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APPROXIMATION PROPERTIES OF MODIFIED GAUSS-WEIERSTRASS INTEGRAL OPERATORS IN EXPONENTIAL WEIGHTED L_p SPACES

Başar Yılmaz

Kirikkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kiriklale, Turkey

Abstract. In this paper, we deal with modified Gauss-Weierstrass integral operators from exponentially weighted spaces $L_{p,a}(\mathbb{R})$ into $L_{p,2a}(\mathbb{R})$. We give the rate of convergence in terms of weighted modulus of continuity. Moreover, we prove weighted approximation of functions belonging to the space $L_{p,a}(\mathbb{R})$ by these operators with the help of a Korovkin type theorem. Finally, we give pointwise approximation of such functions by these operators at generalized Lebesgue points.
Keywords: Gauss-Weierstrass operators, Korovkin type theorem, exponential weighted spaces

1. Introduction

The well-known Gauss-Weierstrass singular integral operators are given by

$$(W_n f)(x) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(x-t)\, e^{-nt^2} dt, \qquad x \in \mathbb{R},\ n \in \mathbb{N},$$

where the function f is selected such that the integrals are finite. These operators were extensively studied by many researchers [3],[4],[5],[7] and [14]. Some approximation problems including Voronovskaya type theorem and quantitative type results have been investigated in L_p and weighted L_p spaces in [9].

In [8], Agratini et al. considered a generalization of the Gauss-Weierstrass singular integral operators defined by

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Corresponding Author: Başar Yılmaz, Kirikkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kiriklale, Turkey | E-mail: basaryilmaz77@yahoo.com
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(1.1) $$\left(W_n^* f\right)(x)=\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f\left(\beta_n(x)+t\right)e^{-nt^2}dt,\quad x\in(-\infty,\infty),\ n\in\mathbb{N},$$

where

(1.2) $$\beta_n(x)=x-\frac{a}{2n},\quad n\geq 1,\ a>0.$$

These operators reproduce not only e_0, where $e_0(t)=1,\ t\in\mathbb{R}$, but also certain exponential functions. In that work, the authors studied these operators in the polynomial weighted continuous functions spaces. They also proved that these operators have better approximation properties than the classical ones. The linear positive operators preserving exponential functions in approximation theory have been intensively studied (see [10],[11],[12],[13],[15],[16] and [17]).

In this paper, we consider the operators W_n^* in the setting of large classes of exponential weighted L_p spaces. Firstly, we show that these operators act from the exponential weighted $L_{p,a}(\mathbb{R})$ space into $L_{p,2a}(\mathbb{R})$, which will be defined below. Then, we get quantitative results for the rate of convergence by the operators in terms of weighted L_p modulus of continuity. Similar result is also given for the derivates of the operators. Furthermore, we obtain weighted approximation by the operators using a weighted Korovkin type theorem. Finally, we investigate a pointwise convergence result by the operators at generalized Lebesgue points.

Below, we recall the definition of exponential weighted space $L_{p,a}(\mathbb{R})$.

Let $a>0$ and $1\leq p<\infty$ be fixed,

$$\nu_a(x)=e^{-ax^2}\text{ for }x\in\mathbb{R},$$

and let $L_{p,a}(\mathbb{R})$ be the space of all functions $f:\mathbb{R}\to\mathbb{R}$ for which $\nu_a f$ is Lebesgue integrable with p-th power over $\mathbb{R}$, where $1\leq p<\infty$, and uniformly continuous and bounded on $\mathbb{R}$. The norm in $L_{p,a}(\mathbb{R})$ is defined by

(1.3) $$\|f\|_{p,a}=\|f(\cdot)\|_{p,a}=\left(\int_{-\infty}^{\infty}\left|\nu_a(x)f(x)\right|^p dx\right)^{1/p},\quad 1\leq p<\infty$$

(see [6]).

