FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 5 (2020), 1393–1398
https://doi.org/10.22190/FUMI2005393N

CESÀRO AND STATISTICAL DERIVATIVE

Fatih Nuray

Abstract. In this study, we introduce the notions of Cesàro, strongly Cesàro and statistical derivatives for real valued functions. These notions are based on the concepts of Cesàro and statistical convergence of a sequence. Then we establish some relationships between strongly Cesàro derivative and statistical derivative.
Keywords: Cesàro derivative; statistical derivative; Cesàro continuity; real valued functions; convergence of a sequence.

1. Introduction

In mathematical analysis, the concepts of limit, continuity and derivative for a function are given respectively. In the literature, the concept of Cesàro limit has been known for many years. Later, Cesàro continuity, statistical limit and statistical continuity concepts were given (see [5]). In [3] strongly sequentially continuous functions were defined and studied. Cesàro derivative and statistical derivative definitions do not appear in the literature. We will introduce the concepts of Cesàro derivative and statistical derivative in this study to fill the gap in the literature.

A sequence $x = (x_k)$ is said to be Cesàro summable to the number u if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = u,$$

in this case we write $(C,1) - \lim x_n = u$, strongly Cesàro summable to the number u if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - u| = 0,$$

Received February 24, 2020; accepted November 19, 2020
2020 *Mathematics Subject Classification.* Primary 40A05; Secondary 40C05, 26A27

in this case we write $[C,1]-\lim x_n = u$, and statistically convergent to the number u if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \quad |x_k - u| \ge \epsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set, in this case we write $st - \lim x_n = u$.

Let (a_n) and (b_n) be two sequences of real numbers such that $(C,1)-\lim a_n = a$ and $(C,1) - \lim b_n = b$. It is known that

$$(C,1) - \lim a_n.b_n = a.b \quad and \quad (C,1) - \lim(a_n + b_n) = a + b.$$

The idea of statistical convergence was introduced by Steinhaus in [13] and Fast in [6] independently and since then has been studied by other authors including [4, 7, 11] and [14]. Recently, the articles [1], [2], [8], [9] and [10] have been published on statistical convergence and its applications.

2. Cesàro Derivative

Very basic finite difference formulas approximates the derivative $f'(x)$ using a sequence $x_n > 0$ such that $\lim_{n\to\infty} x_n = 0$. Two basic formulas for derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x_0 are

$$\lim_{n\to\infty} \tfrac{f(x_0+x_n)-f(x_0)}{x_n} = f'(x_0) \text{ and } \lim_{n\to\infty} \tfrac{f(x_0+x_n)-f(x_0-x_n)}{2x_n} = f'(x_0).$$

The first formula is Newton's difference quotient and determines the slope of a secant line of the graph of f. The second formula is the symmetric difference quotient and determines the slope of a cord of the graph of f. For more detail (see [12]).
With the similar approach we will now define the Cesàro derivative.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(x_0 + x_k) - f(x_0)}{x_k} = w$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.1 as follows:

Definition 2.2. A function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(x_0 + x_k) - f(x_0 - x_k)}{2x_k} = w$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

A function $f : \mathbb{R} \to \mathbb{R}$ is Cesàro continuous at a point x_0 if

$$(C, 1) - \lim f(x_0 + x_n) = f(x_0)$$

holds for each sequence $(x_n) \to 0$.

Theorem 2.1. *Let a function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ then f is Cesàro continuous at the point x_0.*

Proof. Let $\lim x_n = 0$. Clearly

$$f(x_0 + x_n) - f(x_0) = \frac{f(x_0 + x_n) - f(x_0)}{x_n} x_n$$

holds for each $n \in \mathbb{N}$. Since $\lim x_n = 0$ implies $(C, 1) - \lim x_n = 0$, we can write

$$(C, 1) - \lim(f(x_0 + x_n) - f(x_0)) = (C, 1) - \lim \frac{f(x_0 + x_n) - f(x_0)}{x_n}(C, 1) - \lim x_n.$$

Hence, from the assumption we have

$$(C, 1) - \lim f(x_0 + x_n) = f(x_0)$$

so f is Cesàro continuous at the point x_0. □

Definition 2.3. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(x_0 + x_k) - f(x_0)}{x_k} - w \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.3 as follows:

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(x_0 + x_k) - f(x_0 - x_k)}{2x_k} - w \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

It is clear from the definitions of Cesàro and strongly Cesàro derivatives that if a function has a strongly Cesàro derivative at point x_0, it has a Cesàro derivative at that point.

3. Statistical Derivative

In this section, we first give the definition of statistical derivative and then we establish some relationships between the strongly Cesàro derivative and statistical derivative.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \left|\frac{f(x_0+x_k)-f(x_0)}{x_k} - w\right| \ge \epsilon\right\}\right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 3.1 as follows:

Definition 3.2. A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \left|\frac{f(x_0+x_k)-f(x_0-x_k)}{2x_k} - w\right| \ge \epsilon\right\}\right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

If a function has derivative it has statistical derivative but converse may not be true.

