{Spec}(R/\text{Ann}(M)).$$

Since R satisfies ACC on prime ideals, there exists a positive integer k such that for each $i \in \mathbb{N}$, $\psi(N_k) = \psi(N_{k+i})$. Now by Lemma 2.1, we have $N_k = N_{k+i}$ as required. □

Remark 2.1. *([8, Proposition 5.3])) Let S be a multiplicatively closed subset of R, p a prime ideal of R such that $p \cap S = \emptyset$ and let M be an R-module. If P is a classical p-prime submodule of M with $P_s \neq M_s$, then P_s is also a classical p_s-prime submodule of M_s. Moreover if Q is a prime R_s-submodule of M_s, then*

$$Q^c = \{m \in M : f(m) \in Q\}$$

is a classical prime submodule of M.

Let p be a prime ideal of a ring R, M an R-module and $N \leqslant M$. By the saturation of N with respect to p, we mean the contraction of N_p in M and designate it by $S_p(N)$. It is also known that

$$S_p(N) = \{e \in M | es \in N for\ some\ s \in R \setminus p\}.$$

Saturations of submodules were investigated in detail in [17].

Proposition 2.1. *For any nonzero R-module M, the following are equivalent:*

(1) The natural map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ is surjective;

(2) For every $p \in \mathrm{V}(\mathrm{Ann}(M))$, there exists $P \in \mathrm{Cl.Spec}(M)$ such that $(P : M) = p$;

(3) $pM_p \neq M_p$, for every $p \in \mathrm{V}(\mathrm{Ann}(M))$;

(4) $S_p(pM)$, the contraction of pM_p in M, is a classical p-prime submodule of M for every $p \in \mathrm{V}(\mathrm{Ann}(M))$;

(5) $\mathrm{Cl.Spec_p(M)} \neq \emptyset$; for every $p \in \mathrm{V}(\mathrm{Ann}(M))$.

Proof. (1)$\Longleftrightarrow$(2): It is clear by Definition 2.2.

(2)$\Longrightarrow$(3): Let $p \in \mathrm{V}(\mathrm{Ann}(M))$ and let N be a classical p-prime submodule of M. Then N_p is a classical pR_p-prime submodule of M_p by Remark 2.1. Now, since $pM_p \subseteq N_p \subsetneq M_p$, we conclude that $pM_p \neq M_p$.

(3)$\Longrightarrow$(4): Since pR_p is the maximal ideal of R_p and $pM_p \neq M_p$, $pM_p = (pR_p)M_p$ is a pR_p-prime, and therefore classical pR_p-prime, submodule of M_p. Then $S_p(pM) = (pM_p)^c$, the contraction of pM_p in M, is a classical p-prime submodule of M by Remark 2.1.

(4)$\Longrightarrow$(5) and (5)$\Longrightarrow$(2) are easy. □

Proposition 2.2. *Every finitely generated R-module M is classical primeful.*

Proof. If $M = 0$, evidently the results is true. Now, let M be a nonzero finitely generated R-module. Then $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$, so for every $p \in \mathrm{V}(\mathrm{Ann}(M))$, M_p is a nonzero finitely generated module over the local ring R_p. Then by virtue of Nakayama's Lemma, $pM_p \neq M_p$, for every $p \in \mathrm{V}(\mathrm{Ann}(M))$. Therefore by Proposition 2.1, M is classical primeful. □

For every finitely generated module M, $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$. The next proposition proves that the equality holds even if M is only a classical primeful module.

Proposition 2.3. *(see [18, Proposition 3.4])) If M is a classical primeful R-module, then* $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$.

Proof. If $M = (0)$, then $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M)) = \emptyset$. Now let M be a nonzero classical primeful R-module, so $\mathrm{V}(\mathrm{Ann}(M)) \neq \emptyset$. By Proposition 2.1, if $p \in \mathrm{V}(\mathrm{Ann}(M))$, then $S_p(pM)$ is a classical p-prime submodule of M, so $\mathrm{S}_p(pM) \neq M$. Since $\mathrm{S}_p(0) \subseteq \mathrm{S}_p(pM)$, then $M \neq \mathrm{S}_p(0)$, from which we can see that $M_p \neq (0)$. Thus $\mathrm{V}(\mathrm{Ann}(M)) \subseteq \mathrm{Supp}(M)$. The other inclusion is always true.

For every prime, ideal p of R, R_p is always a quasilocal ring. However, for an arbitrary R-module M, M_p is not necessarily a local R_p-module. But by the next proposition, if M is a nonzero classical top classical primeful R-module, then $R/\mathrm{Ann}(M)$ is a quasilocal ring.

