

TRIANGULAR A -STATISTICAL RELATIVE UNIFORM CONVERGENCE FOR DOUBLE SEQUENCES OF POSITIVE LINEAR OPERATORS

Selin Çınar

Faculty of Science and Arts, Department of Mathematics,
Sinop University, 57000 Sinop, Turkey

Abstract. In this paper, we introduce the concept of triangular A -statistical relative convergence for double sequences of functions defined on a compact subset of the real two-dimensional space. Based upon this new convergence method, we prove Korovkin-type approximation theorem. Finally, we give some further developments.

Keywords: positive linear operators, the double sequences, regular matrix, triangular A -statistical convergence, Korovkin theorem.

1. Introduction

Classical Bohman-Korovkin theorem is a well known theorem which has an important place in approximation theory ([13], [16], [21]). This theorem establishes the uniform convergence in the space $C[a, b]$ of all continuous real functions defined on the interval $[a, b]$, for a sequence of positive linear operators (L_n) , assuming the convergence by the test functions $f_r(s) = s^r$, $r = 0, 1, 2$. Moreover, different finite classes of test functions were studied, in both one and multi-dimensional case. Many mathematicians studied and improved this theory by defining positive linear operators via convergence methods on various function spaces ([1], [4], [5], [6], [15], [18], [20], [23], [24], [25], [26], [34]). In recent years, general versions of Korovkin theorem have been studied, in which a more general notion of convergence is used. One of these convergences is the statistical convergence first introduced by Fast and Steinhaus ([17], [30]). Korovkin type approximation theorems have been first studied via the notion of statistical convergence by Gadjiev and Orhan [19]. For

Received March 9, 2020; accepted May 30, 2020.

Corresponding Author: Selin Çınar, Faculty of Science and Arts, Department of Mathematics, Sinop University, 57000 Sinop, Turkey | E-mail: scinar@sinop.edu.tr
2010 *Mathematics Subject Classification.* 40A35; 41A25, 41A36

© 2021 by University of Niš, Serbia | Creative Commons License: CC BY-NC-ND

double sequences of positive linear operators, statistical convergence and some of its generalizations to convergence generated by summability matrix methods were carried on by Demirci and Dirik ([8], [14]). With the help of these studies, triangular A -statistical convergence which is a different kind of statistical convergence was identified by Bardaro et. al. ([2], [3]).

Recently, Demirci and Orhan [11] have defined the statistically relatively uniform convergence by using statistical convergence and the relatively uniform convergence and established its use in the Korovkin-type approximation theory. Also, a type of modular convergence, called relative modular convergence, was introduced in [33] originated by studies in modular spaces and these studies continued ([9], [10], [12]).

Our main aim in this paper is to present a new kind of statistical convergence for double sequence, called triangular A -statistical relative uniform convergence. We will compare this new convergence with triangular A -statistical convergence and obtain more general results.

Now, we begin with the definitions and notations required for this article.

E. H. Moore [22] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, E. W. Chittenden [7] gave the following definition of relatively uniform converge which is equivalent to the definition given by Moore:

A sequence (f_n) of functions, defined on any compact subset of real space, converges relatively uniformly to a limit function f if there exists a function $\sigma(s)$, such that for every $\varepsilon > 0$ there exists an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$|f_n(s) - f(s)| < \varepsilon |\sigma(s)|$$

holds uniformly in s . The sequence (f_n) is said to converge uniformly relatively to the scale function σ or more briefly relatively uniformly. Similarly, Dirik and Şahin [31] gave the following for double sequences of functions:

A double sequence $(f_{i,j})$ of functions, defined on any compact subset of the real two-dimensional space, converges relatively uniformly to a limit function f if there exists a function $\sigma(s, t)$, called a scale function such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $i, j > n_\varepsilon$ the inequality

$$|f_{i,j}(s, t) - f(s, t)| < \varepsilon |\sigma(s, t)|$$

holds uniformly in (s, t) . The double sequence $(f_{i,j})$ is said to converge uniformly relatively to scale function σ or more briefly, relatively uniformly.

