FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. Vol. 35, No 5 (2020), 1381–1391
https://doi.org/10.22190/FUMI2005381S

APPLICATIONS OF MATRIX TRANSFORMATIONS TO ABSOLUTE SUMMABILITY

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Abstract. Rhoades and Savaş [6], [11] established necessary conditions for inclusions of the absolute matrix summabilities under additional conditions. In this paper, we determine necessary or sufficient conditions for some classes of infinite matrices, and using this, we get necessary or sufficient conditions for more general absolute summabilities applied to all matrices.

Keywords: matrix summability; infinite matrices; Cesàro matrices; triangular matrix.

1. Introduction

Let X and Y be two sequence spaces of the space ω, the set of all sequences with real or complex terms. Let $A = (a_{nv})$ be an infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series are convergent for $v, n \geq 0$. If $A(x) \in Y$ for all $x \in X$, then A is called a matrix transformation from X into Y, and denoted by (X, Y).

In many cases, since an infinite matrix can be considered as a linear operator between two sequence spaces, the theory of matrix transformations in sequence spaces has aroused interest for many years, of which purpose is to provide the necessary and sufficient conditions for a matrix to map a sequence space into another.

X is called a BK-space, if it is a Banach space on which all coordinate functionals defined by $p_n(x) = x_n$ are continuous.

Received November 23, 2018; accepted July 22, 2019
2020 *Mathematics Subject Classification.* Primary 40C05, 40D25, 40F05, 46A45

Let Σa_v be a given infinite series with n-th partial sum s_n and let (γ_n) be a sequence of nonnegative numbers. By $(A_n(s))$, we denote the A-transform of the sequence $s=(s_n)$. The series Σx_v is said to be summable $|A,\gamma_n|_k, k\geq 1$, if (see [7])

$$\sum_{n=1}^{\infty}\gamma_n^{k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty. \tag{1.1}$$

Note that, for $\gamma_n=n$, $|A,\gamma_n|_k=|A|_k$ [12], Also, if A is chosen as the matrices of the weighted mean (R,p_n) $(resp.\gamma_n=P_n/p_n)$ and Cesàro mean (C,α) together with $\gamma_n=n$, then, it reduces to the summabilities $|R,p_n|_k$ [8] $\left(resp.|\overline{N},p_n|_k\ \ [1]\right)$ and $|C,\alpha|_k$ [2], respectively. By the weighted and Cesàro matrices we mention

$$a_{nv}=\begin{cases}\frac{p_v}{P_n}, & 0\leq v\leq n\\ 0, & v>n,\end{cases}$$

and

$$a_{nv}=\begin{cases}\frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}, & 0\leq v\leq n\\ 0, & v>n.\end{cases}$$

respectively, where (p_n) is a sequence of positive numbers with $P_n=p_0+p_1+...+p_n\to\infty$, and

$$A_n^{\alpha}=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!},n\geq 1,\ \ A_0^{\alpha}=1$$

$$|A_n^{\alpha}|\leq A(\alpha)n^{\alpha}\text{ for all }\alpha$$

$$A_n^{\alpha}\geq A(\alpha)n^{\alpha}\text{ and }A_n^{\alpha}>0\text{ for }\alpha>-1.$$

Let $A=(a_{nv})$ be a lower triangular matrix, we derive the matrices $\overline{A}=(\overline{a}_{nv})$ and $\widehat{A}=(\widehat{a}_{nv})$ from the matrix A as follows:

$$\begin{aligned}\overline{a}_{00} &= \widehat{a}_{00}=a_{00}\\ \overline{a}_{nv} &= \sum_{r=v}^{n}a_{nr};\ \ n,v=0,1,...\\ \widehat{a}_{nv} &= \overline{a}_{nv}-\overline{a}_{n-1,v},\ \ \overline{a}_{n-1,n}=0.\end{aligned}$$

Then, $\widehat{A}$ is a triangular matrix and has unique inverse which is also triangular (see [13]). We will denote its inverse $\widehat{A}'$. Hence, it can be written that

$$A_n(x)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{r=0}^{n}\left(\sum_{v=r}^{n}a_{nv}\right)x_r=\sum_{v=0}^{n}\overline{a}_{nv}x_v$$

and

$$\widehat{A}_n(x) = A_n(x)-A_{n-1}(x)=\sum_{v=0}^{n}\left(\overline{a}_{nv}-\overline{a}_{n-1,v}\right)x_v=\sum_{v=0}^{n}\widehat{a}_{nv}x_v. \tag{1.2}$$

which means that the summability $|A,\gamma_n|_k$ is equivalent to

$$\sum_{n=0}^{\infty}\gamma_n^{k-1}\left|\widehat{A}_n(x)\right|^k<\infty. \tag{1.3}$$