As usual, for $f\in L_{p,a}(\mathbb{R})$ the weighted modulus of continuity is defined as

(1.4) $$\omega\left(f;L_{p,a}(\mathbb{R});t\right):=\sup_{|h|\leq t}\left\|\Delta_h f(\cdot)\right\|_{p,a}\text{ for }t\geq 0,$$

where

$$\Delta_h f(x):=f(x+h)-f(x).$$

The above ω has the following properties:

$$\omega\left(f;L_{p,a}\left(\mathbb{R}\right);t_1\right)\leq\omega\left(f;L_{p,a}\left(\mathbb{R}\right);t_2\right)\ \text{ for } 0\leq t_1<t_2, \tag{1.5}$$

$$\omega\left(f;L_{p,a}\left(\mathbb{R}\right);\lambda t\right)\leq\left(1+\lambda\right)e^{a(\lambda t)^2}\omega\left(f;L_{p,a}\left(\mathbb{R}\right);t\right)\ \text{ for } \lambda,t\geq 0, \tag{1.6}$$

$$\lim_{t\to 0^+}\omega\left(f;L_{p,a}\left(\mathbb{R}\right);t\right)=0 \tag{1.7}$$

for every $f\in L_{p,a}\left(\mathbb{R}\right)$ (see[1]).

2. Auxiliary results

In this part, we shall give some fundamental properties of the generalized Gauss-Weierstrass integral operators W_n^* in the spaces $L_{p,2\alpha}\left(\mathbb{R}\right)$. Lemma 2.1 can be obtained by elementary calculations.

Lemma 2.1. *The equality*

$$\int_0^\infty x^p e^{-2ax^2p}dx=\frac{1}{2^{\frac{p+3}{2}}}\frac{\Gamma\left(\frac{p+1}{2}\right)}{(ap)^{\frac{p+1}{2}}},$$

where Γ *is the Gamma function and holds for every* $p\in[1,\infty)$ *and* $a>0$.

Lemma 2.2. *(see [8]) If* $W_n^*,\ n\geq 1$, *are the operators given by (1.1), then for each integer* $j\geq 0$, $e_j\left(t\right)=t^j\ t\in\mathbb{R}$, *we have*

$$\left(W_n^*e_j\right)(x)=\beta_n^j(x)+\sum_{s=0}^{\lfloor j/2\rfloor}\frac{(2s-1)!}{(2n)^s}\binom{j}{2s}\beta_n^{j-2s}\left(x\right),\quad p\geq 2\quad ,x\in\mathbb{R}.$$

Also, as particular cases, we have

$$W_n^*e_0=1,\ W_n^*e_1=\beta_n\left(x\right),\ W_n^*e_2=\beta_n^2\left(x\right)+\frac{1}{2n}.$$

This formula shows that $W_n^*f\ \left(n>2a,\ a>0\right)$ is a sequence of linear positive operators from $L_{p,a}\left(\mathbb{R}\right)$ into $L_{p,2a}\left(\mathbb{R}\right)$.

Lemma 2.3. *If* $f\in L_{p,\alpha}\left(\mathbb{R}\right)$, *with* $1\leq p<\infty$, *then for* $n>2a$, *we have*

$$\left\|W_n^*f\right\|_{p,2a}\leq M_n\left\|f\right\|_{p,a}, \tag{2.1}$$

where

$$M_n=\sqrt{\frac{\pi}{n-2a}}e^{\frac{a^3}{2n^2-4an}}.$$

Proof. In view of the definition of the operators W_n^*, we can write

$$\begin{aligned}\|W_n^* f\|_{p,2a} &= \left(\int_{-\infty}^{\infty} \left|e^{-2ax^2}\left(W_n^* f\right)(x)\right|^p dx\right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} e^{-2ax^2 p}\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f\left(\beta_n(x)+t\right)e^{-nt^2}dt\right|^p dx\right)^{1/p}.\end{aligned}$$