Let $A = (a_{i,j})$ be a two-dimensional matrix transformation. For a double sequence $x = (x_{i,j})$ of real numbers, we put

$$(Ax)_i := \sum_{j=1}^{\infty} a_{i,j} x_{i,j},$$

if the series is convergent. We will say that A is regular if $\lim Ax = L$ whenever $\lim x = L$. The well-established characterization for regular two-dimensional matrix transformation is known as the Silverman-Toeplitz conditions [32]:

$$(i) \|A\| = \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$$

$$(ii) \lim_i a_{i,j} = 0 \text{ for each } j \in \mathbb{N},$$

$$(iii) \lim_i \sum_{j=1}^{\infty} a_{i,j} = 1.$$

A double sequence $x = (x_{i,j})$ of real numbers, $i, j \in \mathbb{N}$, the set of all positive integers, is said to be convergent in the Pringsheim's sense or P -convergent if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{i,j} - L| < \varepsilon$ whenever $i, j > N$ and L is called the Pringsheim limit (denoted by $P\text{-}\lim_{i,j} x_{i,j} = L$) [28]. More briefly, we will say that such an x is P -convergent to L . A double sequence is said to be bounded if there exists a positive number K such that $|x_{i,j}| \leq K$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequences need not to be bounded, provided the double sequences converges in Pringsheim's sense for every $(i, j) \in \mathbb{N}^2$.

Let now $A = (a_{n,m,i,j})$ be a four-dimensional matrix and $x = (x_{i,j})$ be a double sequence. Then the double (transformed) sequence, $Ax := ((Ax)_{n,m})$, is denoted by

$$(Ax)_{n,m} = \sum_{i,j=1,1}^{\infty, \infty} a_{n,m,i,j} x_{i,j},$$

where it is assumed that the summation exists as a Pringsheim limit for each $(n, m) \in \mathbb{N}^2$.

Recall that four-dimensional matrix $A = (a_{n,m,i,j})$ is said to be RH -regular if it maps every bounded P -convergent sequence into a P -convergent sequence with the same P -limit. The Robison-Hamilton conditions (see also [29]) state that a four-dimensional matrix $A = (a_{n,m,i,j})$ is RH -regular if and only if

$$(i) P\text{-}\lim_{n,m} a_{n,m,i,j} = 0 \text{ for each } i \text{ and } j,$$

$$(ii) P\text{-}\lim_{n,m} \sum_{i,j}^{\infty, \infty} a_{n,m,i,j} = 1,$$

$$(iii) P\text{-}\lim_{n,m} \sum_{i=1}^{\infty} |a_{n,m,i,j}| = 0 \text{ for each } j \in \mathbb{N},$$

$$(iv) P\text{-}\lim_{n,m} \sum_{j=1}^{\infty} |a_{n,m,i,j}| = 0 \text{ for each } i \in \mathbb{N},$$

$$(v) \sum_{i,j=1,1}^{\infty, \infty} |a_{n,m,i,j}| \text{ is } P\text{-convergent for every } (n, m) \in \mathbb{N}^2,$$

(vi) there exist finite positive integers A and B such that

$$\sum_{i,j>B} |a_{n,m,i,j}| < A$$

for every $(n, m) \in \mathbb{N}^2$.

Let $A = (a_{n,m,i,j})$ be a nonnegative RH -regular summability matrix. If $K \subset \mathbb{N}^2$, then the A -density of K is denoted by

$$\delta_A^2(K) := P - \lim_{n,m} \sum_{(i,j) \in K} a_{n,m,i,j}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{i,j})$ is said to be A -statistically convergent to L and denoted by $st_A^2 \lim_{i,j} x_{i,j} = L$ if, for every $\varepsilon > 0$,

$$P - \lim_{n,m} \sum_{(i,j) \in K(\varepsilon)} a_{n,m,i,j} = 0,$$

where $K(\varepsilon) = \{(i, j) \in \mathbb{N}^2 : |x_{i,j} - L| \geq \varepsilon\}$. If we take $A = C(1, 1)$, then $C(1, 1)$ -statistical convergence coincides with the notion of statistical convergence for double sequences ([27]), where $C(1, 1) = (c_{i,j,n,m})$ is the double Cesàro matrix, defined by $c_{i,j,n,m} = 1/ij$ if $1 \leq n \leq i$, $1 \leq m \leq j$ and $c_{i,j,n,m} = 0$ otherwise. We state the set of all A -statistically convergent double sequences by st_A^2 .

2. Triangular A -Statistical Relative Uniform Convergence

First, we recall some definitions given in [2].

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix, $K \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $K_i := \{j \in \mathbb{N} : (i, j) \in K, j \leq i\}$. Triangular A -density of K , is given by

$$\delta_A^T(K) := \lim_i \sum_{j \in K_i} a_{i,j},$$

provided that the limit on right-hand side exists in \mathbb{R} .