By $|\gamma A|_k$, we define the set of all series summable by $|A,\gamma_n|_k$. Then, a series Σx_v is summable $|A,\gamma_n|_k$ iff $x=(x_v)\in|\gamma A|_k$, i.e.,

$$|\gamma A|_k=\left\{x=(x_v):\widetilde{A}(x)=\left(\widetilde{A}_n(x)\right)\in\ell_k\right\} \tag{1.4}$$

where $\widetilde{A}_n(x)=\gamma_n^{1-1/k}\widehat{A}_n(x)$ for all $n\geq 0$ and ℓ_k is the set of all k-absolutely convergent series.

We note that, since $\widetilde{A}=(\widetilde{a}_{nv})$ is a triangle matrix, it is routine to show that $|\gamma A|_k$ is a BK- space if normed by

$$\|x\|_{|\gamma A|_k}=\left\|\widetilde{A}(x)\right\|_{\ell_k},\ \ 1\leq k<\infty. \tag{1.5}$$

Dealing with the absolute weighted mean summability of infinite series, Bor and Thorpe [1] established sufficient conditions in order that all $\left|\overline{N},p_n\right|_k$ summable series is also summable $\left|\overline{N},q_n\right|_k$, and conversely. The author [10] showed that Bor and Thorpe's conditions are not only sufficient but also necessary for the conclusion. Also, these results of the author [10] were extended by Rhoades and Savaş [6] using a triangle matrix instead of weighted mean matrix as follows.

Theorem 1.1. Let $1<k\leq s<\infty$, (p_n) be a sequence satisfying

$$\sum_{n=v+1}^{\infty}n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k=O\left(\frac{1}{P_n^k}\right). \tag{1.6}$$

Let B be a lower triangular matrix. Then, necessary conditions for Σx_v summable $\left|\overline{N},p_n\right|_k$ to imply Σx_v is summable $|B|_s$ are

$$\frac{P_v\left|b_{vv}\right|}{p_v}=O\left(v^{1/s-1/k}\right),$$

$$\sum_{n=v+1}^{\infty}n^{s-1}\left|\Delta_v\widehat{b}_{nv}\right|^s=O\left(v^{s-s/k}\frac{p_v}{P_v}\right),$$

$$\sum_{n=v+1}^{\infty}n^{s-1}\left|\widehat{b}_{n,v+1}\right|^s=O\left(1\right).$$

This result has also been extended by Savaş [11] to the matrix methods as follows

Theorem 1.2. Let $1 < k \leq s < \infty$, A and B be two lower triangular matrices. A satisfying

$$\sum_{n=v+1}^{\infty} n^{k-1} \left|\Delta_v \widehat{a}_{nv}\right|^k = O\left(\left|a_{vv}\right|^k\right). \tag{1.7}$$

Then necessary conditions for Σx_v summable $|A|_k$ to imply Σx_v is summable $|B|_s$ are

$$|b_{vv}| = O\left(v^{1/s-1/k} \left|a_{vv}\right|\right),$$

$$\sum_{n=v+1}^{\infty} n^{s-1} \left|\Delta_v \widehat{b}_{nv}\right|^s = O\left(v^{s-s/k} \left|a_{vv}\right|^s\right)$$

and

$$\sum_{n=v+1}^{\infty} n^{s-1} \left|\widehat{b}_{n,v+1}\right|^s = O\left(\sum_{n=v+1}^{\infty} n^{k-1} \left|\widehat{a}_{n,v+1}\right|^k\right)^{s/k}.$$

2. Main results

We note that Theorem 1.1 and Theorem 1.2 give necessary conditions for the triangle matrices under the conditions (1.6) and (1.7). In the present paper, we determine necessary or sufficient conditions for a matrix $T \in (|\gamma A|_k, |\phi B|_s)$, $1 \leq k \leq s < \infty$. Also, in the special case, we get some more general results that do not include the conditions (1.6) and (1.7). More precisely, we give the following theorems.

