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NEW TYPE OF ALMOST CONVERGENCE

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Abstract. In [1] for a given sequence (λ_n) with $\lambda_n < \lambda_{n+1} \to \infty$ a new summability method C_λ was introduced which generalizes the classical Cesàro method. In this paper, we introduce some new almost convergence and almost statistical convergence definitions for sequences which generalize the classical almost convergence and almost statistical convergence.
Key words: sequence convergence, almost convergence, summability theory.

1. Introduction

Let (λ_n) be a given real valued sequence such that

$$0 \leq \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$$

and $[\lambda_n]$ denote the integer part of λ_n. The set of such sequences will be denoted Λ. Consider the mean

$$\sigma_n = \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} x_k, \quad n = 1, 2, 3, ...$$

of a given sequence (x_k) of real or complex numbers. If

$$\lim_{n\to\infty} \sigma_n = \ell,$$

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then we say that (x_k) is C_λ-summable to ℓ. In the particular case when $\lambda_n = n$ we see that σ_n is the $(C,1)$ mean of (x_k).Therefore, C_λ-method yields a submethod of the Cesàro method $(C,1)$, and hence it is regular for any λ. C_λ-matrix is obtained by deleting a set of rows from Cesàro matrix. $(C,1)$ and C_λ are equivalent for bounded sequences if and only if $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$. The basic properties of C_λ-method can be found in ([1],[24]).

Summability of matrix submethods was studied in [12] and [28]. The authors of [12] and [28] presented results showing when C_λ is equivalent to the Cesàro method C_1 for bounded sequences. Armitage and Maddox proved inclusion and Tauberian results for the C_λ method in [1]. In [24], inclusion properties of the C_λ method for bounded sequences and its relationship to statistical convergence are studied also a condensation test presented for statistical convergence.

In this study, firstly we will introduce $\hat{C}_\lambda$-almost convergent sequence and prove some inclusion relations. Later we will give definition of $\hat{C}_\lambda$- almost statistically convergent sequence and examine the relationship between $\hat{C}_\lambda$-almost convergence and $\hat{C}_\lambda$-almost statistically convergence. Finally, we will generalize the spaces $[C_\lambda]$ and $[\hat{C}_\lambda]$ to spaces $[C_\lambda(f)]$ and $[\hat{C}_\lambda(f)]$ by using a modulus function f. Thus, it will fill a gap in the literature.

2. Almost Convergence

Let ℓ_∞ be the Banach space of real valued bounded sequences (x_k) with the usual norm $||x|| := sup_k|x_k|$. There exists continuous linear functional $\phi : \ell_\infty \to \mathbb{R}$ called Banach limit if the following conditions hold:

$$\begin{array}{rl}
(i) & \phi(ax_k + by_k) = a\phi(x_k) + b\phi(y_k), \quad a, b \in \mathbb{R} \\
(ii) & \phi(x_k) \geq 0 \quad \text{if} \quad x_k \geq 0, \quad k = 1, 2, 3.... \\
(iii) & \phi(Sx) = \phi(x), \quad Sx = (x_2, x_3, x_4, ...) \\
(iv) & \phi(e) = 1 \quad \text{where} \quad e = (1, 1, 1, ...).
\end{array}$$

A sequence (x_k) in ℓ_∞ is said to be almost convergent if all of its Banach limits are equal. It is well known that any Banach limit of (x_k) lies between $\liminf x_k$ and $\limsup x_k$ [13].

Note that a convergent sequence is almost convergent, and its limit and its generalized limit are identical, but an almost convergent sequence need not be convergent. The sequence (x_k) defined as

$$x_k = \begin{cases} 1, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

is almost convergent to $1/2$ but not convergent.

Lorentz [13] gave the following characterization for almost convergence: A sequence (x_n) is said to be almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} x_{k+i} = \ell$$

uniformly in i.

Maddox[15] has defined strongly almost convergent sequence as follows:
A bounded sequence (x_k) is said to be strongly almost convergent to ℓ if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}|x_{k+i}-\ell|=0$$

uniformly in i.
Readers can refer to recently published articles ([2],[3],[14],[18],[19],[20],[22]) for more information.
Consider the mean

$$\hat{C}_\lambda x=\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}x_{k+i}$$

of a given sequence (x_k) of real numbers and $i=1,2,3,....$

Definition 2.1. A bounded sequence (x_k) is said to be $\hat{C}_\lambda$-almost convergent to ℓ if and only if .

$$\lim_{n\to\infty}\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}x_{k+i}=\ell$$

uniformly in i.