In a similar manner to the natural density, we can give some properties for the triangular A -density:

- i) if $K_1 \subset K_2$, then $\delta_A^T(K_1) \subset \delta_A^T(K_2)$,
- ii) if K has triangular A -density, then $\delta_A^T(\mathbb{N}^2 \setminus K) = 1 - \delta_A^T(K)$.

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. The double sequence $x = (x_{i,j})$ is triangular A -statistically convergent to L provided that for every $\varepsilon > 0$

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} : j \leq i, |x_{i,j} - L| \geq \varepsilon\}$ and this denoted by $st_A^T - \lim_i x_{i,j} = L$. We should note that if we take $A = C_1 := (c_{i,j})$, the Cesàro matrix defined by

$$c_{i,j} := \begin{cases} \frac{1}{i}, & \text{if } 1 \leq j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

then the triangular A -density is called triangular density which is denoted by

$$\delta^T(K) = \lim_i \frac{1}{i} |K_i|$$

where $|K_i|$ be the cardinality of K_i . According to the above definitions triangular A -statistical convergent reduces triangular statistical convergent.

Let S is a compact subset of the real two-dimensional space. By $C(S)$ we define the space of all continuous real valued functions on S and $\|f\|_{C(S)}$ denotes the usual supremum norm of f in $C(S)$. Let f and $f_{i,j}$ belong to $C(S)$.

Definition 2.1. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. A double sequence of fuctions $(f_{i,j})$ is said to triangular A -statistically uniformly convergent to f on S provided that for every $\varepsilon > 0$,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \left\{ j \in \mathbb{N} : j \leq i, \sup_{(s,t) \in S} |f_{i,j}(s,t) - f(s,t)| \geq \varepsilon \right\}$. In this case, we write $f_{i,j} \Rightarrow f (st_A^T)$.

Definition 2.2. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. $(f_{i,j})$ is said to be triangular A -statistically relatively uniformly convergent to f on S if there exists a function $\sigma(s,t)$, $|\sigma(s,t)| > 0$, called a scale function, provided that for every $\varepsilon > 0$,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \left\{ j \in \mathbb{N} : j \leq i, \sup_{(s,t) \in S} \left| \frac{f_{i,j}(s,t) - f(s,t)}{\sigma(s,t)} \right| \geq \varepsilon \right\}$. In this case, we write $f_{i,j} \Rightarrow f (st_A^T, \sigma)$.

It will be observed that triangular A -statistical uniform convergence is the special case of triangular A -statistical relative uniform convergence in which the scale function is a non-zero constant.

Example 2.1. Take $A = C_1$ and $S = [0, 1] \times [0, 1]$. For each $(i, j) \in \mathbb{N}^2$, define $\gamma_{i,j} : S \rightarrow \mathbb{R}$ by

$$\gamma_{i,j}(s,t) = \begin{cases} \frac{2i^2 j^2 st}{1+i^3 j^3 s^2 t^2}, & \text{if } i \text{ and } j \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\|\gamma_{i,j} - \gamma\|_{C(S)} = 1$, this sequence does not triangular statistically uniform convergent to $\gamma = 0$, but triangular statistically relatively uniform convergent to $f = 0$, with a scale function defined by,

$$\sigma(s, t) = \begin{cases} \frac{1}{st}, & \text{if } (s, t) \in (0, 1] \times (0, 1] \\ 0, & \text{if } s = 0 \text{ or } t = 0 \end{cases}$$

clearly, for every $\varepsilon > 0$,

$$\lim_i \frac{1}{i} \left| \left\{ j \in \mathbb{N} : j \leq i, \sup_{(s,t) \in S} \left| \frac{\gamma_{i,j}(s, t) - \gamma(s, t)}{\sigma(s, t)} \right| \geq \varepsilon \right\} \right| = 0.$$

3. A Korovkin-type approximation theorem

Let L be a linear operator from $C(S)$ into itself and is called positive, if $L(f) \geq 0$, for all $f \geq 0$. Also, we denote the value of $L(f)$ at a point $(s, t) \in S$ by $L(f(u, v); s, t)$ or, briefly, $L(f; s, t)$.

