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ON BINOMIAL SUMS WITH THE TERMS OF SEQUENCES $\{g_{kn}\}$ AND $\{h_{kn}\}$

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Abstract. In this paper, we derive sums and alternating sums of products of terms of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial coefficients. For example,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}g_{k(n-ti)}h_{kti}$$
$$=2^{n-m}n^{\underline{m}}g_{kn}-n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{n(1-t)}h_{kt}^{n-m}g_{k(tm+tn-n)},$$

where c is a nonzero real number, t is any integer and m is a nonnegative integer.
Keywords: Binomial sums, alternating sums, generalized Fibonacci numbers, recurrence relation.

1. Introduction

Define the second order linear recursive sequences $\{u_n\}$ and $\{v_n\}$ for $n \geq 1$ and nonzero integers p, q, by

$$u_{n+1} = pu_n + qu_{n-1} \text{ and } v_{n+1} = pv_n + qv_{n-1}$$

with initials $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = p$, respectively.

When $q = 1$, $u_n = U_n$ (the nth generalized Fibonacci number) and $v_n = V_n$ (the nth generalized Lucas number). Also, when $p = q = 1$, $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number). If α and β are the roots of the

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equation $x^2 - px - q = 0$, the Binet formulae of the sequences $\{u_n\}$ and $\{v_n\}$ have the forms

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

respectively, where $\alpha, \beta = \left(p \pm \sqrt{\Delta}\right)/2$ and $\Delta = p^2 + 4q$. From [3], Kılıç and Stanica derived the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0$, $n > 0$. It is clear that

$$u_{k(n+1)} = v_k u_{kn} + (-1)^{k+1} q^k u_{k(n-1)} \text{ and } v_{k(n+1)} = v_k v_{kn} + (-1)^{k+1} q^k v_{k(n-1)},$$

where the initial conditions are 0, u_k, and 2, v_k, respectively. The Binet formulae of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$u_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \text{ and } v_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively. It is clearly seen that $u_{-kn} = (-1)^{kn+1} u_{kn}$ and $u_{2kn} = u_{kn} v_{kn}$.

In [9], Komatsu obtained the two binomial sums of the generalized Fibonacci numbers as follows:

$$\sum_{i=0}^{n} \binom{n}{i} c^i u_i = g_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$g_{n+1} = (pc + 2) g_n + \left(qc^2 - pc - 1\right) g_{n-1} \quad (n \geq 1)$$

with $g_0 = 0$, $g_1 = c$ and

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i} d^i u_i = h_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$h_{n+1} = (pd + 2c) h_n + \left(qd^2 - pcd - c^2\right) h_{n-1} \quad (n \geq 1)$$

with $h_0 = 0$ and $h_1 = d$, where c, d are nonzero real numbers. Also he gave several Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions.

In [1], Cook et al. obtained some binomial summation identities including the terms of the sequence $\{g_n\}$ in [9]. For example,

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i \left(qc^2 - pc - 1\right)^{2n-i} g_{2i+1} = (pc + 2)^{2n} g_{2n+1},$$

and

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(qc^2-pc-1\right)^{2(2n-i)}g_{4i} = c^{2n}(pc+2)^{2n}\left(p^2+4q\right)^n g_{4n}.$$

In [10], Ömür et al. defined the subsequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial sums $g_{kn}=\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}u_{ki}$ and $h_{kn}=\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}v_{ki}$. These subsequences satisfy the following relations

$$g_{k(n+1)}=\left(c^kv_k+2\right)g_{kn}-\left(c^{2k}(-q)^k+c^kv_k+1\right)g_{k(n-1)}$$

and

$$h_{k(n+1)}=\left(c^kv_k+2\right)h_{kn}-\left(c^{2k}(-q)^k+c^kv_k+1\right)h_{k(n-1)}$$

in which $g_0=0$, $g_k=c^ku_k$ and $h_0=2$, $h_k=2+c^kv_k$, respectively. The Binet formulae of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ are

$$g_{kn}=\frac{\left(c^k\alpha^k+1\right)^n-\left(c^k\beta^k+1\right)^n}{\alpha-\beta}\text{ and }h_{kn}=\left(c^k\alpha^k+1\right)^n+\left(c^k\beta^k+1\right)^n,$$

respectively. The authors obtained some binomial summation identities of sequence $\{g_{kn}\}$. For example, for $n>0$,