By a generalization of Minkowski's inequality and making use of substitution $\beta_n(x)+t=u$, the above formula reduces to

$$\begin{aligned}\|W_n^* f\|_{p,2a} &\leq \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(e^{-nt^2 p}\int_{-\infty}^{\infty}|f(u)|^p e^{-2a\left(u+\frac{a}{2n}-t\right)^2 p}du\right)^{1/p} dt \\ &\leq \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(e^{-nt^2 p}\int_{-\infty}^{\infty}|f(u)|^p e^{-au^2 p}e^{2a\left(\frac{a}{2n}-t\right)^2 p}du\right)^{1/p} dt \\ &\leq \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}e^{2a\left(\frac{a}{2n}-t\right)^2}\left(\int_{-\infty}^{\infty}|f(u)|^p e^{-au^2 p}du\right)^{1/p} dt \\ &= \|f\|_{p,a}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2+2a\left(\frac{a}{2n}-t\right)^2}dt \\ &= \|f\|_{p,a}\sqrt{\frac{\pi}{n-2a}}e^{\frac{a^3}{2n^2-4an}}.\end{aligned}$$

Thus, the proof of Lemma 2.3 is completed. □

3. Approximation theorems

Firstly, we shall prove rate of convergence by the operators (1.1) of functions belonging to $L_{p,\alpha}(\mathbb{R})$.

Theorem 3.1. *If $f \in L_{p,a}(\mathbb{R})$, then we have*

$$\|W_n^* f - f\|_{p,2a} \leq \omega\left(f; L_{p,2a}; \frac{a}{2n}\right) + \omega\left(f; L_{p,2a}; \frac{1}{\sqrt{n}}\right)\left(\sqrt{\frac{n}{n-a}}\right)$$

for $n > a$.

Proof. From (1.1), (1.3) and the Minkowski inequality, we get

$$\|W_n^* f - f\|_{p,2a}$$

$$
\begin{aligned}
&= \left(\int_{-\infty}^{\infty}\left|e^{-2ax^2}\left(\left(W_n^* f\right)(x)-f(x)\right)\right|^p dx\right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty}\left|e^{-2ax^2 p}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(f\left(\beta_n(x)+t\right)-f(x)\right)e^{-nt^2}dt\right|^p dx\right)^{1/p} \\
&\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\left(\int_{-\infty}^{\infty}e^{-2ax^2 p}\left|f\left(\beta_n(x)+t\right)-f(x)+f(x+t)-f(x+t)\right|^p dx\right)^{1/p} dt \\
&\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\left(\int_{-\infty}^{\infty}\left|e^{-2ax^2}\left(f\left(\beta_n(x)+t\right)-f(x+t)\right)\right|^p dx\right)^{1/p} dt \\
&\quad +\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\left(\int_{-\infty}^{\infty}\left|e^{-2ax^2}\left(f(x+t)-f(x)\right)\right|^p dx\right)^{1/p} dt \\
&\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\left\|f\left(\beta_n(\cdot)+t\right)-f(\cdot+t)\right\|_{p,2a}dt + \\
&+ \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\left\|f(\cdot+t)-f(\cdot)\right\|_{p,2a}dt.
\end{aligned}
$$

Then by (1.4), we obtain

$$
\left\|W_n^* f-f\right\|_{p,2a} \le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\omega\left(f;L_{p,2a};\frac{a}{2n}\right)dt+\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}\omega\left(f;L_{p,2a};t\right)dt
$$

and from (1.6), we have

$$
\begin{aligned}
\left\|W_n^* f-f\right\|_{p,2a} &\le \omega\left(f;L_{p,2a};\frac{a}{2n}\right)+\omega\left(f;L_{p,2a};\frac{1}{\sqrt{n}}\right)\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(1+\sqrt{n}t\right)e^{-t^2(n-a)}dt \\
&= \omega\left(f;L_{p,2a};\frac{a}{2n}\right)+\omega\left(f;L_{p,2a};\frac{1}{\sqrt{n}}\right)\left(\sqrt{\frac{n}{n-a}}\right) \quad , \quad n>a.
\end{aligned}
$$