Theorem 3.1. [2] *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix and $(L_{i,j})$ be a double sequence of positive linear operators from $C(S)$ into $C(S)$. Then for every $f \in C(S)$ we have*

$$(3.1) \quad st_A^T - \lim_i \|L_{i,j}(f) - f\|_{C(S)} = 0$$

if and only if

$$(3.2) \quad st_A^T - \lim_i \|L_{i,j}(f_r) - f_r\|_{C(S)} = 0 \text{ for every } r = 0, 1, 2, 3,$$

where $f_0(s, t) = 1$, $f_1(s, t) = s$, $f_2(s, t) = t$, $f_3(s, t) = s^2 + t^2$.

Now we have the following Korovkin type approximation theorem for triangular A -statistical relative convergence that is our main theorem.

Theorem 3.2. *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Let $(L_{i,j})$ be a double sequence of positive linear operators from $C(S)$ into $C(S)$. Then, for all $f \in C(S)$ we have*

$$(3.3) \quad L_{i,j}(f) \Rightarrow f \quad (st_A^T, \sigma)$$

if and only if,

$$(3.4) \quad L_{i,j}(f_r) \Rightarrow f_r \quad (st_A^T, \sigma_r) \quad (r = 0, 1, 2, 3),$$

where $f_0(s, t) = 1$, $f_1(s, t) = s$, $f_2(s, t) = t$, $f_3(s, t) = s^2 + t^2$ and $\sigma_r(s, t) = \max\{|\sigma_r(s, t)|; r = 0, 1, 2, 3\}$, $|\sigma_r(s, t)| > 0$ and $\sigma_r(s, t)$ is unbounded, $r = 0, 1, 2, 3$.

Proof. Since each $f_r \in C(S)$ ($r = 0, 1, 2, 3$), (3.3) \implies (3.4) is obvious. Suppose now that (3.4) holds. By continuity of f on the compact set S , we can write $|f(s, t)| \leq M$ where $M := \|f\|_{C(S)}$. Also, since f is continuous on S , for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u, v) - f(s, t)| < \varepsilon$ for all $(u, v) \in S$ satisfying $|u - s| < \delta$ and $|v - t| < \delta$. Hence, we get

$$(3.5) \quad |f(u, v) - f(s, t)| < \varepsilon + \frac{2M}{\delta^2} \left\{ (u - s)^2 + (v - t)^2 \right\}.$$

Since $L_{i,j}$ is linear and positive, we obtain

$$\begin{aligned} |L_{i,j}(f; s, t) - f(s, t)| &= |L_{i,j}(f(u, v) - f(s, t); s, t) \\ &\quad - f(s, t)(L_{i,j}(f_0; s, t) - f_0(s, t))| \\ &\leq \left| L_{i,j} \left(\varepsilon + \frac{2M}{\delta^2} \left\{ (u - s)^2 + (v - t)^2 \right\}; s, t \right) \right| \\ &\quad + M |L_{i,j}(f_0; s, t) - f_0(s, t)| \\ &\leq \left(\varepsilon + M + \frac{2M}{\delta^2} (A^2 + B^2) \right) |L_{i,j}(f_0; s, t) - f_0(s, t)| \\ &\quad + \frac{4M}{\delta^2} A |L_{i,j}(f_1; s, t) - f_1(s, t)| \\ &\quad + \frac{4M}{\delta^2} B |L_{i,j}(f_2; s, t) - f_2(s, t)| \\ &\quad + \frac{2M}{\delta^2} |L_{i,j}(f_3; s, t) - f_3(s, t)| + \varepsilon \end{aligned}$$

where $A := \max |s|$, $B := \max |t|$. Now we multiply the both-sides of the above inequality by $\frac{1}{|\sigma(s, t)|}$,

$$(3.6) \quad \left| \frac{L_{i,j}(f; s, t) - f(s, t)}{\sigma(s, t)} \right| \leq K \left\{ \left| \frac{L_{i,j}(f_0; s, t) - f_0(s, t)}{\sigma(s, t)} \right| + \left| \frac{L_{i,j}(f_1; s, t) - f_1(s, t)}{\sigma(s, t)} \right| \right. \\ \left. + \left| \frac{L_{i,j}(f_2; s, t) - f_2(s, t)}{\sigma(s, t)} \right| + \left| \frac{L_{i,j}(f_3; s, t) - f_3(s, t)}{\sigma(s, t)} \right| \right\} + \frac{\varepsilon}{|\sigma(s, t)|},$$