In this case we write $x_k\to\ell(\hat{C}_\lambda)$. In the particular case when $\lambda_n=n$ we get the definition of almost convergent sequence.

Theorem 2.1. *Let $\{\lambda_n\},\{\nu_n\}\in\Lambda$. If $\lim_{n\to\infty}\frac{\nu_n}{\lambda_n}=1$, then $\hat{C}_\lambda$- almost convergence is equivalent to $\hat{C}_\nu$- almost convergence on ℓ_∞.*

Proof. Let $x\in\ell_\infty$ and consider $M_n:=\max\{\lambda_n,\nu_n\}$ and $m_n:=\min\{\lambda_n,\nu_n\}$. Since $\lim_{n\to\infty}\frac{\nu_n}{\lambda_n}=1$, we can write $\lim_{n\to\infty}\frac{m_n}{M_n}=1$, then for each n and i

$$\begin{aligned}
|\hat{C}_\nu x-\hat{C}_\lambda x| &= |\frac{1}{\nu_n}\sum_{k=1}^{[\nu_n]}x_{k+i}-\frac{1}{\lambda_n}\sum_{k=1}^{[\lambda_n]}x_{k+i}| \\
&= |\frac{1}{M_n}\sum_{k=1}^{M_n}x_{k+i}-\frac{1}{m_n}\sum_{k=1}^{m_n}x_{k+i}| \\
&= |\sum_{k=1}^{m_n}(\frac{1}{M_n}-\frac{1}{m_n})x_{k+i}+\frac{1}{M_n}\sum_{k=m_n+1}^{M_n}x_{k+i}| \\
&\le \sup_{k,i}|x_{k+i}|\sum_{k=1}^{m_n}\frac{M_n-m_n}{M_nm_n}+\sup_{k,i}|x_{k+i}|\frac{M_n-m_n}{M_n} \\
&= \sup_{k,i}|x_{k+i}|\frac{m_n(M_n-m_n)}{M_nm_n}+\sup_{k,i}|x_{k+i}|\frac{M_n-m_n}{M_n} \\
&= 2\sup_{k,i}|x_{k+i}|(1-\frac{m_n}{M_n})\to 0
\end{aligned}$$

as $n \to \infty$ uniformly in i. Hence, if $x \to L(\hat{C}_\lambda)$,

$$0 \leq |\frac{1}{\nu_n}\sum_{k=1}^{\nu_n} x_{k+i} - L| \leq |\hat{C}_\nu x - \hat{C}_\lambda x| + |\hat{C}_\lambda x - L| \to 0$$

as $n \to \infty$ uniformly in i. Similarly, if $x \to L(\hat{C}_\nu)$,

$$0 \leq |\frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n} x_{k+i} - L| \leq |\hat{C}_\lambda x - \hat{C}_{\nu\lambda} x| + |\hat{C}_\nu x - L| \to 0$$

as $n \to \infty$ uniformly in i. Thus, the proof is completed. □

By using similar techniques to Theorem 1 of [1] we can prove following theorem:

Theorem 2.2. *Let* $\{\lambda_n\}, \{\nu_n\} \in \Lambda$.
(i) $\hat{C}_\lambda$ *implies* $\hat{C}_\mu$ *if and only if* $D(\mu) \setminus D(\lambda)$ *is a finite set, where*

$$D(\lambda) = \{[\lambda_n] : \quad n = 1, 2, ...\}.$$

(ii) $\hat{C}_\mu$ *is equivalent* $\hat{C}_\mu$ *if and only if* $D(\lambda) \triangle D(\mu)$ *is a finite set.*

Also by using similar techniques to Theorem 2.2 of [24] we can prove following theorem:

Theorem 2.3. *Let* $\{\lambda_n\} \in \Lambda$. *If* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, *then* $\hat{C}_\lambda$- *almost convergence is equivalent to almost convergence on* ℓ_∞.

Definition 2.2. A bounded sequence (x_k) is said to be strongly $\hat{C}_\lambda$-almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} |x_{k+i} - \ell| = 0$$

uniformly in i.

In this case we write $x_k \to \ell([\hat{C}_\lambda])$. In the particular case when $\lambda_n = n$ we get the definition of strongly almost convergent sequence.

