

STRONG CONVERGENCE THEOREM FOR UNIFORMLY L–LIPSCHITZIAN MAPPING OF GREGUS TYPE IN BANACH SPACES

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Abstract. In this paper, we introduced a new mapping called uniformly L-Lipschitzian mapping of Gregus type, and used the Mann iterative scheme to approximate the fixed point. A Strong convergence result for the sequence generated by the scheme is shown in real Banach space. Our result generalized and unify many recent results in this area of research. In addition, using Java (jdk 1.8.0_101), we give a numerical example to support our claim.

Key words: Mann iterative scheme; uniformly L-Lipschitzian mapping; normalized duality mapping.

1. Introduction

Let E and E^* be a real Banach space and its dual space respectively. The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|x\| = \|f\|\}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

Definition 1.1. [Ofoedu E.U [13]] Let K be a nonempty closed convex subset of a real Banach space E . The mapping $T : K \rightarrow E$ is said to be

i) nonexpansive if for all $x, y \in K$

$$(1.1) \quad \|Tx - Ty\| \leq \|x - y\|.$$

ii) uniformly L–Lipschitzian if there exists $L > 0$ such that, for any $x, y \in K$

$$(1.2) \quad \|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall n \geq 1.$$

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- iii) asymptotically nonexpansive if there exists $k_n \in [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that for any given $x, y \in K$,

$$(1.3) \quad \|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n > 1.$$

- iv) asymptotically pseudocontractive if there exists a sequence $k_n \in [0, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$. and there exists $j(x - y) \in J(x - y)$ such that

$$(1.4) \quad \langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2, \quad n \geq 1.$$

We can easily see from equations (1.2), (1.3), (1.4) that the class of asymptotically non-expansive mappings is a generalization of the class of uniformly L-Lipschitzian mapping. And that every asymptotically nonexpansive mappings are asymptotically pseudocontractive, the reason is shown below,

$$(1.5) \quad \langle T^n x - T^n y, j(x - y) \rangle \leq \|T^n x - T^n y\| \|x - y\| \leq k_n \|x - y\|^2, \quad \forall n \geq 1.$$

But the converse is not always true. The example to show that the converse is not true was constructed by Rhoades [15]. The asymptotically nonexpansive mappings and the asymptotically pseudocontractive mappings were introduced by Goebel and Kirk [4] and Schu [16] respectively.

In 1980, Gregus [5] introduced what is now known as the Gregus fixed point theorem. He proved the following theorem.

Theorem 1.1. Gregus [5] *Let K be a closed convex subset of a Banach space E and $T : K \rightarrow K$ a mapping that satisfies $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$ for all $x, y \in K$ where $0 < a < 1, b, c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.*

The class of mapping introduced by Gregus [5] is a generalization of non-expansive mapping which is a very important mapping in fixed point theorem and applications, because if $a = 1, b = c = 0$ then we have the mapping in (1.1), and if $a = 0, b = c = \frac{1}{2}$ we have the Kannan mappings introduced by Kannan in [6]. This class of mappings have been extended by many authors in various ways and under different conditions on T . For results on these, see [8, 11, 12, 14] and the references therein.

The trend for uniformly L-Lipschitzian mapping and asymptotically pseudocontractive mapping is given below for better understanding the concept we intend to introduce.

In 1991 Schu [16], proved the following result using the modified Mann iterative scheme

Theorem 1.2. Schu [16] *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \rightarrow K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping with a sequence $\{k_n\}$ satisfying the following conditions:*

- (i) $k_n \rightarrow 1$ as $n \rightarrow 1$;
- (ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, where $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ such that $\epsilon < \alpha_n < b$, $\forall n \geq 1$ where $\epsilon > 0$ and $b \in (0, L^{-2}[(1+L^2)^{1/2} - 1])$ are some positive numbers. For any $x_1 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$(1.6) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T in K .