where $K = \max \left\{ \varepsilon + M + \frac{2M}{\delta^2} (A^2 + B^2), \frac{4M}{\delta^2} A, \frac{4M}{\delta^2} B, \frac{2M}{\delta^2} \right\}$ and where $\sigma(s, t) = \max \{ |\sigma_r(s, t)|; r = 0, 1, 2, 3 \}$. Taking the supremum over $(s, t) \in S$, we get

$$(3.7) \quad \sup_{(s,t) \in S} \left| \frac{L_{i,j}(f; s, t) - f(s, t)}{\sigma(s, t)} \right| \leq \sup_{(s,t) \in S} \frac{\varepsilon}{|\sigma(s, t)|} + K \left\{ \sup_{(s,t) \in S} \left| \frac{L_{i,j}(f_0; s, t) - f_0(s, t)}{\sigma_0(s, t)} \right| \right. \\ + \sup_{(s,t) \in S} \left| \frac{L_{i,j}(f_1; s, t) - f_1(s, t)}{\sigma_1(s, t)} \right| \\ + \sup_{(s,t) \in S} \left| \frac{L_{i,j}(f_2; s, t) - f_2(s, t)}{\sigma_2(s, t)} \right| \\ \left. + \sup_{(s,t) \in S} \left| \frac{L_{i,j}(f_3; s, t) - f_3(s, t)}{\sigma_3(s, t)} \right| \right\}.$$

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $\sup_{(s,t) \in S} \frac{\varepsilon}{|\sigma(s,t)|} < r$. Then, setting

$$D_i := \left\{ j \in \mathbb{N} : j \leq i, \left\| \frac{L_{i,j}(f) - f}{\sigma} \right\|_{C(S)} \geq r \right\},$$

$D_i^r := \left\{ j \in \mathbb{N} : j \leq i, \left\| \frac{L_{i,j}(f_r) - f_r}{\sigma_r} \right\|_{C(S)} \geq \frac{r - \sup_{(s,t) \in S} \frac{\varepsilon}{|\sigma(s,t)|}}{4K} \right\}$, $r = 0, 1, 2, 3$. It is easy to see that

$$D_i \subseteq \bigcup_{r=0}^3 D_i^r$$

which gives, for all $i \in \mathbb{N}$, then

$$\sum_{j \in D_i} a_{i,j} \leq \sum_{r=0}^3 \sum_{j \in D_i^r} a_{i,j}.$$

Letting $i \rightarrow \infty$ and using (3.4), we obtain (3.5). The proof is complete. \square

If one replaces the scale function by a non-zero constant, then the Theorem 3.2 reduces to the Theorem 3.1.

We now show that our result Theorem 3.2 is stronger than Theorem 3.1.

Example 3.1. Let consider the following Bernstein operators given by

$$(3.8) \quad B_{i,j}(f; s, t) = \sum_{k=0}^i \sum_{p=0}^j f\left(\frac{k}{i}, \frac{p}{j}\right) \binom{i}{k} \binom{j}{p} s^k (1-s)^{i-k} t^p (1-t)^{j-p}$$

where $(s, t) \in S = [0, 1] \times [0, 1]$; $f \in C(S)$. Also, observe that

$$\begin{aligned} B_{i,j}(f_0; s, t) &= f_0(s, t), \\ B_{i,j}(f_1; s, t) &= f_1(s, t), \\ B_{i,j}(f_2; s, t) &= f_2(s, t), \\ B_{i,j}(f_3; s, t) &= f_3(s, t) + \frac{s - s^2}{i} + \frac{t - t^2}{j}, \end{aligned}$$

where $f_0(s, t) = 1$, $f_1(s, t) = s$, $f_2(s, t) = t$ and $f_3(s, t) = s^2 + t^2$. Using these polynomials, we introduce the following positive linear operators on $C(S)$:

$$(3.9) \quad P_{i,j}(f; s, t) = (1 + \gamma_{i,j}(s, t)) B_{i,j}(f; s, t), \quad (s, t) \in S = [0, 1] \times [0, 1] \text{ and } f \in C(S)$$

where $\gamma_{i,j}(s, t)$ is given in Example 2.1. Now, take $A = C_1$, the Cesàro matrix. Since $\gamma_{i,j} \rightrightarrows \gamma = 0$ (st_T, σ), where

$$\sigma(s, t) = \begin{cases} \frac{1}{st}, & \text{if } (s, t) \in (0, 1] \times (0, 1], \\ 0, & \text{if } s = 0 \text{ or } t = 0. \end{cases}$$

Then, we conclude that

$$P_{i,j}(f_r) \rightrightarrows f_r \text{ (st}_T, \sigma) \text{ (} r = 0, 1, 2, 3 \text{)}.$$

So by our main theorem, Theorem 3.2, we immediately see that

$$P_{i,j}(f) \rightrightarrows f \text{ (st}_T, \sigma) \text{ for } f \in C(S).$$

However, since $(\gamma_{i,j})$ is not triangular statistically uniformly convergent to $\gamma = 0$ on the interval S , we can say that Theorem 3.1 does not work for our operators defined by (3.9).

