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NORMAL CAYLEY GRAPHS OF CERTAIN GROUPS WHICH ARE LOCALLY PRIMITIVE

A. Mahmiani

Abstract. Let $\Gamma = Cay(G, S)$ be the Cayley graph of a finite group G on S. We call Γ X-locally primitive if $R(G) \leq X \leq Aut(\Gamma)$ and X_v acts primitively on $\Gamma(v) = \{u \in G : u\ is\ adjacent\ to\ v\ in\ \Gamma\}$, for all vertices v, where $R(G)$ denotes the right regular representation of G and X_v denotes the stabilizer of v under X. In this paper we consider a certain group of order $8n$ and prove that it is never an X-locally primitive normal connected Cayley graph of valency at least 3.
Keywords: Cayley graph, connected graph, finite group, locally primitive graph.

1. Introduction

In this paper we are concerned with simple connected graphs $\Gamma = (V, E)$, where V and E denote the set of vertices and edges of Γ. An edge joining $u, v \in V$ is denoted by $\{u, v\}$. The group of automorphisms of Γ is a subgroup X of $Aut(\Gamma)$. The graph Γ is called X-vertex transitive (vertex-transitive in the case of $X = A$) if X is transitive on V, and Γ is called X-locally primitive (locally primitive in short) if $X_v = \{g \in X : v_g = v\}$ is primitive on $\Gamma(v)$, for each vertex $v \in V$, where $\Gamma(v)$ denotes the set of all vertices of which are adjacent to v. The degree of a vertex v is denoted by $deg(v)$ and is equal to $|\Gamma(v)|$, A graph is called regular of degree d if the degree of each vertex is equal to d.

Let G be a finite group and S be an inverse closed subset of G, i.e., $S = S^{-1}$, such that $1 \notin S$. The Cayley graph $Cay(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} : g \in G, s \in S\}$. it can be proved that a Cayley graph $Cay(G, S)$ is connected if and only if $G = \langle S\rangle$, i.e., G is generated by S. A Cayley graph $Cay(G, S)$ is always regular of degree $|S|$. For $g \in G$ we define the mapping $\rho_g : G \to G$ by $\rho_g(x) = xg$ for all $x \in G$. Clearly ρ_g is an automorphism of the Cayley graph $Cay(G, S)$ and $R(G) = \{\rho_g : g \in G\}$ is a subgroup of $A = Aut(Cay(G, S))$ isomorphic to G, called the right regular representation of

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G. Since $R(G)$ acts regularly on G we deduce that $Cay(G, S)$ is a vertex-transitive graph.

For two inverse closed subsets S and T of a group G not containing the identity element 1 of G, if there is an automorphism α of G such that $S^\alpha = T$ then S and T are said to be equivalent, and in this case we have $Cay(G, S) \cong Cay(G, T)$.

Let $\Gamma = Cay(G, S)$ be the Cayley graph of a finite group G on S. Let

$$Aut(G, S) = \{\alpha \in Aut(G) : S^\alpha = S\}$$

and set $A = Aut(\Gamma)$. Then $N_A(R(G)) = R(G) \times Aut(G, S)$.

The following definition is initiated in [7].

Definition 1.1. Let $\Gamma = Cay(G, S)$ be the Cayley graph G on a set S. Then Γ is said to be normal if $R(G)$ is a normal subgroup of $Aut(\Gamma)$.

It is proved in [1] that $\Gamma = Cay(G, S)$ is normal if and only if $A = Aut(\Gamma) = R(G) \times Aut(G, S)$, where $\times$ denotes semi-direct product of groups, and in this case $A_1 = Aut(G, S)$ where A_1 is the stabilizer of the identity 1 of G in A. The normality of Cayley graphs have been extensively studied from different views by many authors. In [6] all disconnected normal Cayley graphs are obtained. Therefore, it suffices to study the connected Cayley graphs when one investigates the normality of Cayley graphs. A class of normal Cayley graphs has been studied in [5]. One aspect of studying Cayley graphs is the investigation of their automorphism groups as permutation groups. Locally primitive Cayley graphs have been studied in [2], [3] and [4].

In [4] a complete characterization of locally primitive normal Cayley graphs of metacyclic groups is given and in particular it is proved that if $\Gamma = Cay(G, S)$ is a connected X-locally primitive normal Cayley graph of degree at least 3, where $G \cong Z_m.Z_n$ is a non-abelian meta-cyclic group, and $R(G) \leq X \leq Aut(\Gamma)$ then $n = 2$ and $G \cong D_{2m}$ is a dihedral group. Motivated by this result we consider a class of groups of order $8n$ denoted by V_{8n} where $V_{8n} = \langle a, b : a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle$ and investigate X-locally primitive normal Cayley graphs $\Gamma = Cay(V_{8n}, S)$ of degree at least 3 and prove that Γ is never an X-locally primitive normal connected Cayley graph for all

$$X, R(V_{8n}) \leq X \leq R(V_{8n}) \times Aut(R(V_{8n}, S)).$$

2. Preliminary results

Lemma 2.1. *Let $\Gamma = Cay(G, S), |S|3$ be a connected X-locally primitive normal Cayley graph, where $R(G) \leq X \leq Aut(\Gamma)$. Then*