$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)}=\left(c^kv_k+2\right)^{2n}g_{k(2n+1)},$$

and

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)}$$
$$=c^{2kn}\left(v_k^2+4q^k(-1)^{k+1}\right)^n g_{k(2n+1)}.$$

In [7], Kılıç et al. introduced sums and alternating sums of products of terms of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ as follows: for odd number n,

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(e+fi)} = D^{(n-1)/2}\Big(U^n_{k(b+f)/2}U_{k(n(b+f)/2+a+e)}$$
$$+(-1)^{e+(b-f)/2}U^n_{k(b-f)/2}U_{k(n(b-f)/2+a-e)}\Big),$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^iU_{k(a+bi)}U_{k(e+fi)} = \frac{1}{D}\Big((-1)^nV^n_{k(b+f)/2}V_{k(n(b+f)/2+a+e)}$$
$$-(-1)^{e+n(b-f)/2}V^n_{k(b-f)/2}V_{k(n(b-f)/2+a-e)}\Big),$$

where a, b, e, f are any integers, $b+f \equiv 2 (\bmod\ 4)$ and $D = p^2+4$.

In [5, 6], Kılıç considered and computed the alternating binomial sums of the forms

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n,i,k,t) \text{ and } \sum_{i=0}^{n} \binom{n}{i} g(n,i,k,t),$$

where $f(n,i,k,t)$ is $U_{kti}V_{k(n-ti)}$ and $U_{kti}V_{(k+1)tn-(k+2)ti}$, and, $g(n,i,k,t)$ is $U_{ki}U_{k(tn+i)}$, $U_{ki}V_{k(tn+i)}$, $V_{ki}V_{k(tn+i)}$ and $V_{ki}U_{k(tn+i)}$ for positive integers t and n. For example, for odd k,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_{k(n-ti)} = U_{kt}^{n} \begin{cases} (-1)^t V_{kn(t-1)}D^{(n-1)/2} & \text{if } n \text{ is odd,} \\ U_{kn(t-1)}D^{n/2} & \text{if } n \text{ is even,} \end{cases}$$

where D is defined as before.

In [4],inspired by the works of [5, 6, 8], Kılıç et al. gave rising factorial of the summation index instead of its powers. Clearly, they considered and computed the generalized alternating weighted binomial sums :

$$\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i f(n,i,k,t),$$

where $f(n,i,k,t)$ as before and m is a nonnegative integer and $x^{\underline{m}}$ stands for the falling factorial defined by $x^{\underline{m}} = x(x-1)...(x-m+1)$. These kinds of binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. For example, for any integers k and t,

$$\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i U_{kti}V_{k(n-ti)} = (-1)^{kn(t+1)+m} n^{\underline{m}} U_{kt}^{n-m}$$
$$\times \begin{cases} U_{k(tn+tm-n)}D^{(n-m)/2} & \text{if } n \equiv m (\bmod\ 2), \\ -V_{k(tn+tm-n)}D^{(n-m-1)/2} & \text{if } n \equiv m+1 (\bmod\ 2), \end{cases}$$

where m is a nonnegative integer.

2. Sums of certain products with the terms of $\{g_{kn}\}$ and $\{h_{kn}\}$

In this section, firstly, we will start with some lemmas for further use.

Lemma 2.1. *For any integers m and n, we have*

$$g_{k(m+n)} + g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} h_{km}g_{kn} & \text{if } n \text{ is odd,} \\ g_{km}h_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$g_{k(m+n)} - g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} g_{km} h_{kn} & \text{if } n \text{ is odd,} \\ h_{km} g_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$h_{k(m+n)} - h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} h_{km} h_{kn} & \text{if } n \text{ is odd,} \\ \Delta g_{km} g_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$h_{k(m+n)} + h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} \Delta g_{km} g_{kn} & \text{if } n \text{ is odd,} \\ h_{km} h_{kn} & \text{if } n \text{ is even,} \end{cases}$$

where c is a nonzero real number.