Theorem 2.1. Let A, B be infinite triangle matrix and T be any infinite matrix of complex numbers. Further, let (γ_n) and (ϕ_n) be two sequences of positive numbers. Then, the necessary conditions for $T \in (|\gamma A|_k, |\phi B|_s)$, $1 < k \leq s < \infty$, are

$$\overline{l}_{nr} = \gamma_r^{-1/k^*} \sum_{i=r}^{\infty} t_{ni} \widehat{a}'_{ir} \text{ converges for } n, r \geq 0 \tag{2.1}$$

$$\sup_m \sum_{v=0}^{m} \frac{1}{\gamma_r} \left|\sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr}\right|^{k^*} < \infty \text{ for } n, r \geq 0 \tag{2.2}$$

$$\sum_{n=m}^{\infty} \phi_n^{s-1} \left|\sum_{v=0}^{n} \sum_{i=m}^{\infty} \widehat{b}_{nv} t_{vi} \widehat{a}'_{im}\right|^s = O(\gamma_m^{s/k^*}), \tag{2.3}$$

where k^* is the conjugate of k, i.e., $k^* = k/(k-1)$.

Theorem 2.2. Let A, B be infinite triangle matrix and T be any infinite matrix of complex numbers. Further, let (ϕ_n) be a sequences of positive numbers. Then, the necessary and sufficient conditions for $T \in (|A|, |\phi B|_s), 1 = k \leq s < \infty$, are

$$\overline{l}_{nr} = \sum_{i=r}^{\infty} t_{ni} \widehat{a}'_{ir} \text{ converges for all } n, r \geq 0 \tag{2.4}$$

$$\sup_{m,r} \left| \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right| < \infty \tag{2.5}$$

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \right|^s = O(1). \tag{2.6}$$

Note that for $1 < k \leq s < \infty$, the characterization of the class of all matrices (ℓ_k, ℓ_s) are not known. Hence one can not expect to get a set of necessary and sufficient conditions for Theorem 2.1.

We require the following lemmas for the proof of our theorems.

Lemma A. Let X and Y be BK spaces, and A be an infinite matrix of complex numbers. If A is a matrix transformation from X into Y, *i.e.*, $A \in (X, Y)$, then it is a bounded linear operator [13].

Lemma B. Let $1 < k < \infty$ and A be an infinite matrix of complex numbers. Then

a-) $A \in (\ell, c)$ iff

$$i\text{-}) \lim_n a_{nv} \text{ exists for all } v \geq 0, \text{ and } \quad ii\text{-}) \sup_{n,v} |a_{nv}| < \infty,$$

b-) $A \in (\ell_k, c)$ iff

$$i\text{-}) \ (i) \text{ is satisfied, and } \quad ii\text{-}) \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty,$$

where c is the set of all convergent sequences, and $1/k + 1/k^* = 1$ [13].

Lemma C. Let $1 \leq s < \infty$ and A be an infinite matrix. Then $A \in (\ell_1, \ell_s)$ iff

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^s < \infty$$

where ℓ_s is the set of all s- absolutely convergent sequences [3].

Proof of the Theorem 2.1. Let $1 < k \leq s < \infty$. Suppose, $T \in (|\gamma A|_k, |\phi B|_s)$. Then, $T(x)$ exists and $T(x) \in |\phi B|_s$ for all $x \in |\gamma A|_k$. Now, $x \in |\gamma A|_k$ iff $y = \widetilde{A}(x) \in$

ℓ_k,where $y_n = \widetilde{A}_n(x) = \gamma_n^{1/k^*}\widehat{A}_n(x)$, and $\widehat{A}_n(x)$ is defined by (1.2). By the inverse of (1.2), we have

$$x_n = \sum_{r=0}^{n} \widehat{a}'_{nr}\widehat{A}_r(x) = \sum_{r=0}^{n} \widehat{a}'_{nr}\gamma_r^{-1/k^*} y_r,$$

and so

$$\begin{aligned}\sum_{v=0}^{m} t_{nv}x_v &= \sum_{v=0}^{m} t_{nv}\sum_{r=0}^{v} \widehat{a}'_{vr}\gamma_r^{-1/k^*} y_r \\ &= \sum_{r=0}^{m}\left(\gamma_r^{-1/k^*}\sum_{v=r}^{m} t_{nv}\widehat{a}'_{vr}\right) y_r = \sum_{r=0}^{\infty} l_{mr}^{(n)} y_r \\ &= L_m^{(n)}(y)\end{aligned}$$

where

$$l_{mr}^{(n)} = \begin{cases} \gamma_r^{-1/k^*}\sum_{v=r}^{m} t_{nv}\widehat{a}'_{vr}, & 0 \le r \le m \\ 0, & r > m. \end{cases}$$