In 2000, Chang [1] extended theorem 1.2 from Hilbert space to uniformly smooth Banach space, by proving the following theorem:

Theorem 1.3. Chang [1] *Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E , $T : K \rightarrow K$ be an asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n \rightarrow \infty} k_n = 1$, and $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\alpha_n \subset [0, 1]$ satisfying the following conditions:*

- (i) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0,$$

If there exists a strict increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \Phi(\|x - p\|),$$

for all $x \in K$ and $n \geq 0$, where $p \in F(T)$, then $x_n \rightarrow p$ as $n \rightarrow \infty$.

Recently Ofoedu [13], extended theorem 1.3 from uniformly smooth Banach space to real Banach space and he also dispensed with the boundedness condition imposed by earlier researchers, by stating and proving the following theorem:

Theorem 1.4. Ofoedu [13] *Let E be a real Banach space, K be a nonempty closed and convex subset of E , $T : K \rightarrow K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, +\infty)$, $\lim_{n \rightarrow \infty} k_n = 1$, and let $p \in F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\alpha_n \subset [0, 1]$ satisfying the following conditions:*

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

$$(iii) \sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$$

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 0.$$

If there exists a strict increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \Phi(\|x - p\|)$$

for all $x \in K$ and $n \geq 0$, where $p \in F(T)$, then $x_n \rightarrow p$ as $n \rightarrow \infty$.

Inspired by the above results, we introduce the following concept which generalizes the class of uniformly L-Lipschitzian mappings. The mapping is defined as follows:

Definition 1.2. Let K be a nonempty closed subset of a real Banach space E . The mapping $T : K \rightarrow E$ is said to be uniformly L-Lipschitzian mapping of Gregus type if there exists $L > 0$, and the sequences $a_n, b_n \in [0, \infty)$, with $a_n, b_n \rightarrow 0$ as $n \rightarrow \infty$, such that for any $x, y \in K$,

$$(1.7) \quad \|T^n x - T^n y\| \leq L\|x - y\| + a_n\|x - T^n x\| + b_n\|y - T^n y\|, \quad \forall n \geq 1.$$

If we set $a_n = b_n = 0 \quad \forall n \in \mathbb{N}$, equation (1.7) is reduced to (1.2). Clearly, every uniformly L-Lipschitzian mapping is uniformly L-Lipschitzian mapping of Gregus type, but the converse is not generally true. It suffices to construct an example of a map that is uniformly L-Lipschitzian of Gregus type but not uniformly L-Lipschitzian.

Example 1.1. Let $E = \mathbb{R}$ be the set of real numbers with the usual norm, and let $K = [0, \infty)$. Consider the mapping $T : K \rightarrow K$ defined by

$$Tx = \frac{x^3}{4(1+x)}, \quad \forall x \in K$$

It is easy to see that T is a monotone increasing function satisfying (1.7), but T does not satisfy inequality (1.2). In fact,

$$\begin{aligned} |T^n x - T^n y| &\leq |Tx - Ty| = \left| \frac{x^3}{4+4x} - \frac{y^3}{4+4y} \right| = \frac{1}{4} \left| \frac{x^3}{1+x} - \frac{y^3}{1+y} \right| \\ &= \frac{1}{4} \left| \frac{x-y + x^3(1+y) - x + y - y^3(1+x)}{(1+x)(1+y)} \right| \\ &\leq \frac{1}{4} \left| \frac{x-y}{(1+x)(1+y)} \right| + \left| \frac{x^3(1+y) - x}{4(1+x)(1+y)} \right| + \left| \frac{y - y^3(1+x)}{4(1+x)(1+y)} \right| \\ &\leq \frac{1}{4} |x-y| + \left| x - \frac{x^3}{4(1+x)} \right| + \left| y - \frac{y^3}{4(1+y)} \right| \\ &= \frac{1}{4} |x-y| + |x - Tx| + |y - Ty|. \end{aligned}$$

Therefore,

$$(1.8) \quad |T^n x - T^n y| \leq \frac{1}{4}|x - y| + |x - Tx| + |y - Ty|.$$

Hence, T is uniformly L–Lipschitzian mapping of Gregus type where the sequences $a_n = b_n = 1$, $\forall n \in \mathbb{N}$ and $L = \frac{1}{4}$. But observe that,

$$\frac{x^3}{4(1+x)} > x \quad \forall x > \frac{4 + \sqrt{32}}{2},$$

hence we have that,

$$|T^n x - T^n y| \leq |Tx - Ty| = \left| \frac{x^3}{4+4x} - \frac{y^3}{4+4y} \right| > |x - y|.$$

Thus, T is not a uniformly L–Lipschitzian mapping. We can now say that the class of uniformly L– Lipschitzian mappings of Gregus type properly includes the class of uniformly L–Lipschitzian mappings. Hence, it is more important to study this class of mappings in fixed point theory and applications.