Proposition 2.4. *Let M be a nonzero classical top classical primeful R-module. If M is a semi-local (resp. local) module, then $R/\mathrm{Ann}(M)$ is a quasisemilocal (resp. a quasilocal) ring.*

Proof. Let M be a local module with unique maximal submodule P. Then $p := (P : M) \in \mathrm{Max}(R)$. Now let $\mathrm{Ann}(M) \subseteq q \in \mathrm{Max}(R)$. It is enough to prove $q = p$. To prove this, we note that $S_q(qM)$ is a classical q-prime submodule of M by Proposition 2.1. Now we show that $S_q(qM) \in \mathrm{Max}(M)$. Let $S_q(qM) \subseteq K$ for some submodule K of M. Then we have $q = (S_q(qM) : M) = (K : M)$. Hence $S_q(qM) = K$ by Lemma 2.1. This implies that $S_q(qM) = P$ and therefore $q = p$. For the semi-local case we argue similarly. □

In the rest of this section, we get some properties of classical top modules. First note that every classical top module is a top module([9, Proposition 2.4]). In the next theorem, we introduce some modules that they are classical top modules.

Theorem 2.2. *Let M be an R-module. Then M is a classical top module in each of the following cases:*

(1) M is a multiplication R-module.

(2) M be a module that every classical prime submodule of M is strongly irreducible.

(3) M is an R-module with the property that for any two submodules N and L of M, $(N : M)$ and $(L : M)$ are comaximal.

Proof. (1). Let $P \in \mathcal{V}(N_1 \cap N_2)$ and so $N_1 \cap N_2 \subseteq P$. Since M is compatible, then $(N_1 \cap N_2 : M) \subseteq (P : M)$, so $N_1 \subseteq P$ or $N_2 \subseteq P$. Therefore $P \in \mathcal{V}(N_1)$ or $P \in \mathcal{V}(N_2)$. This implies that M is a classical top module.

(2). Let $P \in \mathcal{V}(N \cap L)$. Since $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L)$, for each submodules N and L of M, then $N \cap L \subseteq P$. Now, since P is strongly irreducible, then $N \subseteq P$ or $L \subseteq P$. Therefore $P \in \mathcal{V}(N) \cup \mathcal{V}(L)$. Thus $\mathcal{C}(M)$ is closed under finite unions. Hence M is a classical top module.

(3). Let P be a classical prime submodule of M with $N \cap L \subseteq P$. Then $(N : M) \cap (L : M) \subseteq (P : M) \in \mathrm{Spec}(R)$. We may assume that $(N : M) \subseteq (P : M)$. Then clearly $(L : M) \nsubseteq (P : M)$ by assumption. Hence $N \subseteq P$. Therefore P is strongly irreducible. This implies that M is a classical top module by (2). □

If Y is a nonempty subset of Cl.Spec(M), then the intersection of the members of Y is denoted by $\mathfrak{T}(Y)$. Thus, if Y_1 and Y_2 are subsets of Cl.Spec(M), then $\mathfrak{T}(Y_1 \cup Y_2) = \mathfrak{T}(Y_1) \cap \mathfrak{T}(Y_2)$. An R-module M is said to be distributive if $(A+B) \cap C = (A \cap C) + (B \cap C)$, for all submodules A, B and C of M(see for example [12]).

Theorem 2.3. *Let M is a classical top module and $\sqrt[cl]{E} = E$ for each submodule E of M. Then M is a distributive module.*

Proof. Let A, B and C be any submodules of M. Then,

$$\begin{aligned}
(A+B) \cap C &= \sqrt[cl]{(A+B) \cap C} \\
&= \cap\{P \in \mathrm{Cl.Spec}(M) | (A+B) \cap C \subseteq P\} \\
&= \cap\{P | P \in \mathcal{V}((A+B) \cap C)\} \\
&= \mathfrak{T}(\mathcal{V}((A+B) \cap C)) \\
&= \mathfrak{T}(\mathcal{V}(A+B) \cup \mathcal{V}(C)) \\
&= \mathfrak{T}((\mathcal{V}(A) \cap \mathcal{V}(B)) \cup \mathcal{V}(C)) \\
&= \mathfrak{T}((\mathcal{V}(A) \cup \mathcal{V}(C)) \cap (\mathcal{V}(B) \cup \mathcal{V}(C))) \\
&= \mathfrak{T}((\mathcal{V}(A \cap C)) \cap (\mathcal{V}(B \cap C))) \\
&= \mathfrak{T}((\mathcal{V}(A \cap C) + (B \cap C))) \\
&= \sqrt[cl]{(A \cap C) + (B \cap C)} \\
&= (A \cap C) + (B \cap C)
\end{aligned}$$

