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CHEBYSHEV'S TYPE INEQUALITY FOR *H*-CONVEX FUNCTIONS AND RELATED MEAN VALUE THEOREMS FOR ASSOCIATED FUNCTIONALS

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Abstract. In this paper we investigate Chebyshev's type inequalities for h-convex functions. These inequalities are obtained by imposing some convenient conditions on h-convex functions. Furthermore, the associated Chebyshev's functional are estimated via mean value theorems.
Keywords: h-convex functions, Chebyshev's type inequalities.

1. Introduction

The well known Chebyshev's inequality was established in (1882-1883) by Chebyshev. Since then it has been studied with enormous interest by various authors (see [2, 3, 5, 4, 7, 11, 10, 12, 16, 17, 20]). Chebyshev's inequality has great importance because it can be applied to any probability distribution in which the mean and variance are defined [13]. Our objective in this paper is to produce its new version and related inequalities for h-convex functions, also we provide estimations of these inequalities (corresponding Chebyshev's functionals) by mean value theorems. The results of this paper will hold for convex functions. Also provide motivation to obtain further Chebyshev's inequalities for other type of convex and related functions.

Theorem 1.1. *Let $f, \varphi : [a, b] \to \mathbb{R}$ be two integrable functions. If f and φ are*

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monotonic in the same direction on $[a,b]$. *Then*

$$\frac{1}{b-a}\int_a^b f(x)\varphi(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b \varphi(x)dx. \tag{1.1}$$

The weighted version of Chebyshev's inequality (see [8, 15]) is stated as follows:

Theorem 1.2. *Let* $f, \varphi : [a,b] \to \mathbb{R}$ *and* $p : [a,b] \to \mathbb{R}_+$ *be integrable functions. If* f *and* φ *are monotonic in the same direction on* $[a,b]$. *Then*

$$\int_a^b p(x)dx \int_a^b p(x)f(x)\varphi(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)\varphi(x)dx. \tag{1.2}$$

Definition 1.1. A function $f : [a,b] \to \mathbb{R}$ is said to be convex, if for every $x, y \in [a,b]$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Lemma 1.1. *[19] A function* $f : I \to \mathbb{R}$ *is convex iff for every* $c \in I$ *the function*

$$\frac{f(x) - f(c)}{x - c}$$

is increasing on I $(x \neq c)$.

A modified Chebyshev's inequality was given by Levin and Steckin under the condition that one function is increasing on half domain and symmetric while other is continuous convex (see [12]). It is restated in the following theorem:

Theorem 1.3. *Let* f *be defined on* $[0,1]$ *and satisfying the conditions:*

(i) f *is decreasing on* $[0, \frac{1}{2}]$.

(ii) $f(x) = f(1-x)$ *for* $x \in [0,1]$.

Then for every continuous convex function φ, *one has the following inequality:*

$$\int_0^1 f(x)\varphi(x)dx \geq \int_0^1 f(x)dx \int_0^1 \varphi(x)dx. \tag{1.3}$$

If f *is decreasing on* $[0, \frac{1}{2}]$, *then inequality (1.3) is reversed.*

In 1984 Toader [25] gave the definition of m-convex functions as follows:

Definition 1.2. The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

.

Lemma 1.2. *[14] A function $f : [0, b] \to \mathbb{R}$ is m-convex iff for every $a \in [0, b]$ the function*

$$\frac{f(x) - mf(a)}{x - ma}$$

is increasing on $[0, ma)$ and $(ma, b]$.

Rehman et. al [21] noted that Levin and Steckin inequality is also valid when one function is decreasing on half domain and symmetric while other is decreasing m-convex and they proved the following result:

Theorem 1.4. *Let $0 \leq a < b$, $m \in (0, 1]$ and $f : [a, \frac{b}{m}] \to \mathbb{R}$ be integrable function and satisfy the conditions:*

(i) f is decreasing on $[a, \frac{ma+b}{2m}]$

(ii) f is symmetric about $\frac{ma+b}{2m}$.

Then for every decreasing m-convex function φ, one has the following inequality:

$$\int_a^{\frac{b}{m}} f(x)\varphi(x)dx \geq \frac{m}{b - ma} \int_a^{\frac{b}{m}} f(x)dx \int_a^{\frac{b}{m}} \varphi(x)dx. \tag{1.4}$$

A generalization of convex functions namely h-convex functions was introduced by Varošanec [26]. Our aim is to extrapolate Theorem 1.3 for the case when one function is h-convex while other is decreasing on half domain. Furthermore general versions of Chebyshev's type inequality are obtained for h-convex functions. These inequalities are estimated by mean value theorems.