1. *$G = \langle S\rangle$, and all elements of S are involutions and conjugate under $Aut(G, S)$.*
2. *$X_1 = Aut(G, S)$ acts faithfully and primitively on S.*

Proof. By normality of F we have $Aut(\Gamma) = R(G) \times Aut(G, S)$ where the action of $Aut(G, S)$ on $R(G)$, or equivalently on G, is by conjugation. By connectedness we have $G = \langle S \rangle$ and by local permittivity condition $X_1 \leq Aut(G, S)$ acts primitively on $\Gamma(1) = S$. Since $G = \langle S \rangle$, the action of X_1 on S is faithful and (2) is proved. Action of X_1 on S is by conjugation and this means that elements of S are conjugate under X_1. Now for $x \in S$ we have $\{x, x^{-1}\} \subseteq S$ and $\{x, x^{-1}\}^{\alpha} = \{(x), (\alpha(x)^{-1})\}$ for all $\alpha \in X_1$. Because of permittivity and $|S| \geq 3$ we obtain $x = x^{-1}$, $\forall x \in S$, implying that all elements of S are involutions and the Lemma is proved. □

Next, let us consider the dihedral group D_8 of order 8. We have $D_8 = \langle a, b : a^2 = b^4 = 1, a^{-1}ba = b^{-1} \rangle = V_8$. Elements of order 4 in D_8 are b and b^3 and elements of order 2 are a, ab^2, ab, ab^3 and b^2. It is well-known that $|Aut(D_8)| = 8$ and that $Aut(D_8) \cong D_8$. We can produce all 8 elements of $Aut(D_8)$ by giving their effects on a and b as follows:

$$f_1 : \begin{cases} a \to a \\ b \to b \end{cases} \quad f_2 : \begin{cases} a \to ab^2 \\ b \to b \end{cases} \quad f_3 : \begin{cases} a \to ab \\ b \to b \end{cases} \quad f_4 : \begin{cases} a \to ab^3 \\ b \to b \end{cases}$$

$$f_5 : \begin{cases} a \to a \\ b \to b^3 \end{cases} \quad f_6 : \begin{cases} a \to ab^2 \\ b \to b^3 \end{cases} \quad f_7 : \begin{cases} a \to ab \\ b \to b^3 \end{cases} \quad f_8 : \begin{cases} a \to ab^3 \\ b \to b^3 \end{cases}$$

We try to find a subset S of $D_8\{1\}$ of cardinality at least 3 such that $\Gamma = Cay(D_8, S)$ is a connected X-locally primitive normal Cayley graph. By Lemma 2.1, $S \subseteq \{a, ab^2, ab, ab^3, b^2\}$, and since $f_i(b^2) = b^2$ for all $1 \leq i \leq 8$, we must have $b^2 / \in S$.

From the other hand since $X_1 \leq Aut(D_8, S)$ acts transitively on S, we must have $|S|||X_1|$ hence from $|X_1|||Aut(D_8, S)|$ we obtain $|X_1||8$ hence$|S||$ 8 from which we deduce $|S| = 4$. Therefore $S \subseteq \{a, ab^2, ab, ab^3\}$ and it 8. If $|X_1| = 4$ then X_1 can not act primitively on S and if $|X_1| = 8$, then $X_1 = Aut(D_8, S) = D_8$ and it is evident that D_8 can not act primitively on a set of size 4.

The above investigations show that $\gamma = Cay(D_8, S)$ is never an X-locally primitive normal connected Cayley graph for any X with $R(D_8) \leq X \leq Aut(\Gamma)$. Therefore in our further discussion about D_{8n} we will assume $n > 1$.

3. Main result

Let V_{8n} denote the following group given by generators and relations:

$$V_{8n} = \langle a, b : a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle$$

It is easy to verify the following facts about V_{8n}:

*Elements of order 2 in V_{8n}, are of the following types:

Type(I): $a^n, b^2, a^n b^2$

Type(II): $a^i, b, i = 1, 3, ..., 2n - 1$
Type(III): $a^i b^3, i = 1, 3, ..., 2n - 1$
Therefore V_{8n} has $2n + 3$ elements of order 2,

⁎$Z(V_{8n}) = 1, b^2$ and

$$G' = \langle a^2, b^2 \rangle, |G'| = 2n,$$

if n is odd and

$$G' = \langle a^2 b^2 \rangle, |G'| = n$$

if n is even.

⁎ Elements of order $2n$ in V_{8n}, are:

$$a^i.(i, 2n) = 1; a^i b^2.(i, 2n) = 1; a^i b^2$$

with $(i, 2n) = 2$ in case n is odd.