Hence M is a distributive module. □

Proposition 2.5. *Let M be a classical top module. Then for every two submodules A and B of M the equality $\sqrt[cl]{A \cap B} = \sqrt[cl]{A} \cap \sqrt[cl]{B}$ holds.*

Proof. By definition, $\sqrt[cl]{A \cap B} = \mathfrak{T}(\mathcal{V}(A \cap B)) = \mathfrak{T}(\mathcal{V}(A) \cup \mathcal{V}(B))$
$= \mathfrak{T}(\mathcal{V}(A)) \cap \mathfrak{T}(\mathcal{V}(B)) = \sqrt[cl]{A} \cap \sqrt[cl]{B}$. □

3. Some properties of topological space Cl.Spec(M)

In this section, we study some properties of topological space Cl.Spec(M). The closure of Y in Cl.Spec(M) with respect to the classical quasi-Zariski topology denoted by $\overline{Y}$.

Lemma 3.1. *Let M be a classical top module and let Y be a nonempty subset of* Cl.Spec(M). *Then $\overline{Y} = \mathcal{V}(\mathfrak{T}(Y))$. Hence, for every $N \leq M$, $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(N)$.*

Proof. Suppose $\mathcal{V}(E)$ is a closed set of Cl.Spec(M) containing Y. Then for every classical prime submodule P in Y, $E \subseteq P$. Therefore $E \subseteq \mathfrak{T}(Y)$ and so $\mathcal{V}(\mathfrak{T}(Y)) \subseteq \mathcal{V}(E)$. Since $Y \subseteq \mathcal{V}(\mathfrak{T}(Y))$, then $\mathcal{V}(\mathfrak{T}(Y))$ is the smallest closed subset of Cl.Spec(M) containing Y. Thus $\overline{Y} = \mathcal{V}(\mathfrak{T}(Y))$.

Finally, since $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \overline{\mathcal{V}(N)}$, and since $\mathcal{V}(N)$ is a closed subset of Cl.Spec(M), then $\overline{\mathcal{V}(N)} = \mathcal{V}(N)$. Consequntly $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(N)$. □

Let X be a topological space and let x and y be two points of X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1- space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if the singleton set $\{x\}$ is a closed set, for any x in X.

Theorem 3.1. *Let M be an R-module. Then* Cl.Spec(M) *is T_1-space if and only if each classical prime submodule is maximal in the family of all classical prime submodules of M. i.e,* Max(M) = Cl.Spec(M).

Proof. Let P be maximal in Cl.Spec(M) with respect inclution. Then $\overline{\{P\}} = \mathcal{V}(\mathfrak{T}(\{P\})) = \mathcal{V}(P)$, but P is maximal in Cl.Spec(M), so $\overline{\{P\}} = \{P\}$. Then $\{P\}$ is a closed set in Cl.Spec(M). Thus Cl.Spec(M) is a T_1 - space, and vice versa. □

Definition 3.1. Let X be a topological space and Y $\subseteq$ X. Then:

(1) X is irreducible if $X \neq \emptyset$ and for every decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$, $i = 1, 2$, we have $A_1 = X$ or $A_2 = X$.

(2) Y is irreducible if Y is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets F, G which are closed in X and satisfy $Y \subseteq F \cup G$, then $Y \subseteq F$ or $Y \subseteq G$[10, Ch. II, p. 119].

Lemma 3.2. *Let M be an R-module. Then for every $P \in$* Cl.Spec(M), *$\mathcal{V}(P)$ is irreducible.*

Proof. Let $\mathcal{V}(P) \subseteq Y_1 \cup Y_2$, for some closed sets Y_1 and Y_2. Since $P \in \mathcal{V}(P)$, either $P \in Y_1$ or $P \in Y_2$. Suppose that $P \in Y_1$. Then $Y_1 = \cap_{i\in I}(\cup_{j=1}^{n_i}\mathcal{V}(N_{ij}))$, for some I, $n_i(i \in I)$ and $N_{ij} \leq M$. Then for all $i \in I$, $P \in \cup_{j=1}^{n_i}\mathcal{V}(N_{ij})$. Thus for all $i \in I$, $\mathcal{V}(P) \subseteq \cup_{j=1}^{n_i}\mathcal{V}(N_{ij}))$, so $\mathcal{V}(P) \subseteq Y_1$. Thus $\mathcal{V}(P)$ is irreducible. □

M. Behboodi and M. R. Haddadi show that if $Y \subseteq \text{Spec}(M)$ and $\mathfrak{T}(Y)$ is a prime submodule of M and $\mathfrak{T}(Y) \in \overline{Y}$, then Y is irreducible([6, Theorem 3.4]). In the next proposition, we extend this fact to classical prime submodules.