□

Also, the following theorem is obvious from the formula

$$
\left(W_n^* f\right)^{(r)}(x)=\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}f^{(r)}\left(\beta_n(x)+t\right)e^{-nt^2}dt.
$$

Theorem 3.2. *If $f^{(r)} \in L_{p,a}(\mathbb{R})$ with fixed $a > 0$ and $r \in \mathbb{N}$, then we have*

$$\left\| W_n^* f^{(r)} - f^{(r)} \right\|_{p,2a} \leq \omega\left(f; L_{p,2a}; \frac{a}{2n}\right) + \omega\left(f; L_{p,2a}; \frac{1}{\sqrt{n}}\right)\left(\sqrt{\frac{n}{n-a}}\right)$$

for $n > a$.

Let ω be a positive continuous function on the whole real axis satisfying the condition

$$\int_{\mathbb{R}} t^{2p}\omega(t)\,dt < \infty,$$

where $p \in [1, \infty)$ is fixed. Let also $L_{p,\omega}(\mathbb{R})$ denote the linear space of measurable, p-absolutely integrable functions on $\mathbb{R}$ with respect to the weight function ω, that is

$$L_{p,\omega}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R};\ \ \|f\|_{p,\omega} := \left(\int_{\mathbb{R}} |f(t)|^p \omega(t)\,dt \right)^{\frac{1}{p}} < \infty \right\}.$$

In [2], the authors obtained the following weighted Korovkin type approximation theorem for any function $f \in L_{p,\omega}(\mathbb{R})$,

Theorem 3.3. *(see [2]) Let $(L_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of positive linear operators from $L_{p,\omega}(\mathbb{R})$ into $L_{p,\omega}(\mathbb{R})$, satisfying the conditions*

$$\lim_{n\to\infty} \|L_n e_j - e_j\|_{p,\omega} = 0,\ \ j = 0, 1, 2.$$

Then for every $f \in L_{p,\omega}(\mathbb{R})$, we have

$$\lim_{n\to\infty} \|L_n f - f\|_{p,\omega} = 0.$$

Our aim is to study the weighted approximation by the sequence of operators W_n^* in the norm $L_{p,2a}(\mathbb{R})$. We consider a weight commonly used in defining spaces of functions with exponential growth. If we choose $\omega(x) = e^{-2ax^2p}$, $x \in \mathbb{R}$, we can give the following theorem.

Theorem 3.4. *If $f \in L_{p,a}(\mathbb{R})$, then we have*

$$\lim_{n\to\infty} \|W_n^* f - f\|_{p,2a} = 0.$$

Proof. According to Theorem 3.3, for the proof, it is sufficient to show that the conditions

$$\lim_{n\to\infty} \|W_n^* e_j - e_j\|_{p,2a} = 0, \qquad j = 0, 1, 2 \tag{3.1}$$

are satisfied. Since $W_n^* e_0 = 1$ the first condition of (3.1) is fulfilled for $j = 0$. Considering Lemma 2.2, we have

$$\begin{aligned} \|W_n^* e_1 - e_1\|_{p,2a} &= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2} \left[(W_n^* e_1)(x) - x \right] \right|^p dx \right)^{1/p} \\ &= \frac{a}{2n} \left(2 \int_0^{\infty} e^{-2ax^2 p} dx \right)^{1/p}. \end{aligned}$$

Then, we get

$$\|W_n^* e_1 - e_1\|_{p,2a} = \frac{a}{n} 2^{\frac{1}{p}-1} \left(\sqrt{\frac{\pi}{2ap}} \right)^{1/p},$$

and the second condition of (3.1) holds for $j = 1$ as $n \to \infty$. Finally, from Lemma 2.2, we obtain

$$\begin{aligned} \|W_n^* e_2 - e_2\|_{p,2a} &= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2 p} \left[(W_n^* e_2)(x) - x^2 \right] \right|^p dx \right)^{1/p} \\ &= \left(\int_{-\infty}^{\infty} e^{-2ax^2 p} \left| \frac{a^2}{4n^2} + \frac{1}{2n} - \frac{ax}{n} \right|^p dx \right)^{1/p}. \end{aligned}$$