In particular, If we let x to be any point in K and $y \in F(T)$ then, from (1.8) we have,

$$|T^n x - T^n y| \leq \frac{1}{2}|x - 0| + \frac{1}{4}|x - Tx| + \frac{1}{4}|0 - T0|.$$

but

$$|T^n x - T^n y| = |T^n x - 0| \leq |Tx - 0| = \left| \frac{x^3}{4+4x} - 0 \right| > |x - 0|,$$

for $x > \frac{4+\sqrt{32}}{2}$, $y = 0 \in F(T)$.

It is our aim in this paper to consider the iterative scheme in (1.6) and prove a strong convergence theorem for the newly introduced uniformly L–Lipschitzian mappings of Gregus type to a unique fixed point in real Banach spaces.

2. Preliminaries

We shall need the following Proposition and lemmas in the main theorem.

Proposition 2.1. *Let K be a nonempty closed convex subset of a Banach space and $T : K \rightarrow K$ be a Uniformly L–Lipschitzian mapping of Gregus Type with $a_n \in (0, \frac{1}{2})$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$. Then for any $x_0 \in K$, let $\{x_n\}$ be an iterative sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geq 0$, there exists an $M > 0$ such that for $p \in F(T)$, the following hold:

- i.) $\|T^n x_n - x_n\| \leq M\|x_n - p\|$
- ii.) $\|T^n x_{n+1} - x_{n+1}\| \leq M\|x_{n+1} - p\|$

Proof. Since T is uniformly L -Lipschitzian mapping of Gregus Type, we have,

$$\begin{aligned}
 \|x_n - T^n x_n\| &\leq \|x_n - p\| + \|T^n x_n - p\| = \|x_n - p\| + \|T^n x_n - T^n p\| \\
 &\leq \|x_n - p\| + L\|x_n - p\| + a_n\|x_n - T^n x_n\| + b_n\|p - T^n p\| \\
 &= (1 + L)\|x_n - p\| + a_n\|x_n - T^n x_n\| \\
 (2.1) \qquad &\leq \frac{(1 + L)}{1 - a_n}\|x_n - p\|.
 \end{aligned}$$

Since, $a_n \in [0, 1/2)$ we have that, $-a_n > -\frac{1}{2}$. Hence, $1 - a_n > 1 - \frac{1}{2}$, this implies that

$$\frac{1}{1 - a_n} < 2.$$

Therefore,

$$\frac{(1 + L)}{1 - a_n} < 2(1 + L).$$

Let $M = 2(1 + L) > 1$, (2.1) becomes

$$(2.2) \qquad \|T^n x_n - x_n\| \leq M\|x_n - p\|.$$

Using similar procedure we can easily get that

$$(2.3) \qquad \|T^n x_{n+1} - x_{n+1}\| \leq M\|x_{n+1} - p\|.$$

This completes the proof. \square

Lemma 2.1. Mogbademu [9] *Let E be a normed linear space then for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

Lemma 2.2. C. Moore and B.V Nnoli [10] *Let $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing function with $\Phi(0) = 0$ and let $\{\theta_n\}, \{\lambda_n\}, \{\sigma_n\}$ be any nonnegative real sequences such that $\sigma_n = o(\lambda_n)$, $\sum_{i=0}^{\infty} \lambda_n = \infty$, and $\lim_{n \rightarrow \infty} \lambda_n = 0$. Suppose that*

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \Phi(\theta_{n+1}) + \sigma_n, \quad n \geq 1,$$

then $\lim_{n \rightarrow \infty} \theta_n = 0$.