Theorem 3.1. *a) If a function $f : \mathbb{R} \to \mathbb{R}$ has strongly Cesàro derivative at a point $x_0 \in \mathbb{R}$ then it has statistical derivative at the point x_0.*

b) If $\left(\frac{f(x_0+x_k)-f(x_0)}{x_k}\right)$ is bounded for each $k \in \mathbb{N}$ and f has statistical derivative at a point $x_0 \in \mathbb{R}$ then f has strongly Cesàro derivative at the point x_0.

Proof. Let's write y_k instead of $\frac{f(x_0+x_k)-f(x_0)}{x_k}$ for simplicity.

a) Let f has strongly Cesàro derivative at a point $x_0 \in \mathbb{R}$. For an arbitrary $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}|y_k - w| &= \left(\frac{1}{n}\sum_{\substack{k=1\\|y_k-w|\ge\epsilon}}^{n}|y_k - w| + \frac{1}{n}\sum_{\substack{k=1\\|y_k-w|<\epsilon}}^{n}|y_k - w|\right)\\
&\ge \frac{1}{n}\sum_{\substack{k=1\\|y_k-w|\ge\epsilon}}^{n}|y_k - w|\\
&\ge \frac{1}{n}|\{1 \le k \le n : |y_k - w| \ge \epsilon\}|\epsilon.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \leq k \leq n : |y_k - w| \geq \epsilon\}| = 0$$

that is, f has a statistical derivative at the point x_0.

b) Now suppose that f has a statistical derivative at the point x_0 and bounded, since $\left(\frac{f(x_0+x_k)-f(x_0)}{x_k}\right)$ is bounded for each $k \in \mathbb{N}$, say $|y_k - w| \leq K$ for all k. Given $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}|y_k - w| &= \frac{1}{n}\left(\sum_{k=1 |y_k-w|\geq\epsilon}^{n}|y_k - w| + \sum_{k=1 |y_k-w|<\epsilon}^{n}|y_k - w|\right) \\
&\leq \frac{1}{n}\left(K\sum_{k=1 |y_k-w|\geq\epsilon}^{n}1 + \sum_{k=1 |y_k-w|<\epsilon}^{n}|y_k - w|\right) \\
&\leq K\frac{1}{n}|\{1 \leq k \leq n : \quad |y_k - w| \geq \epsilon\}| + \frac{1}{n}\sum_{k=1}^{n}\epsilon
\end{aligned}$$

hence we have,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}|y_k - w| = 0,$$

that is f has strongly Cesàro derivative at the point x_0. □

REFERENCES

1. T. Acar and S. A. Mohiuddine: *Statistical (C, 1)(E, 1) summability and Korovkin's theorem.* Filomat **30**(2) (2016) 387–393.
2. C. Belen, and S. A. Mohiuddine: *Generalized weighted statistical convergence and application.* Appl. Math. Comput. 219 (2013) 9821–9826.
3. A. Borichev, R. Deville and E. Matheron: *Strongly sequentially continuous functions.* Quaestiones Mathematicae, **24** (2001), 535–548.
4. J. S. Connor: *The statistical and strong p-Cesáro of sequences.* Analysis, **8**(1988), 47–63.
5. J. Connor, and K.G. Grosse-Erdmann: *Sequential definitions of continuity for real functions.* Rocky Mountain J. Math., bf 33, 1, (2003), 93–121.
6. H. Fast: *Sur la convergence statistique* Cooloq. Math. **2**(1951), 241–244.
7. J. A. Fridy: *On statistical convergence* Analysis, **5**(1985), 301–313.
8. U. Kadak and S. A. Mohiuddine: *Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems.* Results Math. **73**(1) (2018), Article 9.

9. S. A. Mohiuddine: *Statistical weighted A-summability with application to Korovkin's type approximation theorem*. J. Inequal. Appl. (2016) 2016:101.
10. S. A. Mohiuddine and B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM **113**(3) (2019) 1955–1973.
11. T. Šalát: *On statistically convergent sequences of real numbers*. Math. Slovaca, **30**(1980), 139–150.
12. S. Pedersen and J. P. Sjoberg: *Sequential Derivatives*, arXiv: 1801.04039v1 [math. CA] 12 Jan 2018.
13. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math., **2**(1951), 73–74.
14. I. J. Schoenberg: *The integrability of certain functions and related summability methods*. Amer. Math. Monthly, **66** (1959), 361–375.

Fatih Nuray
Faculty of Science
Department of Mathematics
Afyon Kocatepe University
Afyonkarahisar, Turkey
fnuray@aku.edu.tr