Lemma 3.1. $|Aut(V_{8n})| = 4n\varphi(2n)$ *if* $n > 1$ *and* $|Aut(V_8)| = 8$.

Proof. . It is clear that $V_8 \cong D_8$ and $Aut(V_8) = D_8$, hence we will assume $n > 1$. For $f \in Aut(V_{8n})$ it is enough to define $f(a)$ and $f(b)$ so that $f(a)$ and $f(b)$ are elements of order $2n$ and 4 respectively. Since $Z(V_{8n}) = \{1, b^2\}$, we must have $f(b^2) = b^2$, hence $f(b) = a^i$ leads to contradiction. Since the order of a is equal to $2n$, we can define $f(a) = a^k, (k, 2n) = 1$,and in this case for $f(b)$ we have the following possibilities: $f(b) = a^l b$ or $a^l b^3, l$ even, $0 \leq 1 \leq 2n$. It can be verified that such an f is indeed an automorphism of f giving the total of $2n\varphi(n)$ automorphisms. Similarly we have to define $f(a) = a^k b^2, (k, 2n) = 1$, and $f(b) = a^l b$ or $a^l b^3$ where 1 is even, $0 \leq 1 \leq 2n$. We will obtain $2n\varphi(n)$ automorphism in this cases well. If n is odd, $a^k b^2$ with $(k, 2n) = 2$ is also art element of order $2n$, hence we may define $f(a) = a^k b^2$ and $f(b) = a^l b$ or $a^l b^3$, l even, $0 \leq 1 \leq 2n$. But in this case it can be verified that $f(ab) = f(a)f(b)$ is not an element of order 2, hence if can not he extended to an automorphism of V_{8n}. Therefore if $n > 1$ there are $4n\varphi(n)$ automorphisms of V_{8n} and the Lemma is proved. □

Lemma 3.2. *If n is even involutions of* V_{8n}*, generate a proper subgroup of* V_{8n}*, of order 4n, and if n is odd, the involutions of Type (II) and (III) generate a proper subgroup of* V_{8n} *of order* $4n$.

Proof. Using the relations in defining definition of V_{8n}, we may use induction to prove:
$ba^k = a^{-k}b$ if k is even, $ba^k = a^{-k}b^{-1}$if k is odd.

Therefore choosing involutions of Type (II) or (III) and multiplying them we obtain:

Type (II) Type (II): $a^k b a^l b = a^{k-1}, k, l$ odd,
Type (II) Type (III): $a^k b a^l b^3 = a^{k-1} b^2, k, l$ odd,
Type (III) Type (III): $a^k b^3 a^l b^3 = a^{k-1}, k, l$ odd.

Therefore a^{2k} and $a^{2k}b^2, 0 \leq k < n$, are also generated by involutions of Type (II) and (III). But if n is even, then a^n, b^2 and $a^n b^2$, which are involutions of Type (I), are also generated. Therefore involutions of Type (II) and (III) generate a subgroup of V_{8n}, with order $4n$. If n is odd, then $\{a^{2k}, a^{2k}b^2 : 0 \leq k < n\}$ does not include a n and $a^n b^2$, hence the subgroup generated by involutions of Type (II) and (III) generate a proper subgroup of V_{8n}, with order $4n$. □

Theorem 3.1. $\Gamma = Cay(V_{8n}, S)$ *is never a connected X-locally primitive normal Cayley graph for any subset* $S \subseteq V_{8n}$, *of cardinality at least 3, where*

$$R(V_{8n}) \leq X \leq Aut(\Gamma).$$

Proof. Since $V_8 = D_8$, by previous section such an S does not exist. Hence we may assume $n > 1$. By Lemma 2.1 all elements of S should be involutions and $V_{8n} = \langle S \rangle$. By Lemma 3.2 if n is even such an S does not exist, hence we may assume that n is odd. By Lemma 3.2 elements of S can not be only of Type (II) or (III), hence S must contain at least one involution of Type (I). Also $b^2 \in S$ because $f(b^2) = b^2$, for all $f \in Aut(\Gamma)$. If a $n \in S$, then by Lemma 3.1 either $f(a) = a^k, (k, 2n) = 1$, or $f(a) = a^k b^2, (k, 2n) = 1$, implying $f(a^n) = a^{kn} = a^n$ or $f(a^n) = a^n b^2$respectively. Of course $f(a) = a^k$ is not possible and if $f(a^n) = a^n b^2$, then $f(a^n b^2) = a^{kn} a^{2n} b^2 = a^n$. Therefore $\{a^n b^2, a^n\}$ is invariant under all $f \in Aut(\Gamma)$, and also it is part of S contradicting the fact that $|S| \geq 3$. Similarly $a^n b^2 \in S$ leads to a contradiction, and the theorem is proved. □

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Anehgaldi Mahmiani
Islamic Azad University
Gonbad Kavoos branch
Gonbad Kavoos, Iran
mahmiani_a@yahoo.com