Proposition 3.1. *Let M be a classical top module and $Y \subseteq \text{Cl.Spec}(M)$. Then $\mathfrak{T}(Y)$ is a classical prime submodule of M if and only if Y is an irreducible space.*

Proof. Let $P = \mathfrak{T}(Y)$ be a classical prime submodule of M and $P \in Y$, so $\overline{Y} = \mathcal{V}(P)$. If $Y \subseteq Y_1 \cup Y_2$, for closed sets Y_1 and Y_2, then $\overline{Y} \subseteq Y_1 \cup Y_2$. Since $\mathcal{V}(P) \subseteq Y_1 \cup Y_2$ and by Lemma 3.2, $\mathcal{V}(P)$ is irreducible, then $\mathcal{V}(P) \subseteq Y_1$ or $\mathcal{V}(P) \subseteq Y_2$. Now, since $Y \subseteq \mathcal{V}(P)$, then either $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus Y is irreducible. For the converse, we can apply [6, Theorem 3.4]. □

Corollary 3.1. *Let M be a classical top module. Then for every classical prime submodule P, $\mathcal{V}(P)$ is an irreducible subspace of* Cl.Spec(M). *Consequently, $\mathcal{V}(N)$ is irreducible if and only if $\sqrt[Cl]{N}$ is a classical prime submodule.*

Proof. First note that $\mathfrak{T}(\mathcal{V}(P)) = \bigcap\{P | P \in \mathcal{V}(P)\} = \sqrt[cl]{P} = P$. Then $\mathcal{V}(P)$ is an irreducible subspace of Cl.Spec(M), by Proposition 3.1. Finnaly, it is enough to note that $\sqrt[Cl]{N} = \mathfrak{T}(\mathcal{V}(N))$. □

Proposition 3.2. *Let M be a classical top R-module, $\overline{R} = R/\text{Ann}(M)$ and let $\psi : \text{Cl.Spec}(M) \longrightarrow \text{Spec}(R/\text{Ann}(M))$ be the natural map of* Cl.Spec(M). *Then ψ is continuous in the classical quasi-Zariski topology.*

Proof. It suffices to prove that $\psi^{-1}(\text{V}(\overline{I})) = \mathcal{V}(IM)$, for every $I \in \text{V}(\text{Ann}(M))$. Let $P \in \text{Cl.Spec}(M)$, then $P \in \psi^{-1}(\text{V}(\overline{I}))$, so $\psi(P) \in \text{V}(\overline{I})$, therefore $\overline{(P : M)} \in \text{V}(\overline{I})$. Then $\overline{(P : M)} \in \text{Spec}(\overline{R})$ and $\overline{I} \subseteq \overline{(P : M)}$, so $(P : M) \in \text{Spec}(R)$ and $I/\text{Ann}(M) \subseteq (P : M)/\text{Ann}(M)$. Hence $(P : M) \in \text{Spec}(R)$ and $\text{Ann}(M) \subseteq I \subseteq (P : M)$. Now, since $IM \subseteq (P : M)M \subseteq P$, then $P \in \mathcal{V}(IM)$, which it shows that $\psi^{-1}(\text{V}(\overline{I})) \subseteq \mathcal{V}(IM)$. In similar way, we can show $\mathcal{V}(IM) \subseteq \psi^{-1}(\text{V}(\overline{I}))$ and hence

$$\psi^{-1}(\text{V}(\overline{I})) = \mathcal{V}(IM).\square$$

Lemma 3.3. *Let M be a classical top R-module, $\overline{R} = R/\text{Ann}(M)$ and let ψ be the natural map of* Cl.Spec(M). *If M is classical primeful, then ψ is both closed and open; more precisely, for every submodule N of M, $\psi(\mathcal{V}(N)) = \text{V}(\overline{(N : M)})$ and*

$$\psi(\text{Cl.Spec}(M) \setminus \mathcal{V}(N)) = \text{Cl.Spec}(R/\text{Ann}(M)) \setminus (\text{V}(\overline{(N : M)}).$$

Proof. First we show that $\psi(\mathcal{V}(N)) = \text{V}(\overline{(N : M)})$, for every $N \leq M$, whenever M is classical primeful. Since ψ is continuous, as we have seen in Proposition 3.2,

$$\psi^{-1}(\text{V}(\overline{(N : M)})) = \mathcal{V}((N : M)M) = \mathcal{V}(N).$$

Hence, $\psi(\mathcal{V}(N)) = \psi \circ \psi^{-1}(\text{V}(\overline{(N : M)}) = \text{V}(\overline{(N : M)}$, since ψ is surjective and M is classical primeful. Consequently:

$$\psi(\text{Cl.Spec}(M) \setminus \mathcal{V}(N)) = \text{Spec}(R/\text{Ann}(M)) \setminus (\text{V}(\overline{(N : M)}).\square$$