4. Rates of Triangular A -Statistical Relative Uniform Convergence

In this section, using the notion of triangular A -statistical relative uniform convergence we study the rate of convergence of positive linear operators with the help of modulus of continuity.

Let $f \in C(S)$. Then the modulus of continuity of f , defined to be

$$w(f; \delta) = \sup \left\{ |f(u, v) - f(s, t)| : (u, v), (s, t) \in S \text{ and } \sqrt{(u-s)^2 + (v-t)^2} \leq \delta \right\}$$

for $\delta > 0$.

Then we hold the following result.

Theorem 4.1. *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Let $(L_{i,j})$ be a double sequence of positive linear operators acting from $C(S)$ into itself. Assume that the following conditions hold:*

$$(a) L_{i,j}(f_0) \rightrightarrows f_0 \text{ (st}_A^T, \sigma_0),$$

$$(b) w(f, \delta) \rightrightarrows 0 \text{ (st}_A^T, \sigma_1), \text{ where } \delta := \delta_{i,j} = \sqrt{\|L_{i,j}(\varphi)\|_{C(S)}}, \text{ with } \varphi(u, v) = (u-s)^2 + (v-t)^2.$$

Then we have, for all $f \in C(S)$,

$$L_{i,j}(f) \rightrightarrows f \text{ (st}_A^T, \sigma)$$

where

$$\sigma(s, t) = \max \{ |\sigma_0(s, t)|, |\sigma_1(s, t)|, |\sigma_0(s, t)\sigma_1(s, t)| \},$$

$|\sigma_i(s, t)| > 0$ and $\sigma_i(s, t)$ is unbounded for $i = 0, 1$.

Proof. Let $f \in C(S)$ and $(s, t) \in S$ be fixed. Using linearity and positivity of $L_{i,j}$

we have, for any $(i, j) \in \mathbb{N}^2$ and $\delta > 0$,

$$\begin{aligned}
& |L_{i,j}(f; s, t) - f(s, t)| \\
&= |L_{i,j}(f(u, v) - f(s, t); s, t) - f(s, t)(L_{i,j}(f_0; s, t) - f_0(s, t))| \\
&\leq L_{i,j}(|f(u, v) - f(s, t)|; s, t) + M|L_{i,j}(f_0; s, t) - f_0(s, t)| \\
&\leq L_{i,j}\left(\left[1 + \frac{\sqrt{(u-s)^2 + (v-t)^2}}{\delta}\right]w(f; \delta); s, t\right) \\
&\quad + M|L_{i,j}(f_0; s, t) - f_0(s, t)| \\
&\leq w(f; \delta)|L_{i,j}(f_0; s, t) - f_0(s, t)| + \frac{w(f; \delta)}{\delta^2}L_{i,j}(\varphi; s, t) + w(f; \delta) \\
&\quad + M|L_{i,j}(f_0; s, t) - f_0(s, t)|,
\end{aligned}$$

where $M = \|f\|_{C(S)}$. Taking the supremum over $(s, t) \in S$ in both sides of the above inequality, we obtain, for any $\delta > 0$,

$$\begin{aligned}
\left\|\frac{L_{i,j}f - f}{\sigma}\right\|_{C(S)} &\leq \frac{w(f, \delta_{i,j})}{\|\sigma_1\|_{C(S)}} \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)} + \frac{w(f, \delta_{i,j})}{\|\sigma_1\|_{C(S)}\delta^2} \left\|\frac{L_{i,j}\varphi}{\sigma_1}\right\|_{C(S)} \\
&\quad + \frac{w(f, \delta_{i,j})}{\|\sigma_1\|_{C(S)}} + M \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)}.
\end{aligned}$$