Proof. By the Binet formulae of $\{g_{kn}\}$ and $\{h_{kn}\}$, the claimed equalities are obtained. $\square$

We recall some facts for the readers convenience: For any real numbers m and n,

$$(2.1) \qquad (m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

and

$$(2.2) \qquad (m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} - n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} + n^{t-2i}\right) + \binom{t}{t/2} (-1)^{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

where t is a positive integer.

Lemma 2.2. *For any integers r and s, we have*

$$\sum_{i=0}^{n} \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} h_{k(2ri+s)} = h_{k(rn+s)} h_{kr}^n,$$

$$\sum_{i=0}^{n} \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} g_{k(2ri+s)} = g_{k(rn+s)} h_{kr}^n,$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} g_{k(2ri+s)} = \begin{cases} -\Delta^{(n-1)/2} g_{kr}^n h_{k(rn+s)} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} g_{kr}^n g_{k(rn+s)} & \text{if } n \text{ is even,} \end{cases}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{r(n-i)}h_{k(2ri+s)} \tag{2.3}$$
$$=\begin{cases} -\Delta^{(n+1)/2}g_{kr}^{n}g_{k(rn+s)} & \text{if } n \text{ is odd,}\\ \Delta^{n/2}g_{kr}^{n}h_{k(rn+s)} & \text{if } n \text{ is even,}\end{cases}$$

where c is a nonzero real number.

Proof. From (2.1), (2.2) and Lemma 2.1, the proof is obtained. □

Lemma 2.3. *[2]Let n and m be integers such that $0 \le m < n$. For $z \neq -1$,*

$$\sum_{k=0}^{n}\binom{n}{k}k^{\underline{m}}z^k = z^m n^{\underline{m}}(1+z)^{n-m} \ .$$

Theorem 2.1. *Let a, b and e be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}$$
$$=\frac{1}{\Delta}\Bigg(h_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-an}$$
$$-2^nh_{k(b-e)}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{e}\Bigg),$$

$$\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}h_{k(ai+b)}h_{k(ai+e)}$$
$$=h_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-an}$$
$$+2^nh_{k(b-e)}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{e},$$

and

$$\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}g_{k(ai+b)}h_{k(ai+e)}$$
$$=g_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-an}$$
$$+2^ng_{k(b-e)}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{e},$$

where c is a nonzero real number.

Proof. Consider that

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}\\
&=\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}\\
&\times\frac{\left[\left(c^{k}\alpha^{k}+1\right)^{ai+b}-\left(c^{k}\beta^{k}+1\right)^{ai+b}\right]\left[\left(c^{k}\alpha^{k}+1\right)^{ai+e}-\left(c^{k}\beta^{k}+1\right)^{ai+e}\right]}{(\alpha-\beta)^{2}}\\
&=\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}\\
&\times\Big(\left(c^{k}\alpha^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+b}\left(c^{k}\beta^{k}+1\right)^{ai+e}\\
&\qquad+\left(c^{k}\beta^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+e}\left(c^{k}\beta^{k}+1\right)^{ai+b}\Big)\\
&=\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(2ai+b+e)}\\
&\qquad-\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e}h_{k(b-e)}.
\end{aligned}$$

From Lemma 2.2, the desired result is obtained. Similarly, the other cases are given. □

Theorem 2.2. *Let a, b and e be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}
=\begin{cases}-\Delta^{(n-1)/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is odd},\\ \Delta^{(n-2)/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is even},\end{cases}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(ai+b)}h_{k(ai+e)}
=\begin{cases}-\Delta^{(n+1)/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is odd},\\ \Delta^{n/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is even},\end{cases}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}h_{k(ai+d)}$$

$$=\begin{cases} -\Delta^{(n-1)/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is odd,}\\ \Delta^{n/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is even,}\end{cases}$$

where c is a nonzero real number.