Corollary 3.2. *Let M be a classical top R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let ψ be the natural map of* Cl.Spec(M). *Then ψ is bijective if and only if it is a homeomorphism.*

Proof. This follows from Proposition 3.2 and Lemma 3.3. □

Proposition 3.3. *Let M be a classical top R-module and let Y be a subset of* Cl.Spec(M). *If Y is irreducible, then $T = \{(P : M) | P \in Y\}$ is an irreducible subset of* Spec(R), *with respect to Zariski topology.*

Proof. Let $\overline{R} = R/\mathrm{Ann}(M)$, ψ the natural map of Cl.Spec(M) and let Y be a subset of Cl.Spec(M). Since ψ is continuous by proposition 3.2, Then $\psi(Y) = \overline{Y}$ is an irreducible subset of Spec$(R/\mathrm{Ann}(M))$. Therefore

$$\mathfrak{T}(\overline{Y}) = (\mathfrak{T}(Y) : M)/\mathrm{Ann}(M) \in \mathrm{Spec}(R/\mathrm{Ann}(M)).$$

Therefore $\mathfrak{T}(T) = (\mathfrak{T}(Y) : M)$ is a prime ideal of R, then by Proposition 3.1, T is an irreducible subset of Spec(R). □

Clearly the next lemma is true(see for example [8], page 10).

Lemma 3.4. *If $\{P_i\}_{i\in I}$ is a chain of classical prime submodules of an R-module M, then $\bigcap_{i\in I} P_i$ is a classical prime submodule of M.*

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$, where $Cl(\{y\})$ is the closure of $\{y\}$ in Y. Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0-space.

Theorem 3.2. *Let M be a classical primeful R-module. If M is a classical top module, then a subset Y of* Cl.Spec(M) *is an irreducible closed subset if and only if $Y = \mathcal{V}(P)$, for some $P \in$* Cl.Spec(M). *Thus every irreducible closed subset of* Cl.Spec(M) *has a generic point.*

Proof. By Corollary 3.1, for every $P \in$ Cl.Spec(M), $Y = \mathcal{V}(P)$ is an irreducible closed subset of Cl.Spec(M). Conversely, if Y is an irreducible closed subset of Cl.Spec(M), then $Y = \mathcal{V}(N)$, for some $N \leq M$. Now, since $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt[Cl]{N})$, then $\mathfrak{T}(Y) = \mathfrak{T}(\mathcal{V}(N)) = \sqrt[Cl]{N}$ is a classical prime submodule of M by Lemma 3.4. Then $\mathcal{V}(\mathfrak{T}(Y)) = \mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(\sqrt[Cl]{N})$, so by Theorem 3.1, $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt[Cl]{N})$, with $\sqrt[Cl]{N} \in$ Cl.Spec(M). □

A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed. In the next theorem, we show that there exists a bijection map from the set of irreducible components of Cl.Spec(M) to the set of minimal classical prime submodules of M.

Theorem 3.3. *Let M be a classical top R-module. Then the map $\mathcal{V}(P) \longmapsto P$ is a bijection from the set of irreducible components of* Cl.Spec(M) *to the set of minimal classical prime submodules of M.*

Proof. Let Y be an irreducible component of Cl.Spec(M). By Theorem 3.2, each irreducible component of Cl.Spec(M) is a maximal element of the set $\{\mathcal{V}(Q)|Q \in \text{Cl.Spec}(M)\}$, so for some $P \in \text{Cl.Spec}(M)$, $Y = \mathcal{V}(P)$. Obviously, P is a minimal classical prime submodule of M. Suppose T is a classical prime submodule of M that $T \subseteq P$, then $\mathcal{V}(P) \subseteq \mathcal{V}(T)$, so $P = T$. Now, let P be a minimal classical prime submodule of M, so for every $Q \in \text{Cl.Spec}(M)$, $P \subseteq Q$. Then for all $Q \in \text{Cl.Spec}(M)$, $\mathcal{V}(Q) \subseteq \mathcal{V}(P)$. Thus $\mathcal{V}(P)$ is a maximal irreducible subset of Cl.Spec(M). □

Theorem 3.4. *Consider the following statements for a nonzero classical top primeful R-module M:*

1. Cl.Spec(M) *is an irreducible space.*
2. Supp(M) *is an irreducible space.*
3. $\sqrt{\text{Ann}(M)}$ *is a prime ideal of R.*
4. Cl.Spec$(M) = \mathcal{V}(pM)$, *for some $p \in \text{Supp}(M)$.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). In addition, if M is a multiplication module, then all of the four statements are equivalent.