Now, if take $\delta := \delta_{i,j} = \sqrt{\|L_{i,j}(\varphi)\|}$, then we may write

$$\left\|\frac{L_{i,j}f - f}{\sigma}\right\|_{C(S)} \leq \frac{w(f, \delta)}{\|\sigma_1\|} \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)} + 2\frac{w(f, \delta)}{\|\sigma_1\|} + M \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)}$$

and hence,

$$(4.1) \quad \left\|\frac{L_{i,j}f - f}{\sigma}\right\|_{C(S)} \leq K \left\{ \frac{w(f, \delta)}{\|\sigma_1\|} \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)} + \frac{w(f, \delta)}{\|\sigma_1\|} + \left\|\frac{L_{i,j}f_0 - f_0}{\sigma_0}\right\|_{C(S)} \right\}$$

where $K = \max\{2, M\}$. For a given $r > 0$, define the following sets:

$$\begin{aligned}
 T & : = \left\{ j \in \mathbb{N} : j \leq i, \left\| \frac{L_{i,j}(f) - f}{\sigma} \right\|_{C(S)} \geq r \right\}, \\
 T_1 & : = \left\{ j \in \mathbb{N} : j \leq i, \frac{w(f, \delta)}{\|\sigma_1\|} \left\| \frac{L_{i,j}f_0 - f_0}{\sigma_0} \right\|_{C(S)} \geq \frac{r}{3K} \right\}, \\
 T_2 & : = \left\{ j \in \mathbb{N} : j \leq i, \frac{w(f, \delta)}{\|\sigma_1\|} \geq \frac{r}{3K} \right\}, \\
 T_3 & : = \left\{ j \in \mathbb{N} : j \leq i, \left\| \frac{L_{i,j}f_0 - f_0}{\sigma_0} \right\|_{C(S)} \geq \frac{r}{3K} \right\}.
 \end{aligned}$$

It follows from (4.1) that

$$T \subset T_1 \cup T_2 \cup T_3.$$

Also, define the sets:

$$\begin{aligned}
 T_4 & : = \left\{ j \in \mathbb{N} : j \leq i, \frac{w(f, \delta)}{\|\sigma_1\|} \geq \sqrt{\frac{r}{3K}} \right\}, \\
 T_5 & : = \left\{ j \in \mathbb{N} : j \leq i, \left\| \frac{L_{i,j}f_0 - f_0}{\sigma_0} \right\|_{C(S)} \geq \sqrt{\frac{r}{3K}} \right\}.
 \end{aligned}$$

Then observe that $T_1 \subset T_4 \cup T_5$. So we have $T \subset T_2 \cup T_3 \cup T_4 \cup T_5$.

Therefore, using (a) and (b), the proof is complete. \square

5. CONCLUSION

If we take $A = C_1 = (c_{i,j})$, the Cesàro matrix defined by

$$c_{i,j} := \begin{cases} \frac{1}{i}, & \text{if } 1 \leq j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

then triangular A -statistical relative uniform convergence reduces to the concept of triangular statistical relative convergence. Furthermore, if we take $A = C_1$ and the scale function by a non-zero constant, then triangular A -statistical relative uniform convergence reduces to the triangular statistical uniform convergence.

If one replaces the scale function by a non-zero constant, then the triangular A -statistical relative uniform convergence reduces to the triangular A -statistical uniform convergence.

REFERENCES

1. T. ACAR, S. A. MOHIUDDINE: *statistical $(C, 1)(E, 1)$ summability and Korovkin's theorem*. Filomat **30(2)** (2016), 387-393.