Definition 2.3. A bounded sequence (x_k) is said to be p-strongly $\hat{C}_\lambda$-almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} |x_{k+i} - \ell|^p = 0$$

uniformly in i where $0 < p < \infty$.

In this case we write $x_k \to \ell([\hat{C}_\lambda]_p)$. In the particular case when $\lambda_n = n$ we get the strongly p-almost convergent sequence definition.

3. Almost Statistical Convergence

The natural density of a set A of positive integers is defined if limit exists by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in A\}|,$$

where $|k \leq n : k \in A|$ denotes the number of elements of A not exceeding n.

Statistical convergence, as it has recently been investigated, was defined by Fast [7]. Schoenberg [27] established some fundamental properties of the concept and studied as a summability method. The more recent times interest in statistical convergence arose after Fridy published his paper [8], and since then there have been many generalizations of the original concept(see [4]-[6],[9]-[11],[16],[21]).
A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0,$$

holds. In this case, we write $st - \lim x_k = \ell$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \ell$, then $st - \lim x_k = \ell$. The converse does not hold, in general. If a sequence $x = (x_k)$ is strongly Cesàro convergent to ℓ, then $x = (x_k)$ is statistically convergent to ℓ and the converse is also true when $x = (x_k)$ is a bounded sequence.

Definition 3.1. [24] A sequence $x = (x_k)$ is said to be C_λ-statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}|\{0 \leq k \leq [\lambda_n] : |x_k - \ell| \geq \epsilon\}| = 0,$$

holds.

In the particular case when $\lambda_n = n$, C_λ- statistically convergence coincide with statistically convergence.

Definition 3.2. A sequence $x = (x_k)$ is said to be C_λ- almost statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}|\{0 \leq k \leq [\lambda_n] : |x_{k+i} - \ell| \geq \epsilon\}| = 0$$

holds uniformly in i.

In the particular case when $\lambda_n = n$ we get the definition of almost statistically convergent sequences was defined in [26].

Theorem 3.1. *If $x_k \to \ell([C_\lambda])$ then $x_k \to \ell(S_\lambda)$. The converse is true if (x_k) is bounded.*

Proof. Let $x_k \to \ell([C_\lambda])$. For an arbitrary $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| &= \left(\frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell| + \frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}|x_k-\ell|\right)\\
&\geq \frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell|\\
&\geq \frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}|\epsilon.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}| = 0$$

that is, $x_k \to \ell(S_\lambda)$.
Now suppose that $x_k \to \ell(S_\lambda)$ and x_k is bounded, since x_k is bounded, say $|x_k-\ell| \leq M$ for all k. Given $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| &= \frac{1}{1+\lambda_n}\left(\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell| + \sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}|x_k-\ell|\right)\\
&\leq \frac{1}{1+\lambda_n}\left(M\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}1 + \epsilon\sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}1\right)\\
&\leq M\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}|\\
&+ \epsilon\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|<\epsilon\}|
\end{aligned}$$

hence we have,

$$\lim_{x\to\infty}\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| = 0.$$

□

The proofs of the following theorems are similar to that of Theorem 3.1, so we state them without of proof.

Theorem 3.2. *Let $0 < p < \infty$. If $x_k \to \ell([C_\lambda]_p)$ then $x_k \to \ell(S_\lambda)$. The converse is true if (x_k) is bounded.*

Theorem 3.3. *Let $0 < p < \infty$. If $x_k \to \ell([\hat{C}_\lambda]_p)$ then $x_k \to \ell(\hat{S}_\lambda)$. The converse is true if (x_k) is bounded.*

Theorem 3.4. *Let $(\lambda_n) \in \Lambda$ with $\limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} < \infty$. If $x_k \to \ell(\hat{S}_\lambda)$ then $x_k \to \ell(\hat{S})$.*