Proof. (1) $\Longrightarrow$ (2): By Proposition 3.2, the natural map ψ is continuous and by assumption ψ is surjective. Therefore $Im(\psi) = \text{Spec}(R/\text{Ann}(M))$ is also irreducible. Now by Proposition 2.3, $\text{Supp}(M) = \text{V}(\text{Ann}(M))$ is homeomorphic to $\text{Spec}(R/\text{Ann}(M))$. Therefore Supp$(M)$ is an irreducible space.

(2) $\Longrightarrow$ (3): By Proposition 3.1, $\mathfrak{T}(\text{Supp}(M))$ is a prime ideal of R. Then $\mathfrak{T}(\text{Supp}(M)) = \mathfrak{T}(\text{V}(\text{Ann}(M))) = \sqrt{\text{Ann}(M)}$ is a prime ideal of R.

(3) $\Longrightarrow$ (4) Let $a \in \sqrt{\text{Ann}(M)}$. So for some integer $n \in N$, $a^nM = 0$. Therefore for every classical prime submodule P of M, $a \in (P : M)$. Then for each $P \in \text{Cl.Spec}(M)$, $\text{Ann}(M) \subseteq \sqrt{\text{Ann}(M)} \subseteq (P : M)$. Since M is classical primeful, there exists a classical prime submodule Q of M such that $(Q : M) = \sqrt{\text{Ann}(M)}$. Then,

$$\begin{aligned}\text{Cl.Spec}(M) &= \{P \in \text{Cl.Spec}(M)|(Q : M) \subseteq (P : M)\}\\ &= \mathcal{V}((Q : M)M)\\ &= \mathcal{V}(\sqrt{\text{Ann}(M)}M).\end{aligned}$$

It is clear that $p := \sqrt{\text{Ann}(M)} \in \text{Supp}(M)$. Therefore $\text{Cl.Spec}(M) = \mathcal{V}(pM)$.

Now, let M be a multiplication module and let $\text{Cl.Spec}(M) = \mathcal{V}(pM)$, for some $p \in \text{Supp}(M)$. Since M is classical primeful, there exists $P \in \text{Cl.Spec}(M)$, such that $(P : M) = p$. Since M is multiplication, we have $\text{Cl.Spec}(M) = \mathcal{V}(pM) = \mathcal{V}((P : M)M) = \mathcal{V}(P)$. This implies that Cl.Spec(M) is an irreducible space by Corollary 3.1. □

Let M be an R-module. For each subset N of M, we denote $\mathrm{Cl.Spec}(M) - \mathcal{V}(N)$ by $\mathcal{U}(N)$. Further for each element $m \in M$, $\mathcal{U}(\{m\})$ is denoted by $\mathcal{U}(m)$. Hence

$$\mathcal{U}(m) = \mathrm{Cl.Spec}(M) - \mathcal{V}(\{m\}).$$

Moreover, for any family $\{N_i\}_{i \in I}$ of submodules of M, we have

$$\mathcal{U}(\sum_{i \in I} N_i) = \mathcal{U}(\bigcup_{i \in I} N_i).$$

Theorem 3.5. *Let M be a classical top module. Then for every $m \in M$, the sets $\mathcal{U}(m)$ form a base for* $\mathrm{Cl.Spec}(M)$.

Proof. Let $\mathcal{U}(N)$ be an open set in $\mathrm{Cl.Spec}(M)$, where N is a submodule of M. Then:

$$\begin{aligned}\mathcal{U}(N) = \mathcal{U}(\bigcup_{n \in N} \{n\}) &= \mathrm{Cl.Spec}(M) - \mathcal{V}(\bigcup_{n \in N} \{n\}) \\ &= \mathrm{Cl.Spec}(M) - \bigcap_{n \in N} \mathcal{V}(\{n\}) \\ &= \bigcup_{n \in N} (\mathrm{Cl.Spec}(M) - \mathcal{V}(\{n\})) \\ &= \bigcup_{n \in N} \mathcal{U}(n)\end{aligned}$$

Then for every $m \in M$, the sets $\mathcal{U}(m)$ form a base of $\mathrm{Cl.Spec}(M)$. □

For a submodule N of an R-module M, we put:

$$\mathcal{FG}(N) := \{L | L \subseteq N \textit{ and } L \textit{ is finitely generated}\}$$

Lemma 3.5. *Let M be an R-module and N be a submodule of M. Then $\mathcal{V}(N) = \bigcap_{L \in \mathcal{FG}(N)} \mathcal{V}(L)$ and $\mathcal{U}(N) = \bigcup_{L \in \mathcal{FG}(N)} \mathcal{U}(L)$.*