From triangle inequality

$$\begin{aligned} \|W_n^* e_2 - e_2\|_{p,2a} \leq 2^{1+1/p} \left(\frac{a^2}{4n^2} + \frac{1}{2n} \right) \left(\int_0^{\infty} e^{-2ax^2 p} dx \right)^{1/p} + \\ + 2^{1+1/p} \frac{a}{n} \left(\int_0^{\infty} e^{-2ax^2 p} x^p dx \right)^{1/p}, \end{aligned}$$

and using Lemma 2.1, we get

$$\|W_n^* e_2 - e_2\|_{p,2a} = \left(\frac{a^2}{4n^2} + \frac{1}{2n} \right) 2^{1+1/p} \left(\sqrt{\frac{\pi}{2pa}} \right)^{1/p} + 2^p \frac{a}{n} \frac{1}{2^{\frac{p+1}{2p}}} \frac{\Gamma\left(\frac{p+1}{2}\right)^{1/p}}{(ap)^{\frac{p+1}{2p}}}$$

and the third condition of (3.1) holds for $j = 2$ as $n \to \infty$. Thus, the proof is completed. □

Here, we give a pointwise convergence result at the points called as *generalized weighted p-Lebesgue point* which is consistent with exponential weighted space $L_{p,a}(\mathbb{R})$.

Theorem 3.5. *If x is a generalized weighted p-Lebesgue point of the function $f \in L_{p,a}(\mathbb{R})$, i.e.; for $x \in \mathbb{R}$ the condition*

$$\lim_{h\to 0}\left(\frac{1}{h}\int_0^h \left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|^p dt\right)^{\frac{1}{p}}=0, \tag{3.2}$$

holds, where β_n and α are given by (1.2), then we have

$$\lim_{n\to\infty} W_n^*(f;x)=f(x). \tag{3.3}$$

Proof. We observe that

$$\begin{aligned}W_n^*(f;x) &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(\beta_n(x)+t)\,e^{-nt^2}dt, \quad x\in(-\infty,\infty),\ n\in\mathbb{N}\\ &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{0} f(\beta_n(x)+t)\,e^{-nt^2}dt+\sqrt{\frac{n}{\pi}}\int_0^{\infty} f(\beta_n(x)+t)\,e^{-nt^2}dt\\ &= \sqrt{\frac{n}{\pi}}\int_0^{\infty}\left[f(\beta_n(x)+t)+f(\beta_n(x)-t)\right]e^{-nt^2}dt.\end{aligned}$$

Hence by the fact $\sqrt{\frac{n}{\pi}}\int\limits_{-\infty}^{\infty} e^{-nt^2}dt=1$, we get

$$\begin{aligned}W_n^*(f;x)-f(x) &= \sqrt{\frac{n}{\pi}}\int_0^{\infty}\left[f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)\right]e^{-nt^2}dt\\ &= \sqrt{\frac{n}{\pi}}\int_0^{\infty}\left(\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right)e^{-t^2(n-2a)}dt.\end{aligned}$$

Since $f\in L_{p,a}(-\infty,\infty)$, $1\le p<\infty$ and if $\frac{1}{p}+\frac{1}{p'}=1$, then by Hölder's inequality

$$\begin{aligned}\left|W_n^*(f;x)-f(x)\right| &\le \int_0^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|\\ &\times\left(\sqrt{\frac{n}{\pi}}e^{-t^2(n-2a)}\right)^{\frac{1}{p}}\left(\sqrt{\frac{n}{\pi}}e^{-t^2(n-2a)}\right)^{\frac{1}{p'}}dt\end{aligned}$$

$$\le\left(\sqrt{\frac{n}{\pi}}\int_0^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|^p e^{-t^2(n-2a)}dt\right)^{\frac{1}{p}} \tag{3.4}$$

$$\times \left(\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-t^2(n-2a)} dt \right)^{\frac{1}{p'}} .$$

