FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 4 (2020), 1199–1204
https://doi.org/10.22190/FUMI2004199B

A NEW STUDY ON ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper, we have generalized a known theorem dealing with $\varphi-\mid C,\alpha,\mid_k$ summability factors of infinite series to the $\varphi-\mid C,\alpha,\beta\mid_k$ summability method under weaker conditions. Also, some new and known results have been obtained.
Keywords: summability factors; infinite series; Cesàro mean; Hölder's inequality; Minkowsk's inequality; almost increasing sequences.

1. Introduction

A positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n), that is (see [8])

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1.1}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta}=0 \quad \texttt{for} \quad n>0. \tag{1.2}$$

Let $(\omega_n^{\alpha,\beta})$ be a sequence defined by (see [5])

$$\omega_n^{\alpha,\beta}=\begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha=1,\beta>-1,\\ \max_{1\leq v\leq n}\left|t_v^{\alpha,\beta}\right|, & 0<\alpha<1,\beta>-1. \end{cases} \tag{1.3}$$

Received December 28, 2019; accepted March 27, 2020
2020 *Mathematics Subject Classification.* Primary 40D15, 26D15 ; Secondary 40F05, 40G05

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C,\alpha,\beta\mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty. \tag{1.4}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C,\alpha,\beta\mid_k$ summability is the same as $\mid C,\alpha,\beta\mid_k$ summability (see [9]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - \mid C,\alpha,\beta\mid_k$ summability reduces to $\mid C,\alpha,\beta;\delta\mid_k$ summability (see [7]). If we take $\beta = 0$, then we have $\varphi - \mid C,\alpha\mid_k$ summability (see [1]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C,\alpha\mid_k$ summability (see [10]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C,\alpha;\delta\mid_k$ summability (see [11]).

2. Known Result

The following theorem is known dealing with the $\varphi - \mid C,\alpha\mid_k$ summability factors of infinite series.

Theorem 2.1 ([3]). Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2.4}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\,|\varphi_n|^k)$ is non increasing and if the sequence (ω_n^α) defined by (see [13])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1\leq v\leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \tag{2.5}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(|\varphi_n|\,\omega_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{2.6}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C,\alpha\mid_k$, $k \geq 1$ and $(\alpha + \epsilon) > 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 for $\varphi - \mid C, \alpha, \beta \mid_k$ summability method under weaker conditions by using an almost increasing sequence instead of a positive non-decreasing sequence. Now we shall prove the following theorem.

Theorem 3.1. Let $0 < \alpha \leq 1$ and let (X_n) be an almost increasing sequence. Let there be sequences (β_n) and (λ_n) such that conditions (2.1)-(2.4) of Theorem 2.1 are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non increasing and if the sequence $(\omega_n^{\alpha,\beta})$ defined by (1.3), satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(|\varphi_n|\,\omega_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \tag{3.1}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$, and $(\alpha+\beta)k + \epsilon > 1$.

Remark. It should be noted that, obviously every increasing sequence is almost increasing. However, the converse need not be true (see [12]).

We need the following lemmas for the proof of our theorem.

Lemma 3.1 ([5]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{3.2}$$

Lemma 3.2 ([4]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (2.3) is satisfied

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty \tag{3.3}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.4}$$

4. Proof of Theorem 3.1. Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by (1.1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 3.1, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \omega_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \omega_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned}$$

To complete the proof of the theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad \text{for} \quad r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} \mid \varphi_n T_{n,1}^{\alpha,\beta} \mid^k \quad & \leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\beta})^{-k} \mid \varphi_n \mid^k \{\sum_{v=1}^{n-1} A_v^{\alpha+\beta} \omega_v^{\alpha,\beta} \mid \Delta\lambda_v \mid\}^k \\
& \leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\beta)k} \mid \varphi_n \mid^k \sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (\omega_v^{\alpha,\beta})^k \beta_v . \{\sum_{v=1}^{n-1} \beta_v\}^{k-1} \\
& = \quad O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k \beta_v \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} \mid \varphi_n \mid^k}{n^{(\alpha+\beta)k+\epsilon}} \\
& = \quad O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k \beta_v v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1} \frac{1}{n^{(\alpha+\beta)k+\epsilon}} \\
& = \quad O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k v^{\epsilon-k} \mid \varphi_v \mid^k \beta_v \int_v^{\infty} \frac{dx}{x^{(\alpha+\beta)k+\epsilon}} \\
& = \quad O(1) \sum_{v=1}^{m} v\beta_v v^{-k} (\omega_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
& = \quad O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{-k} (\omega_r^{\alpha,\beta} \mid \varphi_r \mid)^k \\
& \quad\quad + O(1) m\beta_m \sum_{v=1}^{m} v^{-k} (\omega_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
& = \quad O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
& = \quad O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
& = \quad O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by the hypotheses of Theorem 3.1 and Lemma 3.2. Since, $\mid \lambda_n \mid = O(1)$ by (2.4), finally we have that

$$\begin{aligned}
& \sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k = O(1) \sum_{n=1}^{m} |\lambda_n| \, |\lambda_n|^{k-1} n^{-k} (\omega_n^{\alpha,\beta} \mid \varphi_n \mid)^k \\
= \quad & O(1)) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{-k} (\omega_v^{\alpha,\beta} |\varphi_v|)^k
\end{aligned}$$

$$+O(1)|\lambda_m|\sum_{n=1}^{m} n^{-k}(\omega_n^{\alpha,\beta}|\varphi_n|)^k = O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m$$

$$= \quad O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of Theorem 3.1 and Lemma 3.2. This completes the proof of Theorem 3.1. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series. If we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series. Also, if we take (X_n) as a positive non-decreasing sequence and $\beta = 0$, then we obtain Theorem 2.1.

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