3. The Main Result

Theorem 3.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E , $T : K \rightarrow K$ be a Uniformly L -Lipschitzian mapping of Gregus Type with $F(T) \neq \emptyset$ where $F(T) = \{x \in K : Tx = x\}$ and $p \in F(T)$. Let $\{k_n\} \in [0, \infty)$ be a sequence of real numbers such that $k_n \rightarrow 1$ as $n \rightarrow \infty$, and let $\alpha_n \in [0, 1]$ satisfying the following:*

- i) $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$
- ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

For any $x_0 \in K$, define a sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geq 0$. If there exists a strictly increasing function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\Phi(0) = 0$ such that

$$(3.1) \quad \langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \Phi(\|x - p\|)$$

for all $x \in K$. Then

- i) $\{x_n\}_{n \geq 0}$ is bounded;
- ii) $x_n \rightarrow p$ as $n \rightarrow \infty$ where p is a unique fixed point of T .

Proof. This proof shall be divided into two steps. In step 1, we will show boundedness, while in step 2 we will show that the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T say p .

Step 1: Let $k = \sup\{k_n : n \geq 1\}$, since T is Uniformly L-Lipschitzian of Gregus Type and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing real valued function with $\Phi(0) = 0$, for $x \in K$, $p \in F(T)$ we obtain

$$\Phi(\|x - p\|) \leq (k + L + \alpha_n M)\|x - p\|^2.$$

Taking limit as $n \rightarrow \infty$ we have

$$\Phi(\|x - p\|) \leq (k + L)\|x - p\|^2.$$

Assume that $x_1 \neq T x_1$ for some $x_1 \in K$ such that

$$(k + L)\|x_1 - p\|^2 \in R(\Phi),$$

we denote that $a_0 = (k + L)\|x_1 - p\|^2$, where $R(\Phi)$ is the range of Φ . Indeed, if $\Phi(a) \rightarrow \infty$ as $a \rightarrow \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0, \infty)\} = a_1 < +\infty$ with $a_1 < a_0$, then for $p \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \rightarrow p$ as $n \rightarrow \infty$ with $u_n \neq p$, thus there exists an $n_0 \in \mathbb{N}$ such that

$$(k + L)\|u_n - p\|^2 < \frac{a_1}{2}$$

for $n \geq n_0$. We redefine $x_1 = u_{n_0}$ and $(k + L)\|x_1 - p\|^2 \in R(\Phi)$.

Set $R = \Phi^{-1}(a_0)$. Then we obtain $\|x_1 - p\| \leq R$.

Denote

$$B_1 = \{x \in K : \|x - p\| \leq R\}, B_2 = \{x \in K : \|x - p\| \leq 2R\}$$

Now, we show that $x_n \in B_1$ using mathematical induction for any $n \geq 1$. If $n = 1$, then $x_1 \in B_1$. Suppose that the result is true for some n , that is $x_n \in B_1$. Now we show that $x_{n+1} \in B_1$. Suppose that, $x_{n+1} \notin B_1$, that is, $x_{n+1} > R$. Denote

$$(3.2) \quad \tau_0 = \min \left\{ 1, \frac{1}{2M}, \frac{1}{2LM}, \frac{\Phi(R)}{16R(M(l+3)R)}, \frac{\Phi(R)}{16R^2} \right\}.$$

Since a_n, b_n, α_n and $k_n - 1 \rightarrow 0$ as $n \rightarrow \infty$. We can let $0 \leq a_n, b_n, \alpha_n, k_n - 1 \leq \tau_0$ for any $n \geq 1$. We obtain the following:

$$(3.3) \quad \begin{aligned} \|x_{n+1} - p\| &\leq \|x_n - p\| + \alpha_n(T^n x_n - x_n) \\ &\leq R + \alpha_n MR \\ &\leq 2R \end{aligned}$$

Using Proposition 2.1 we have the following,

$$(3.4) \quad \begin{aligned} \|T^n x_n - T^n x_{n+1}\| &\leq L\alpha_n \|x_n - x_{n+1}\| + a_n \|x_n - T^n x_n\| + b_n \|x_{n+1} - T^n x_{n+1}\| \\ &\leq (L\alpha_n + a_n)M \|x_n - p\| + b_n M \|x_{n+1} - p\| \\ &\leq (L\alpha_n + a_n)MR + 2b_n MR \\ &\leq \tau_0 MR(L+3) \\ &\leq \frac{\Phi(R)}{16R}. \end{aligned}$$