Since the integral at the last row is convergent for all $n > 2a$, we have

$$|W_n^* (f; x) - f(x)|^p \le$$
$$\sqrt{\frac{n}{\pi}} \int_0^{\infty} \left| \frac{f(\beta_n(x)+t) + f(\beta_n(x)-t) - 2f(x)}{e^{2\alpha t^2}} \right|^p e^{-t^2(n-2a)} dt.$$

Let

$$F(t) := \int_0^t \left| \frac{f(\beta_n(x)+\xi) + f(\beta_n(x)-\xi) - 2f(x)}{e^{2a\xi^2}} \right|^p d\xi. \tag{3.5}$$

Then

$$dF(t) = \left| \frac{f(\beta_n(x)+t) + f(\beta_n(x)-t) - 2f(x)}{e^{2at^2}} \right|^p dt.$$

Suppose that x is a generalized p-Lebesgue point of the function f. According to conditions (3.2) and (3.5), we shall write

$$\lim_{h \to 0} \frac{F(h)}{h} = 0.$$

In this case, for every $\varepsilon > 0$ there exist a $\delta > 0$ such that when

$$F(h) \le \frac{\varepsilon}{B} h \tag{3.6}$$

for all $h \le \delta$. Let

$$B = \left(\frac{\delta e^{-\delta^2(n-2a)} 2\sqrt{(n-2a)} + 1}{2\sqrt{(n-2a)}} \right) \sqrt{n}. \tag{3.7}$$

We can split the right-hand side of the last inequality into two parts:

$$\begin{aligned} |W_n^* (f; x) - f(x)|^p &\le \sqrt{\frac{n}{\pi}} \int_0^{\delta} \left| \frac{f(\beta_n(x)+t) + f(\beta_n(x)-t) - 2f(x)}{e^{2at^2}} \right|^p e^{-t^2(n-2a)} dt \\ &\quad + \sqrt{\frac{n}{\pi}} \int_{\delta}^{\infty} \left| \frac{f(\beta_n(x)+t) + f(\beta_n(x)-t) - 2f(x)}{e^{2at^2}} \right|^p e^{-t^2(n-2a)} dt \\ &= I_1 + I_2. \end{aligned}$$

To complete the proof, we have to show that

$$\lim_{n \to \infty} I_1 = \lim_{n \to \infty} I_2 = 0.$$

We consider I_1. Using integration by parts and (3.6), we find that

$$\begin{aligned}
I_1 &= \sqrt{\frac{n}{\pi}}\int_0^{\delta} e^{-t^2(n-2a)}dF(t) \\
&= \sqrt{\frac{n}{\pi}}\, e^{-t^2(n-2a)}F(t)\Big|_0^{\delta} + 2\sqrt{\frac{n}{\pi}}\int_0^{\delta} tF(t)(n-2a)e^{-t^2(n-2a)}dt \\
&\leq \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}F(\delta) + 2\sqrt{\frac{n}{\pi}}(n-2a)\int_0^{\delta} tF(t)e^{-t^2(n-2a)}dt \\
&= \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}F(\delta) + 2\sqrt{\frac{n}{\pi}}\frac{\varepsilon}{B}(n-2a)\int_0^{\delta} t^2e^{-t^2(n-2a)}dt \\
&\leq \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}\frac{\varepsilon}{B}\delta + 2\sqrt{\frac{n}{\pi}}\frac{\varepsilon}{B}(n-2a)\frac{\Gamma(\frac{3}{2})}{2(n-2a)^{\frac{3}{2}}} \qquad n-2a>0 \\
&= \sqrt{n}e^{-\delta^2(n-2a)}\frac{\varepsilon}{B}\delta + \frac{\sqrt{n}}{2}\frac{\varepsilon}{B}\frac{1}{\sqrt{(n-2a)}} \\
&\leq \frac{\varepsilon}{B}\sqrt{n}\left(\frac{2\delta e^{-\delta^2(n-2a)}\sqrt{(n-2a)}+1}{2\sqrt{(n-2a)}}\right).
\end{aligned}$$