Proof. Consider that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}$$
$$=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}$$
$$\times\frac{\left[\left(c^{k}\alpha^{k}+1\right)^{ai+b}-\left(c^{k}\beta^{k}+1\right)^{ai+b}\right]\left[\left(c^{k}\alpha^{k}+1\right)^{ai+e}-\left(c^{k}\beta^{k}+1\right)^{ai+e}\right]}{(\alpha-\beta)^{2}}$$
$$=\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}$$
$$\times\Big(\left(c^{k}\alpha^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+b}\left(c^{k}\beta^{k}+1\right)^{ai+e}$$
$$+\left(c^{k}\beta^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+e}\left(c^{k}\beta^{k}+1\right)^{ai+b}\Big)$$
$$=\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(2ai+b+e)}$$
$$-\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e}h_{k(b-e)}.$$

From (2.3), the desired result is obtained. Similarly, using Lemma 2.2, the other cases can be obtained. □

Theorem 2.3. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}g_{k(n-ti)}h_{kti}=(-1)^{m}n^{\underline{m}}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{n(1-t)}g_{kt}^{n-m}$$
$$\times\begin{cases} -\Delta^{(n-m)/2}g_{k(tn+tm-n)} & \text{if } n\equiv m \pmod 2,\\ \Delta^{(n-m-1)/2}h_{k(tn+tm-n)} & \text{if } n\equiv m+1 \pmod 2,\end{cases}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i g_{kti}h_{k(n-ti)} = (-1)^m n^{\underline{m}}\left(c^{2k}(-q)^k + c^k v_k + 1\right)^{n(1-t)} g_{kt}^{n-m}$$
$$\times\begin{cases} \Delta^{(n-m)/2} g_{k(tn+tm-n)} & \text{if } n \equiv m (\text{mod } 2), \\ -\Delta^{(n-m-1)/2} h_{k(tn+tm-n)} & \text{if } n \equiv m+1 (\text{mod } 2), \end{cases}$$

where c is a nonzero real number and m is a nonnegative integer.

Proof. Observe that

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i g_{k(n-ti)}h_{kti}$$
$$= \frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i\left(\left(c^k\alpha^k+1\right)^{n-ti} - \left(c^k\beta^k+1\right)^{n-ti}\right)$$
$$\times\left(\left(c^k\alpha^k+1\right)^{ti} + \left(c^k\beta^k+1\right)^{ti}\right)$$
$$= \frac{\left(c^k\alpha^k+1\right)^n}{\alpha-\beta}\left(\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i + \sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{ti}\right)$$
$$-\frac{\left(c^k\beta^k+1\right)^n}{\alpha-\beta}\left(\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i + \sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{ti}\right),$$

which by Lemma 2.3, equals

$$\frac{\left(c^k\alpha^k+1\right)^n}{\alpha-\beta}\left\{(-1)^m\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm} n^{\underline{m}}\left(1-\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^t\right)^{n-m}\right\}$$
$$-\frac{\left(c^k\beta^k+1\right)^n}{\alpha-\beta}\left\{(-1)^m\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm} n^{\underline{m}}\left(1-\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^t\right)^{n-m}\right\}$$
$$= (-1)^m n^{\underline{m}}\frac{1}{\alpha-\beta}\left\{\left(c^k\alpha^k+1\right)^n\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm}\left(1-\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^t\right)^{n-m}\right.$$
$$\left. -\left(c^k\beta^k+1\right)^n\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm}\left(1-\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^t\right)^{n-m}\right\}$$
$$= (-1)^m n^{\underline{m}}(\alpha-\beta)^{n-m-1}$$
$$\times\left\{\left(c^k\alpha^k+1\right)^n\left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm}\left(\frac{\left(c^k\alpha^k+1\right)^t-\left(c^k\beta^k+1\right)^t}{(\alpha-\beta)\left(c^k\alpha^k+1\right)^t}\right)^{n-m}\right.$$
$$\left. -(-1)^{n-m}\left(c^k\beta^k+1\right)^n\left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm}\left(\frac{\left(c^k\alpha^k+1\right)^t-\left(c^k\beta^k+1\right)^t}{(\alpha-\beta)\left(c^k\beta^k+1\right)^t}\right)^{n-m}\right\},$$

which by the Binet formulae, gives us the claimed result. □

Similar to the proof method of Theorem just above, we have the following results without proof.