This implies that $T(x)$ exists for all $x \in |\gamma A|_k$ iff $L^{(n)}(y)$ exists for $y \in \ell_k$, or equivalently, $L^{(n)} = \left(l_{mr}^{(n)}\right) \in (\ell_k, c)$. So, it follows from Lemma B that $T(x)$ exists iff (2.1) and (2.2) are satisfied. Further,

$$\begin{aligned}T_n(x) &= \sum_{v=0}^{\infty} t_{nv}x_v = \sum_{r=0}^{\infty} \lim_{m\to\infty} l_{mr}^{(n)} y_r \\ &= \sum_{r=0}^{\infty} \overline{l}_{nr} y_r = \overline{L}_n(y),\end{aligned}$$

which means $T(x) = \overline{L}(y)$. On the other hand, since $x \in |\phi B|_s$ iff $\widetilde{B}_n(x) \in \ell_s$, $T(x) \in |\phi B|_s$ iff $\widetilde{B}_n(T(x)) \in \ell_s$, *i.e.*, $C(y) \in \ell_s$, where

$$c_{nr} = \sum_{v=0}^{n} \widetilde{b}_{nv}\overline{l}_{vr} \text{ for } n, r \ge 0,$$

because, for each $n \ge 0$,

$$\begin{aligned}C_n(y) &= \sum_{v=0}^{\infty} c_{nr} y_r = \sum_{r=0}^{\infty}\left(\sum_{v=0}^{n} \widetilde{b}_{nv}\overline{l}_{vr}\right) y_r \\ &= \sum_{v=0}^{n} \widetilde{b}_{nv}\overline{L}_v(y) = \sum_{v=0}^{n} \widetilde{b}_{nv}T_v(x) \\ &= \widetilde{B}_n(T(x)).\end{aligned}$$

Also, it can be seen that $C = \widetilde{B}.\overline{L}$. So, by combining the above calculations we get $C \in (\ell_k, \ell_s)$. On the other hand, since ℓ_k is BK space for $k \ge 1$, then, by

Lemma A , the matrix C defines a bounded linear operator $L_C : \ell_k \to \ell_s$ such that $L_C(x) = (C_n(x))$ for all $x \in \ell_k$, and so there exists a constant M such that

$$\|L_C(x)\|_{\ell_s} \leq M \|x\|_{\ell_k} \text{ for all } x \in \ell_k. \tag{2.7}$$

Now in particular we put $x_m = 1$ and $x_n = 0$ for $n \neq m$. Then, we obtain

$$C_n(x) = \begin{cases} 0, & n < m \\ c_{nm}, & n \geq m \end{cases}$$

and

$$\|L_C(x)\|_{\ell_s} = \left(\sum_{n=m}^{\infty} \left| \phi_n^{1/s^*} \gamma_m^{-1/k^*} \sum_{v=0}^{n} \sum_{i=m}^{\infty} \widehat{b}_{nv} t_{vi} \widehat{a}'_{im} \right|^s \right)^{1/s}.$$

So, it follows from (2.7) that (2.3) holds. This completes the proof.

Proof of the Theorem 2.2. Let $1 = k \leq s < \infty$. Then, $T \in (|A|, |\phi B|_s)$ iff $T(x)$ exists and $T(x) \in |\phi B|_s$ for all $x \in |A|$. Now, $x \in |A|$ iff $y \in \ell$, where $y_n = \widehat{A}_n(x)$ and $\widehat{A}_n(x)$ is defined by (1.2). Then, by the inverse of (1.2), we have

$$x_n = \sum_{r=0}^{n} \widehat{a}'_{nr} \widehat{A}_r(x) = \sum_{r=0}^{n} \widehat{a}'_{nr} y_r,$$

and so

$$\begin{aligned} \sum_{v=0}^{m} t_{nv} x_v &= \sum_{v=0}^{m} t_{nv} \sum_{r=0}^{v} \widehat{a}'_{vr} \gamma_r^{-1/k^*} y_r \\ &= \sum_{r=0}^{m} \left(\sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right) y_r = \sum_{r=0}^{\infty} l_{mr}^{(n)} y_r \\ &= L_m^{(n)}(y) \end{aligned}$$

where

$$l_{mr}^{(n)} = \begin{cases} \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr}, & 0 \leq r \leq m \\ 0, & r > m. \end{cases}$$