Using (3.7), we have, for all $\varepsilon > 0$,

$$I_1 < \varepsilon.$$

For I_2, we can easily see that

$$\begin{aligned}
&\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p \\
\leq\; & 2^p\left(\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)}{e^{2at^2}}\right|^p + 2^p\left|\frac{f(x)}{e^{2\alpha t^2}}\right|^p\right) \\
=\; & 2^{2p}\left(\left|\frac{f(\beta_n(x)+t)}{e^{2at^2}}\right|^p + \left|\frac{f(\beta_n(x)-t)}{e^{2at^2}}\right|^p + \left|\frac{f(x)}{e^{2at^2}}\right|^p\right).
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
&\sqrt{\frac{n}{\pi}}\int_{\delta}^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p e^{-2t^2(2n-a)}dt \\
\leq\; & \sqrt{\frac{n}{\pi}}e^{-\delta^2(n+\alpha)}2^{2p}\left(\int_{\delta}^{\infty}\left|\frac{f(\beta_n(x)+t)}{e^{2at^2}}\right|^p dt + \int_{\delta}^{\infty}\left|\frac{f(\beta_n(x)-t)}{e^{2at^2}}\right|^p dt\right)
\end{aligned}$$

$$+\sqrt{\frac{n}{\pi}}2^{2p}\left|f(x)\right|^{p}\int_{\delta}^{\infty}e^{-t^{2}(n+2a(p-1))}dt.$$

Making use of the substitutions

$$\beta_{n}\left(x\right)+t=u,\ \beta_{n}\left(x\right)-t=w\text{ and }v=t^{2}\left(n+2a(p-1)\right), \tag{3.8}$$

to the above integrals, respectively, we get

$$\int_{-\infty}^{\infty}\left|\frac{f\left(u\right)}{e^{2a(u+\frac{a}{2n}-x)^{2}}}\right|^{p}du\leq\int_{-\infty}^{\infty}\frac{\left|f\left(u\right)\right|^{p}}{e^{\alpha u^{2}p+2\alpha(\frac{a}{2n}-x)^{2}p}}du=\left\|f\right\|_{p,a}^{p}e^{-2ap\left(\frac{a}{2n}-x\right)^{2}}\leq\left\|f\right\|_{p,a}^{p} \tag{3.9}$$

from (3.8) and (3.9), we can write the following inequality.

$$\begin{aligned}\left|W_{n}^{*}\left(f;x\right)-f(x)\right|^{p} &\leq \sqrt{\frac{n}{\pi}}e^{-\delta^{2}(2n-a)}2^{2p+1}\left\|f\right\|_{p,a}^{p}\\ &+\sqrt{\frac{n}{\pi}}\frac{2^{2p-1}\left|f(x)\right|^{p}}{(n+2a(p-1))}\int_{\delta^{2}n+2a(p-1)}^{\infty}\frac{1}{\sqrt{\frac{v}{n+2a(p-1)}}}e^{-v}dv.\end{aligned}$$

We get that

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}e^{-t^{2}(2n-a)}=0\ \text{ and }\ \lim_{n\to\infty}\int_{\delta^{2}(n+2a(p-1))}^{\infty}\frac{1}{\sqrt{\frac{v}{n+2a(p-1)}}}e^{-v}dv=0. \tag{3.10}$$

If we take the limit of both sides of the last inequality, we find

$$\lim_{n\to\infty}I_{2}=0$$

by (3.10). Therefore, for a large n, we obtain

$$\left|W_{n}^{*}\left(f;x\right)-f(x)\right|<\varepsilon$$

and the proof is completed. □

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