Proof. Suppose that $P \in \mathcal{V}(N)$. If $L \in \mathcal{FG}(N)$, then $L \subseteq N \subseteq P$. Then $P \in \mathcal{V}(L)$, and $\mathcal{V}(N) \subseteq \bigcap_{L \in \mathcal{FG}(N)} \mathcal{V}(L)$. Now, let for every $L \in \mathcal{FG}(N)$, $P \in \mathcal{V}(L)$ and $P \notin \mathcal{V}(N)$. Since $N \nsubseteq P$, then there exists $x \in N \setminus P$. Then $Rx \subseteq N$ and Rx is finitely generated, hence $Rx \in \mathcal{FG}(N)$. Therefore $x \in Rx \subseteq P$, a contradiction. Thus $\bigcap_{L \in \mathcal{FG}(N)} \mathcal{V}(L) \subseteq \mathcal{V}(N)$. □

Theorem 3.6. *Let M be a classical top R-module. Then every quasi-compact open subset of* $\mathrm{Cl.Spec}(M)$ *is of the form $\mathcal{U}(N)$, for some finitely generated submodule N of M.*

Proof. Suppose $\mathcal{U}(B) = \text{Cl.Spec}(M) \setminus \mathcal{V}(B)$ is a quasi-compact open subset of Cl.Spec(M). Then by Lemma 3.5, $\mathcal{U}(B) = \bigcup_{L \in \mathcal{FG}(B)} \mathcal{U}(L)$. Now, since $\mathcal{U}(B)$ is quasi-compact, then every open covering of $\mathcal{U}(B)$ has a finite subcovering, therefore $\mathcal{U}(B) = \mathcal{U}(L_1) \cup ... \cup \mathcal{U}(L_n) = \mathcal{U}(\sum_{i=1}^{n} L_i)$. □

Proposition 3.4. *Let M be a classical top R-module. If* Spec(R) *is a T_1-space, then* Cl.Spec(M) *is also a T_1-space.*

Proof. Suppose Q is a classical prime submodule of M. Then $Cl(\{Q\}) = \mathcal{V}(Q)$. If $P \in \mathcal{V}(Q)$, then by Theorem 3.1, every prime ideal of R is a maximal ideal, so $(Q : M) = (P : M)$, then by Lemma 2.1, $Q = P$. Therefore $Cl(\{Q\}) = \{Q\}$ and this implies that Cl.Spec(M) is a T_1-space. □

Definition 3.2. A topological space X is Noetherian provided that the open (respectively, closed) subsets of X satisfy the ascending (respectively, descending) chain condition (see for example [3], page 79 or [10], §4.2).

Theorem 3.7. *An R-module M has Noetherian calssical spectrum if and only if the ACC for classical radical submodules of M holds.*

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an ascending chain of classical radical submodules of M. Since for all $i \in \mathbb{N}$, $\sqrt[Cl]{N_i} = N_i$, then equivalently

$$\sqrt[Cl]{N_1} \subseteq \sqrt[Cl]{N_2} \subseteq \sqrt[Cl]{N_3} \subseteq \dots$$

is an ascending chain of classical radical submodules of M. Then equivalently

$$\mathfrak{T}(\mathcal{V}(N_1)) \subseteq \mathfrak{T}(\mathcal{V}(N_2)) \subseteq \mathfrak{T}(\mathcal{V}(N_3)) \subseteq \dots$$

is an ascending chain of classical radical submodules of M. Therefore

$$\mathcal{V}(N_1) \supseteq \mathcal{V}(N_2) \supseteq \mathcal{V}(N_3) \supseteq \dots$$

is a descending chain of closed sets $\mathcal{V}(N_i)$ of Cl.Spec(M). Now, R-module M has Noetherian spectrum if and only if Cl.Spec(M) is a Noetherian topological space if and only if there exists a positive integer k such that $\mathcal{V}(N_k) = \mathcal{V}(N_{k+n})$ for each $n = 1, 2, ...$ if and only if $\sqrt[Cl]{N_k} = \sqrt[Cl]{N_{k+n}}$ if and only if $N_k = N_{k+n}$ if and only if the ACC for classical radical submodules of M holds.□

Theorem 3.8. *Let M be a classical top R-module such that* Cl.Spec(M) *is a Noetherian space. Then the following statements are true.*

1. *Every ascending chain of classical prime submodules of M is stationary.*
2. *The set of minimal classical prime submodules of M is finite. In particular,* Cl.Spec$(M) = \bigcup_{i=1}^{n} \mathcal{V}(P_i)$*, where P_i are all minimal classical prime submodules of M.*

Proof. (1). Suppose $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ is an ascending chain of classical prime submodules of M. Therefore $\mathcal{V}(N_1) \supseteq \mathcal{V}(N_2) \supseteq \dots$ is a descending chain of closed subsets of Cl.Spec(M), which is stationary by assumption. There exists an integer $k \in \mathbb{N}$ such that $\mathcal{V}(N_k) = \mathcal{V}(N_{k+i})$, for each $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $N_k = N_{k+i}$.