Let us consider the following estimate, using Lemma 2.1

$$(3.5) \quad \begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n T^n x_n - p\|^2 \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T^n x_n - p, j(x_{n+1} - p) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p) \rangle \\ &\quad + 2\alpha_n \langle T^n x_{n+1} - p, j(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|T^n x_n - T^n x_{n+1}\| \|x_{n+1} - p\| \\ &\quad + 2\alpha_n [k_n \|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)] \\ &\leq (1 - \alpha_n)^2 R^2 + 2\alpha_n k_n \|x_{n+1} - p\|^2 + \end{aligned}$$

$$(3.6) \quad 2\alpha_n \frac{\Phi(R)}{16R} 2R - 2\alpha_n \Phi(R).$$

Since $\alpha_n, k_n \rightarrow 1$ as $n \rightarrow \infty$. Thus, without loss of generality, we let $1 - 2\alpha_n k_n > 0$ for any $n \geq 1$, since

$$(3.7) \quad \frac{1}{1 - 2\alpha_n k_n} \leq 1 + \frac{2\alpha_n k_n}{1 - 2\alpha_n k_n},$$

from (3.6) we have that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq R^2 + \frac{2\alpha_n}{1 - 2\alpha_n k_n} \left[(k_n - 1) + \frac{\alpha_n}{2} \right] R^2 + \\
 &\quad \frac{\alpha_n \Phi(R)}{4(1 - 2\alpha_n k_n)} - \frac{2\alpha_n}{1 - 2\alpha_n k_n} \Phi(R) \\
 &\leq R^2 + \frac{\alpha_n \Phi(R)}{4(1 - 2\alpha_n k_n)} + \frac{\alpha_n \Phi(R)}{4(1 - 2\alpha_n k_n)} - \frac{2\alpha_n}{1 - 2\alpha_n k_n} \Phi(R) \\
 &= R^2 - \frac{3\alpha_n}{2(1 - 2\alpha_n k_n)} \Phi(R) \\
 (3.8) \quad &\leq R^2,
 \end{aligned}$$

which is a contradiction. Hence, the sequence $\{x_n\}$ is bounded.

Step 2: Here, we intend to show that x_n converges uniquely to $p \in F(T)$. Firstly, let us show that p is unique.

Now, we show that p is unique. Suppose for contradiction there exists $p, q \in F(T)$, where $p \neq q$, such that the sequence $\{x_n\}$ converges to p, q hence, we have that,

$$\begin{aligned}
 \|p - q\| &= \|T^n p - T^n q\| \\
 &\leq L\|p - q\| + a_n\|p - T^n p\| + b_n\|q - T^n q\| \\
 &= L\|p - q\|
 \end{aligned}$$

Therefore,

$$0 \leq \|p - q\| \leq 0.$$

Hence, $p = q$ is a contradiction.

Next, we show that the sequence $\{x_n\}$ converge to a unique fixed point p of T .

Since $\|x_n - p\|$ is bounded, there exists $M_* > 0$ such that $\|x_n - p\|^2 < M_*$. Hence, from equation (3.5) we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|T^n x_n - T^n x_{n+1}\| \|x_{n+1} - p\| \\
 &\quad + 2\alpha_n [k_n \|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)] \\
 &\leq \|x_n - p\|^2 - \alpha_n \Phi(\|x_{n+1} - p\|) \\
 (3.9) \quad &\quad + 2\alpha_n (k_n - 1) M_*^2 + \alpha_n^2 M_*^2 + 2\alpha_n M_* M (L\alpha_n + a_n + b_n) M_*
 \end{aligned}$$

Comparing Lemma 2.2 with (3.9) we can let $\theta_{n+1}^2 = \|x_{n+1} - p\|^2$, $\theta_n^2 = \|x_n - p\|^2$, $\lambda_n = \alpha_n$ and $\sigma_n = 2\alpha_n (k_n - 1) M_*^2 + \alpha_n^2 M_*^2 + 2\alpha_n M_* M (L\alpha_n + a_n + b_n) M_*$. From condition (i), we have that $\sum_{n=1}^\infty \lambda_n = \sum_{n=1}^\infty \alpha_n = \infty$, therefore, $\|x_n - p\| \rightarrow 0$ as $n \rightarrow \infty$. \square

Example 3.1. Consider $T = \frac{x^3}{4+4x}$, we have shown in Example 1.1, that T is uniformly L-Lipschitzian of Gregus type, and we can easily check that the fixed point of T is $p = 0$.