Theorem 2.4. *Let k and t be any integers. Then*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} g_{k(n-ti)} h_{kti}\\&= 2^{n-m} n^{\underline{m}} g_{kn} - n^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{n(1-t)} h_{kt}^{n-m} g_{k(tm+tn-n)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} g_{kti} h_{k(n-ti)}\\&= 2^{n-m} n^{\underline{m}} g_{kn} + n^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{n(1-t)} h_{kt}^{n-m} g_{k(tm+tn-n)},\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.5. *Let k and t be any integers. Then*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{i} g_{kti} h_{kn(t+1)-ki(t+2)}\\&= n^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{m} h_{k}^{n-m} g_{k(nt-m)} + n^{\underline{m}} h_{k(t+1)}^{n-m} g_{km(t+1)},\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{i} g_{kn(t+1)-ki(t+2)} h_{kti}\\&= n^{\underline{m}} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{m} h_{k}^{n-m} g_{k(nt-m)} - n^{\underline{m}} h_{k(t+1)}^{n-m} g_{km(t+1)},\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.6. *Let k and t be any integers. Then*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} (-1)^{i} \left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{i} g_{kti} h_{kn(t+1)-ki(t+2)}\\&= (-1)^{m} n^{\underline{m}} \Delta^{(n-m)/2}\\&\times\begin{cases}\begin{array}{c}\left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{m}\\ \times g_{k}^{n-m} g_{k(tn-m)} + g_{k(t+1)}^{n-m} g_{km(t+1)}\end{array} & \text{if } n \equiv m \pmod 2,\\[2ex] \begin{array}{c}\Delta^{-1/2}\Big\{\left(c^{2k}(-q)^{k} + c^{k}v_{k} + 1\right)^{m}\\ \times g_{k}^{n-m} h_{k(tn-m)} - g_{k(t+1)}^{n-m} h_{km(t+1)}\Big\}\end{array} & \text{if } n \equiv m+1 \pmod 2,\end{cases}\end{aligned}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kn(t+1)-ki(t+2)}h_{kti}$$
$$=(-1)^{m}n^{\underline{m}}\Delta^{(n-m)/2}$$
$$\times\begin{cases}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{m} \\ \times g_{k}^{n-m}g_{k(tn-m)}-g_{k(t+1)}^{n-m}g_{km(t+1)} & \text{if } n\equiv m(\text{mod } 2),\\ \Delta^{-1/2}\Big\{\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{m} \\ \times g_{k}^{n-m}h_{k(tn-m)}+g_{k(t+1)}^{n-m}h_{km(t+1)}\Big\} & \text{if } n\equiv m+1(\text{mod } 2),\end{cases}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.7. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kti}h_{kn-ki(t+2)}$$
$$=n^{\underline{m}}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt}h_{k(t+1)}^{n-m}g_{k(mt+nt+m)}-n^{\underline{m}}h_{k}^{n-m}g_{km},$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kn-ki(t+2)}h_{kti}$$
$$=-n^{\underline{m}}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt}h_{k(t+1)}^{n-m}g_{k(mt+nt+m)}-n^{\underline{m}}h_{k}^{n-m}g_{km},$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.8. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kti}h_{kn-ki(t+2)}$$
$$=(-1)^{m}n^{\underline{m}}\Delta^{(n-m)/2}$$
$$\times\begin{cases}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt} \\ \times g_{k(t+1)}^{n-m}g_{k(mt+nt+t)}-g_{k}^{n-m}g_{km} & \text{if } n\equiv m(\text{mod } 2),\\ -\Delta^{-1/2}\Big\{\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt} \\ \times g_{k(t+1)}^{n-m}h_{k(mt+nt+t)}-g_{k}^{n-m}h_{km}\Big\} & \text{if } n\equiv m+1(\text{mod } 2),\end{cases}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kn-ki(t+2)}h_{kti}$$
$$=(-1)^{m}n^{\underline{m}}\Delta^{(n-m)/2}$$
$$\times\begin{cases}-\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt} \\ \times g_{k(t+1)}^{n-m}g_{k(mt+nt+t)}-g_{k}^{n-m}g_{km} & \text{if } n\equiv m (\text{mod } 2),\\ \Delta^{-1/2}\left\{\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt}\right. \\ \left.\times g_{k(t+1)}^{n-m}h_{k(mt+nt+t)}+g_{k}^{n-m}h_{km}\right\} & \text{if } n\equiv m+1 (\text{mod } 2),\end{cases}$$

where c is a nonzero real number and m is a nonnegative integer.

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