(2). This follows from Theorem 3.3 and the fact that if X is a Noetherian space, then the set of irreducible components of X is finite(see for example [10, Proposition 10]). □

Recall that if M is a Noetherian module, then each open subset of Spec(M) is quasi-compact(see for example [15, Theorem 3.3]). The next theorem shows that the same result is true for Cl.Spec(M) in Noetherian classical top modules.

Theorem 3.9. *Let M be a Noetherian classical top module. Then each open subset of* Cl.Spec(M) *is quasi-compact.*

Proof. Let for every submodule N of $M, \mathcal{U}(N)$ be an open subset of Cl.Spec(M). Also, let $\{\mathcal{U}(n_i)\}_{n_i \in N}$ be a basic open cover for $\mathcal{U}(N)$. We show that there exist a finite subfamily of $\{\mathcal{U}(n_i)\}_{n_i \in N}$ which covers Cl.Spec(M). Since $\mathcal{U}(N) \subseteq \bigcup_{n_i \in N} \mathcal{U}(n_i) = \mathcal{U}(\bigcup_{n_i \in N} n_i)$, then for every submodule K of M that is generated by the set $A = \{n_i\}_{i \in I}$, $\mathcal{U}(N) \subseteq \mathcal{U}(K)$. Since M is a Noetherian module, then $K =< k_1, k_2, ..., k_n >$, for some $k_i \in K$, therefore $b_i = \sum_{j=1}^{n} r_{ij} n_{ij}$, where $i = 1, ..., n$ and $n_{ij} \in A$. Thus there exists the subset $\{n_{i1}, ..., n_{in}\} \subseteq A$ such that $K =< n_{i1}, ..., n_{in} >$. So $\mathcal{U}(N) \subseteq \mathcal{U}(K) = \mathcal{U}(< n_{i1}, ..., n_{in} >)$. Then

$$\mathcal{U}(N) \subseteq \mathcal{U}(\bigcup_{i=1}^{n} n_i) = \bigcup_{i=1}^{n} \mathcal{U}(n_i).$$

consequently, $\mathcal{U}(N)$ is quasi-compact. □

Recall that a function Φ between two topological spaces X and Y is called an open map if, for any open set U in X, the image $\Phi(U)$ is open in Y. Also, Φ is called a homeomorphism if it has the following properties

(i) Φ is a bijection;

(ii) Φ is continuous;

(iii) Φ is an open map

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster's characterization [15], a topology τ on a set X is spectral if and only if the following axioms hold:

(i) X is a T_0-space.

(ii) X is quasi-compact and has a basis of quasi-compact open subsets.

(iii) The family of quasi-compact open subsets of X is closed under finite intersections.

(iv) X is a sober space; i.e., every irreducible closed subset of X has a generic point.

For any ring R, it is is well-known that Spec(R) satisfies these conditions(cf. [10], Chap. II, 4.1 - 4.3]). We show that Cl.Spec(M) is necessarily a spectral space in the classical quasi-Zariski topology for every module M.

We remark that any closed subset of a spectral space is spectral for the induced topology.

Theorem 3.10. *Let M be a classical top primful R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let ψ be the natural map of* Cl.Spec(M). *Then ψ is a homeomorphism.*

Proof. It is clear by Lemma 2.1, Proposition 3.2, Lemma 3.3 and Corollary 3.2. □

Corollary 3.3. *Let M be a classical top primful R-module. Then* Cl.Spec(M) *with classical quasi-Zariski topology is a spectral space.*

Lemma 3.6. *Let M be a classical top R-module. Then the following statements are equivalent:*

(a) the natural map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ *is injective.*

(b) Cl.Spec(M) *is a T_0-space.*

Proof. We recall that a topological space is T_0 if and only if the closures of distinct points are distinct. Now, the result follows from

$$P = Q \Longleftrightarrow \mathcal{V}(P) = \mathcal{V}(Q), \quad \forall P, Q \in \mathrm{Cl.Spec}(M). \square$$

Corollary 3.4. *Let M be a Noetherian classical primeful top module. Then the following statements are holed:*

(i) Cl.Spec(M) *is a T_0-space.*

(ii) Cl.Spec(M) *is quasi-compact and has a basis of quasi-compact open subsets.*

(iii) The family of quasi-compact open subsets of Cl.Spec(M) *is closed under finite intersections.*

(iv) Cl.Spec(M) *is a sober space; i.e., every irreducible closed subset of* Cl.Spec(M) *has a generic point.*

Proof. It is clear by Lemma 3.6, Theorem 3.5, Theorem 3.9, Theorem 3.2. □

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