Proof. Assume (x_k) is $x_k \to \ell(\hat{S}_\lambda)$ and $\limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} < \infty$. Consider $\Gamma = \mathbb{N}\backslash\Lambda := \{\nu_n\}$. If Γ is finite , then $\hat{S}$ is equivalent to $\hat{S}_\lambda$. Now assume that Γ is infinite. Then there exists an K such that $n \geq K, \nu_n > \lambda_1$. Since Γ and Λ are disjoint, for $n \geq K$, there exists an integer m such that $\lambda_m < \nu_n < \lambda_{m+1}$. We write $\nu_n = \lambda_{m+j}$ where $0 < j < \lambda_{m+1} - \lambda_m$. Then, for $n \geq K$,

$$\begin{aligned}
& \frac{1}{\nu_n}|\{k \leq \nu_n : |x_{k+i} - \ell| \geq \epsilon\}| \\
= \quad & \frac{1}{\lambda_{m+j}}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{1}{\lambda_{m+j}}|\{\lambda_{m+1} \leq k \leq \lambda_{m+j} : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
\leq \quad & \frac{1}{\lambda_m}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{1}{\lambda_{m+j}}|\{1 \leq k \leq \lambda_{m+1} : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
= \quad & \frac{1}{\lambda_m}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{\lambda_{m+1}}{\lambda_{m+j}}\frac{1}{\lambda_{m+1}}|\{1 \leq k \leq \lambda_{m+1} : \quad |x_{k+i} - \ell| \geq \epsilon\}|.
\end{aligned}$$

Since, $0 < \frac{\lambda_{m+1}}{\lambda_{m+j}} < \frac{\lambda_{m+1}}{\lambda_m}$ and $\frac{\lambda_{m+1}}{\lambda_m}$ is bounded, then $\frac{\lambda_{m+1}}{\lambda_{m+j}}$ is bounded too. Thus, we see that $\frac{1}{n}|\{1 \leq k \leq n : \quad |x_{k+i} - \ell| \geq \epsilon\}|$ may be partitioned into two disjoint subsequences each having the common limit zero uniformly in i. Hence, we get $x_k \to \ell(\hat{S})$. $\square$

4. Convergence with respect to a modulus function

The notion of a modulus function was introduced by [23]. Ruckle [25] used the idea of a modulus function to construct the sequence space

$$L(f) = \{(x_k) : \quad \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK-space, and Ruckle proved that the intersection of all such $L(f)$ space is ϕ, the space of finite sequences, thereby answering negatively a question of A. Wilansky: "Is there a smallest FK-space in which the set $\{e_1, e_2, ...\}$ of unit vectors is bounded?" [17].

A real valued function f defined on $[0,\infty)$ is called a modulus function if it has following properties:

1. $f(x) \geq 0$ for each x,
2. $f(x) = 0$ if and only if $x = 0$,
3. $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
4. f is increasing,
5. $\lim_{x\to 0^+} f(x) = 0$.

Since $|f(x) - f(y)| \leq f(x-y)$,(see [17]), it follows from conditions (3) and (5) that f is continuous on $[0,\infty)$.

Many new sequence spaces are defined by using the modulus function in the summability theory. Sequence spaces defined in this way generalize known sequence spaces. By using a modulus function f firstly Ruckle [25] defined the sequence space

$$L(f) = \{(x_k) : \quad \sum_{k=1}^{\infty} f(|x_k|) < \infty\},$$

which generalization of the space

$$\ell_1 = \{(x_k) : \quad \sum_{k=1}^{\infty} |x_k| < \infty\}$$

and later Maddox [17] introduced following sequence spaces which are generalizations of the classical spaces of strongly summable sequences

$$w_0(f) = \{(x_k) : \quad \frac{1}{n}\sum_{k=1}^{n} f(|x_k|) = 0\},$$

$$w(f) = \{(x_k) : \quad \frac{1}{n}\sum_{k=1}^{n} f(|x_k - \ell|) = 0 \quad \text{for real number} \quad \ell\},$$

$$w_\infty(f) = \{(x_k) : \quad \sup_n \frac{1}{n}\sum_{k=1}^{n} f(|x_k|) < \infty\}.$$

If we take $f(x) = x^p$ $(0 < p < 1)$ then the space $L(f)$ is the familiar space l_p.
It is known that,

$$\ell_1 \subset L(f), \quad w_0 \subset w_0(f), \quad w \subset w(f), \quad \text{and} \quad w_\infty \subset w_\infty(f).$$

Apart from these spaces, there are many sequence spaces defined using the modulus function in the literature. For example, Connor [5] introduced strongly A-summable sequences with respect to a modulus function.