Now, take $\alpha_n = 1/2$, and the initial guess value $x_0 = 0.5, 1.0, 1.5$ and 2.0 . In table 3.1 and figure 3.1, we give a numerical example using Java 2.7, to support our claim that the sequence x_n converges uniquely to its fixed point $p = 0$ for the uniformly L–Lipschitzian mapping of Gregus type T .

Table 3.1: Numerical Example for the uniformly L–Lipschitzian mapping of Gregus type using the iteration in (1.6), $Tx = \frac{x^3}{4(1+x)}$.

S/N	0.5	1	1.5	2
0				
1	0.260416667	0.5625	0.91875	1.333333333
2	0.131959801	0.295488281	0.509897366	0.793650794
3	0.066233649	0.150233556	0.265923844	0.43166398
4	0.033150889	0.075485267	0.134818761	0.222854726
5	0.016579852	0.037792625	0.0676793	0.112558723
⋮	⋮	⋮	⋮	⋮
25	1.58E-08	3.61E-08	6.46E-08	1.08E-07
26	7.91E-09	1.80E-08	3.23E-08	5.39E-08
27	3.95E-09	9.01E-09	1.62E-08	2.69E-08
28	1.98E-09	4.51E-09	8.08E-09	1.35E-08
⋮	⋮	⋮	⋮	⋮
55	1.47E-17	3.36E-17	6.02E-17	1.00E-16
56	7.36E-18	1.68E-17	3.01E-17	5.02E-17
57	N/A	8.40E-18	1.50E-17	2.51E-17
58	N/A	N/A	7.52E-18	1.25E-17
59	N/A	N/A	N/A	6.27E-18

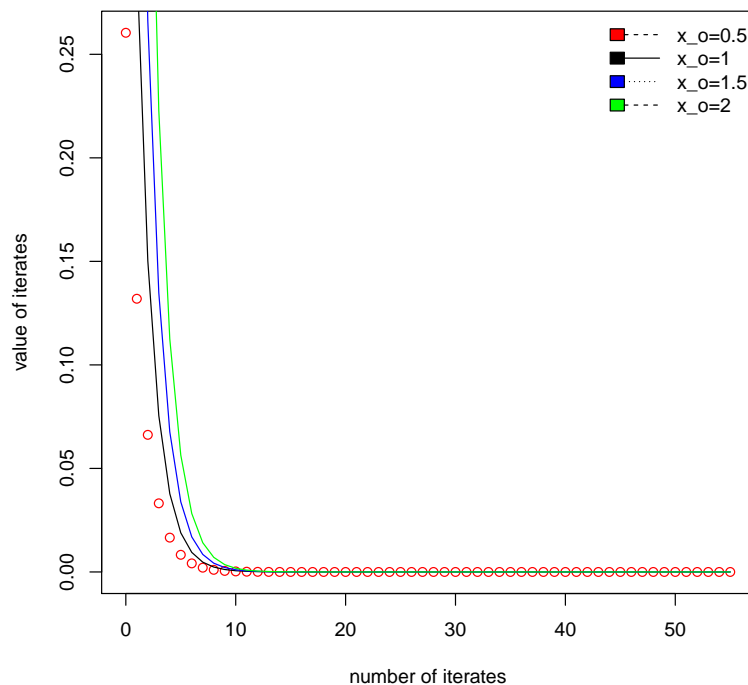


FIG. 3.1: Convergence behaviour of Modified Mann iteration process to the fixed point $p = 0$ with initial guess values taken at $x_0 = 0.5, 1.0, 1.5$ and 2.0 .

Competing Interest

The authors declares that they have no competing interest.

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