2. C. BARDARO, A. BOCCUTO, K. DEMIRCI, I. MANTELLINI, S. ORHAN: *Triangular A -statistical approximation by double sequences of positive linear operators*. Results Math. **68** (2015), 271-291.
3. C. BARDARO, A. BOCCUTO, K. DEMIRCI, I. MANTELLINI, S. ORHAN: *Korovkin-type theorems for modular $\Psi - A$ -statistical convergence*. J. Funct. Spaces Article ID 160401, **2015** (2015), p. 11.
4. C. BARDARO and I. MANTELLINI: *Korovkin's theorem in modular spaces*. Commentationes Math. **47** (2007), 239-253.
5. C. BARDARO and I. MANTELLINI: *A Korovkin Theorem in multivariate modular function spaces*. J. Funct. Spaces Appl. **7** (2009), 105-120.
6. C. BELEN and S. A. MOHIUDDINE: *Generalized weighted statistical convergence and application*. Appl. Math. Comput. **219** (2013), 9821-9826.
7. E. W. CHITTEN: *On the limit functions of sequences of continuous functions converging relatively uniformly*. Trans. Amer. Math. Soc. **20** (1919), 179-184.
8. K. DEMIRCI, F. DIRIK: *A Korovkin type approximation theorem for double sequences of positive linear operators of two variables in A -statistical sense*. Bull. Korean Math. Soc. **47**(4) (2010) 825-837.
9. K. DEMIRCI, B. KOLAY: *A -Statistical Relative Modular Convergence of Positive Linear Operators*. Positivity **21** (2017), 847-863.
10. K. DEMIRCI, S. ORHAN: *Statistical approximation by double sequences of positive linear operators on modular spaces*. Positivity **19** (2015) 23-36.
11. K. DEMIRCI, S. ORHAN: *Statistically relatively uniform convergence of positive linear operators*. Results. Math. **69** (2016) 359-367.
12. K. DEMIRCI, S. ORHAN: *Statistical relative approximation on modular spaces*. Results. Math. **71** (2017) 1167-1184.
13. R. A. DEVORE, G. G. LORENTZ: *Constructive Approximation (Grund. Math. Wiss. 303)*. Springer Verlag, Berlin, 1993.
14. F. DIRIK, K. DEMIRCI: *Korovkin-type approximation theorem for functions of two variables in statistical sense*. Turk. J. Math. **34** (2010) 73-83.
15. K. DONNER: *Korovkin Theorems in L^p Spaces*. J. Funct. Anal **42**(1) (1981) 12-28.
16. S. M. EISENBERG: *Korovkin's Theorem*. Bull. Malays. Math. Soc. **2**(2) (1979) 13-29.
17. H. FAST: *Sur la Convergence Statistique*. Colloq. Math. **2** (1951) 241-244.
18. P. GARRANCHO: *A general Korovkin result under generalized convergence*. Constr. Math. Anal. **2**(2) (2019) 81-88.
19. A. D. GADJIEV, C. ORHAN: *Some approximation theorems via statistical convergence*. Rocky Mountain J. Math. **32** (2002) 129-138.
20. U. KADAK, S. A. MOHIUDDINE: *Generalized statistically almost convergence based on the difference operator which includes the (p, q) -gamma function and related approximation theorems*. Results. Math. **73**(1) (2018) 1-31.
21. P. P. KOROVKIN: *Linear Operators and Approximation Theory*. Hindustan Publ. Co., Delhi, 1960.
22. E. H. MOORE: *An Introduction to a Form of General Analysis*. The New Hawen Mathematical Colloquim, Yale University Press, New Hawen, 1910.

23. S. A. MOHIUDDINE: *Statistical weighted A -summability with application to Korovkin's type approximation theorem*. J. Inequal. Appl. **2016**(1) (2016) 1-13.
24. S. A. MOHIUDDINE, B. A. S. ALAMRI: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM **113**(3) (2019) 1955-1973.
25. S. A. MOHIUDDINE, A. ASIRI, B. HAZARIKA: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*. Int. J. Gen. Syst. **48**(5) (2019) 492-506.
26. S. A. MOHIUDDINE, B. HAZARIKA, M. A. ALGHAMDI: *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems*. Filomat **33**(14) (2019) 4549-4560.
27. F. MÓRICZ: *Statistical Convergence of multiple sequences*. Arch. Math. **81**(1) (2003) 82-89.
28. A. PRINGSHEIM: *Zur theorie der zweifach unendlichen zahlenfolgen*. Math. Ann. **53** (1900) 289-321.
29. G. M. ROBISON: *Divergent double sequences and series*. Amer. Math. Soc. Transl. **28** (1926) 50-57.
30. H. STEINHAUS: *Sur la Convergence Ordinaire et la Convergence Asymptotique*. Colloq. Math. **2** (1951) 73-74.
31. P. ŞAHİN, F. DIRİK: *Statistical Relative Uniform Convergence of Double Sequence of Positive Linear Operators*. Appl. Math. **17** (2017) 207-220.
32. T. O. TEOPLITZ: *Über allgemeine lineare Mittelbildungen*. Prace Math. Fiz. **22** (1911) 113-119.
33. B. YILMAZ, K. DEMIRCI, S. ORHAN: *Relative Modular Convergence of Positive Linear Operators*. Positivity **20** (2016), 565-577.
34. D. E. WULBERT: *Convergence of Operators and Korovkin's theorem*. J. Approx. Theory **1** (1968) 381-390.