In this section, by using a modulus function f, we will introduce the sequence spaces $[C_\lambda(f)]$ and $[\widehat{C}_\lambda(f)]$ which are generalization of the sequence spaces $[C_\lambda]$ and $[\widehat{C}_\lambda]$ and we are going to show that

$$[C_\lambda] \subset [C_\lambda(f)] \quad \text{and} \quad [\widehat{C}_\lambda] \subset [\widehat{C}_\lambda(f)]$$

holds.

Definition 4.1. Let (x_k) be a sequence of real or complex numbers, f be a modulus function and $(\lambda_n) \in \Lambda$ be a sequence. If

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_k - \ell|) = 0,$$

then we say that (x_k) is $[C_\lambda]$-summable to ℓ with respect to f and $\lambda = (\lambda_n)$.

The space of all sequences $[C_\lambda]$-summable to ℓ with respect to f and $\lambda = (\lambda_n)$ will be denoted by $[C_\lambda(f)]$.

Theorem 4.1. *For any modulus function f we have $[C_\lambda] \subset [C_\lambda(f)]$ holds for $\lambda = (\lambda_n) \in \Lambda$, that is, (x_k) is $[C_\lambda]$-summable to ℓ then (x_k) is $[C_\lambda]$-summable to ℓ with respect to the modulus function f.*

Proof. If (x_k) is $[C_\lambda]$-summable to ℓ, then we have

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} |x_k - \ell| = 0.$$

Let $\epsilon > 0$ and choose θ with $0 < \theta < 1$ such that $f(t) < \epsilon$ holds for $0 \le t \le \theta$. Now since for $|x_k - \ell| > \theta$,

$$|x_k - \ell| \le \frac{|x_k - \ell|}{\theta} < 1 + [\frac{|x_k - \ell|}{\theta}]$$

and

$$f(|x_k - \ell|) \le (1 + [\frac{|x_k - \ell|}{\theta}])f(1) < 2f(1)\frac{|x_k - \ell|}{\theta},$$

we can write

$$\begin{aligned} \sum_{k=0}^{[\lambda_n]} f(|x_k - \ell|) &= \sum_{\substack{k=0 \\ |x_k-\ell|\le\theta}}^{[\lambda_n]} f(|x_k - \ell|) + \sum_{\substack{k=0 \\ |x_k-\ell|>\theta}}^{[\lambda_n]} f(|x_k - \ell|) \\ &\le \epsilon[\lambda_n + 1] + \frac{2}{\theta} f(1)([\lambda_n] + 1)\frac{1}{[\lambda_n]+1} \sum_{k=0}^{[\lambda_n]} |x_k - \ell|. \end{aligned}$$

Hence, (x_k) is $[C_\lambda]$-summable to ℓ with respect to the modulus function f. $\square$

Definition 4.2. Let (x_k) be a sequence of real or complex numbers, f be a modulus function and $\lambda = (\lambda_n) \in \Lambda$. If

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) = 0$$

holds uniformly in i, then we say that (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f.

The space of sequences $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f will be denoted by $[\widehat{C}_\lambda(f)]$.

Theorem 4.2. *If $(x_k) \to \ell[\widehat{C}_\lambda(f)]$ and $(x_k) \to \ell'[\widehat{C}_\lambda(f)]$ then $\ell = \ell'$.*

Proof. Let $(x_k) \to \ell[\widehat{C}_\lambda(f)]$ and $(x_k) \to \ell'[\widehat{C}_\lambda(f)]$. Then given $\epsilon > 0$, for all $i \in \mathbb{N}$ there exists $n > n_0$ such that

$$\frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) < \frac{\epsilon}{2}$$

and

$$\frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell'|) < \frac{\epsilon}{2}.$$

From these and the following inequality

$$f(|\ell - \ell'|) = f(|x_{k+i} + \ell - \ell' - x_{k+i}|) \leq f(|x_{k+i} - \ell|) + f(|x_{k+i} - \ell'|)$$

we can write

$$\begin{aligned} f(|\ell - \ell'|) &\leq \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) + \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell'|) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $i \in \mathbb{N}$. Since $\epsilon > 0$ is arbitrary, we have $\ell = \ell'$ by the properties (2) and (5) of modulus function. $\square$

Theorem 4.3. *For any modulus function f we have $[\widehat{C}_\lambda] \subset [\widehat{C}_\lambda(f)]$, that is, (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ then (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f.*

The proof of the theorem similar to the Theorem 4.1, so we omit it.

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