This implies that $T(x)$ exists for all $x \in |A|$ iff $L^{(n)}(y) \in (\ell, c)$, or equivalently, by Lemma B, (2.4) and (2.5) are satisfied. Further, we have

$$\begin{aligned} T_n(x) &= \sum_{v=0}^{\infty} t_{nv} x_v = \sum_{r=0}^{\infty} \lim_{m \to \infty} l_{mr}^{(n)} y_r \\ &= \sum_{r=0}^{\infty} \overline{l}_{nr} y_r = \overline{L}_n(y), \end{aligned}$$

which also means $T(x) = \overline{L}(y)$. On the other hand, since $T(x) = \overline{L}(y)$, then, $T(x) \in |\phi B|_s$ iff $C(y) \in \ell_s$, where

$$c_{nr} = \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \text{ for } n, r \geq 0,$$

because,

$$\begin{aligned} C_n(y) &= \sum_{r=0}^{\infty} c_{nr} y_r = \sum_{r=0}^{\infty} \left(\sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \right) y_r \\ &= \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{L}_v(y) = \sum_{v=0}^{n} \widetilde{b}_{nv} T_v(x) \\ &= \widetilde{B}_n(T(x)). \end{aligned}$$

Thus it follows from Lemma C that

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \right|^s = O(1),$$

which completes the proof.

We note that in the special case $T = I$, identity matrix, then $I \in (|\gamma A|_k, |\phi B|_s)$ means that if a series is summable $|A, \gamma_n|_k$, then it is also summable $|B, \phi_n|_s$, and also, conditions (2.1), (2.2) hold and (2.3) reduces to

$$\phi_m^{s-1} \left| \frac{b_{mm}}{a_{mm}} \right|^s + \sum_{n=m+1}^{\infty} \phi_n^{s-1} \left| \sum_{i=m}^{n} \widehat{b}_{ni} \widehat{a}'_{im} \right|^s = O(\gamma_m^{s/k^*}).$$

So, as consequences of Theorem 2.1-2.2, we have many results. Now we list some of them.

Corollary 2.3. Let A and B be infinite triangle matrix of complex numbers. Further, let (γ_n) and (ϕ_n) be two sequences of positive numbers.

a-) If $1 < k \leq s < \infty$, then, the necessary conditions in order that a series by summable $|A, \gamma_n|_k$ is also summable $|B, \phi_n|_s$ are

$$\phi_m^{1/s^*} \left| \frac{b_{mm}}{a_{mm}} \right| = O(\gamma_m^{1/k^*}) \tag{2.8}$$

$$\sum_{n=m+1}^{\infty} \phi_n^{s-1} \left| \sum_{i=m}^{n} \widehat{b}_{ni} \widehat{a}'_{im} \right|^s = O(\gamma_m^{s/k^*}). \tag{2.9}$$

b-) If $1 = k \leq s < \infty$, then, the necessary and sufficient conditions in order that a series by summable $|A|$ is also summable $|B, \phi_n|_s$ are that (2.8) and (2.9) with $k = 1$ are satisfied.

Let us take $\phi_n = \gamma_n = n$ for all n. Since $|A, \gamma_n|_k = |A|_k$ and $|B, \phi_n|_s = |B|_s$, then, Corollary 2.3 reduces to the following result which do not include the additional condition (1.7) of Theorem 1.2.

Corollary 2.4. Let $1 < k \leq s < \infty$, A and B be triangle matrix of complex numbers. Then necessary conditions in order that a series by summable $|A|_k$ is also summable $|B|_s$ are

$$m^{1/k-1/s}\left|\frac{b_{mm}}{a_{mm}}\right| = O(1)$$

and

$$\sum_{n=m+1}^{\infty} n^{s-1}\left|\sum_{i=m}^{n}\widehat{b}_{ni}\widehat{a}'_{im}\right|^s = O\left(m^{s/k^*}\right).$$

If $1 = k \leq s < \infty$, by Theorem 2.2, these conditions with $k = 1$ are also necessary and sufficient for the conclusion to satisfy.

Also, if we put $A = I$ and $\gamma_v = v$ for all $v \geq 1$, then the summability $|A, \gamma_n|_k$ is equivalent to the condition

$$\sum_{n=1}^{\infty} n^{k-1}|x_n|^k < \infty.$$

Hence the following result is deduced by theorem 2.1, which is due to Sarıgöl [9].

Corollary 2.5. Let $1 \leq s < \infty$ and B be triangle matrix of complex numbers. Then, the necessary and sufficient conditions in order that an absolutely convergent series is also summable $|B|_s$ are

$$\sum_{n=v}^{\infty} n^{s-1}\left|\widehat{b}_{nv}\right|^s = O(1).$$

Further, if A and B are the matrix of weighted means (R, p_n) and (R, q_n) then, it is easily seen that $\widehat{a}_{nv} = p_n P_{v-1}/P_n P_{n-1}, 1 \leq v \leq n$, and zero otherwise, $\widehat{a}'_{vv} = P_v/p_v$, $\widehat{a}'_{v,v-1} = -P_{v-2}/p_{v-1}$ and $\widehat{a}'_{n,v} = 0$ for $n \neq v, v+1$, and also, $\widehat{b}_{nv} = q_n Q_{v-1}/Q_n Q_{n-1}, 1 \leq v \leq n$, and zero otherwise. So the following result follows immediately from Theorem 2.2, of which sufficiency for the case $\phi_v = \gamma_v = v$ and $k = s$ is due to Orhan and Sarıgöl [5].

Corollary 2.6. Let $1 = k \leq s < \infty$ and B be triangle matrix of complex numbers. Then, necessary and sufficient conditions in order that a series by summable $|R, p_n|$ is also summable $|R, q_n|_s$ are

$$v^{1-1/s}\left|\frac{P_v q_v}{p_v Q_v}\right| = O(1)$$

and

$$\left|Q_{v-1}\frac{P_v}{p_v} - Q_v\frac{P_{v-1}}{p_v}\right|^s \sum_{n=v+1}^{\infty} n^{s-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^s = O(1).$$

Let A and B be Cesàro matrices (C,α) and (C,β), respectively. In this case, it is well known that $\widehat{a}_{nv}=vA_{n-v}^{\alpha-1}/nA_n^{\alpha}$, $\widehat{b}_{nv}=vA_{n-v}^{\beta-1}/nA_n^{\beta}$, and $\widehat{a}'_{nv}=vA_{n-v}^{-\alpha-1}A_v^{\alpha}/n$. So, (2.1) is equivalent to

$$v^{\alpha-\beta+1/k-1/s}=O\left(1\right),$$

or $\beta\geq\alpha+1/k-1/s$. Also, since (see, Lemma 5, Mehdi [4])

$$\sum_{n=v}^{\infty}\frac{1}{n}\left|\frac{A_{n-r}^{\beta-\alpha-1}}{A_n^{\beta}}\right|^s=\begin{cases}O(v^{-s\beta-1}),\ s\left(\beta-\alpha-1\right)<-1\\ O(v^{-s\beta-1}\log v),\ s\left(\beta-\alpha-1\right)=-1\\ O(v^{-s(\alpha+1)}),\ s\left(\beta-\alpha-1\right)>-1\end{cases}$$

we have

$$\begin{aligned}E_v &= \sum_{n=v}^{\infty}n^{s-1}\left|\sum_{r=v}^{n}\widehat{b}_{nr}\widehat{a}'_{rv}\right|^s=(vA_v^{\alpha})^s\sum_{n=v}^{\infty}n^{s-1}\left|\frac{1}{nA_n^{\beta}}\sum_{r=v}^{n}A_{n-r}^{\beta-1}A_{r-v}^{-\alpha-1}\right|^s\\ &= (vA_v^{\alpha})^s\sum_{n=v}^{\infty}\frac{1}{n}\left|\frac{A_{n-v}^{\beta-\alpha-1}}{A_n^{\beta}}\right|^s=O\left(v^{s-s/k}\right).\end{aligned}$$

In fact, since $\beta\geq\alpha+1/k-1/s$, it is clear that $s\left(\beta-\alpha-1\right)+s+1-s/k\geq 0$. So, it is easy to see from Mehdi's lemma that (2.8) is satisfied, because, E_v is equal to $O(1)v^{-s(\beta-\alpha-1)-1-s+s/k}$, $O(1)v^{-s(\beta-\alpha-1)-1-s+s/k}\log v$ and $O(1)v^{-s+s/k}$ for $s\left(\beta-\alpha-1\right)<-1$, $s\left(\beta-\alpha-1\right)=-1$ and $s\left(\beta-\alpha-1\right)>-1$, respectively. So Theorem 2.1 reduces to the following result of which sufficiency was proved by Flett [2].

Corollary 2.7. Let $1<k\leq s<\infty$, and $\alpha>-1$. Then, necessary conditions in order that a series by summable $\left|C,\alpha\right|_k$ is also summable $\left|C,\beta\right|_s$ are $\beta\geq\alpha+1/k-1/s$.

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