Definition 1.3. Let $h : [0, b] \to \mathbb{R}$ be a non-negative function and $(0, 1) \subseteq [0, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be h-convex, if f is non-negative for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$, one has

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y). \tag{1.5}$$

Remark 1.1. Particular value of h in inequality (1.5) gives us the following functions:

1. $h(\lambda) = \lambda$ gives the convex functions.
2. $h(\lambda) = 1$ gives the P-functions.
3. $h(\lambda) = \lambda^s$ and $\lambda \in (0, 1)$ gives the s-convex functions of second sense.
4. $h(\lambda) = \frac{1}{\lambda}$ and $\lambda \in (0, 1)$ gives the Godunova-Levin functions.
5. $h(\lambda) = \frac{1}{\lambda^s}$ and $\lambda \in (0, 1)$ gives the s-Godunova-Levin functions of second sense.

Definition 1.4. [26] A function $h : J \to \mathbb{R}$ is said to be super-multiplicative function if

$$h(xy) \geq h(x)h(y) \tag{1.6}$$

for all $x, y \in J$, when $xy \in J$.

For some recent results by different authors due to h-convex functions we refer the reader to (see [1, 9, 18, 24, 23, 26, 28, 27]). The results of this paper are described in three sections. In Section 2 first we will construct some h-convex and monotone functions which are further utilized to obtain Chebyshev's type inequalities. In Section 3 the error estimations of Chebyshev's type inequalities are analyzed by using mean value theorems. In Section 4 we will discuss special cases of results given in Sections 2 and 3.

2. Main Results

The following Lemmas for h-convex functions are similar to [6, Lemma 2.1, 2.2]. They are used to prove Chebyshev's type inequalities for h-convex functions.

Lemma 2.1. *Let $h:[0,b]\to\mathbb{R}$, $(0,1)\subseteq[0,b]$ be non negative function such that $h(\lambda)+h(1-\lambda)\le 1$ for all $\lambda\in(0,1)$. Also let $0\le a<2bh\left(\frac{1}{2}\right)$ and $f:[a,2bh(\frac{1}{2})]\to\mathbb{R}$ be h-convex function. Then*

$$F(x):=f(x)+f\left(a+2bh\left(\frac{1}{2}\right)-x\right) \tag{2.1}$$

satisfies the following two conditions:

(i) F is h-convex function on $[a,2bh(\frac{1}{2})]$.

(ii) For all $x,y\in[a,b]$,

$$F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right)\le F(x)\le F(a)=F\left(2bh\left(\frac{1}{2}\right)\right). \tag{2.2}$$

Proof. (i) Suppose $x,y\in[a,b]$ and $\lambda\in(0,1)$. Then using the h-convexity of f, one has

$$\begin{aligned}
F(\lambda x+(1-\lambda)y) &= f(\lambda x+(1-\lambda)y)\\
&+ f\left(a+2bh\left(\frac{1}{2}\right)-\lambda x-(1-\lambda)y\right)\\
&\le h(\lambda)f(x)+h(1-\lambda)f(y)+h(\lambda)f\left(a+abh\left(\frac{1}{2}\right)-x\right)\\
&+ h(1-\lambda)f\left(a+2bh\left(\frac{1}{2}\right)-y\right)\\
&= h(\lambda)F(x)+h(1-\lambda)F(y).
\end{aligned}$$

Hence F is an h-convex function.
(ii). With the assumption $h(\lambda)+h(1-\lambda)\le 1$, let us consider

$$F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right) = F\left(\frac{1}{2}\left(a+2bh\left(\frac{1}{2}\right)-x\right)+\frac{1}{2}x\right)$$

$$
\begin{aligned}
&\leq h\left(\frac{1}{2}\right)F(x)+h\left(\frac{1}{2}\right)F\left(a+2bh\left(\frac{1}{2}\right)-x\right)\\
&\leq h\left(\frac{1}{2}\right)F(x)+\left(1-h\left(\frac{1}{2}\right)\right)F\left(a+2bh\left(\frac{1}{2}\right)-x\right)\\
&= F(x)
\end{aligned}
$$

and

$$
\begin{aligned}
F(x) &= F\left(\frac{(x-a)}{2bh\left(\frac{1}{2}\right)-a}\cdot 2bh\left(\frac{1}{2}\right)+\left(\frac{2bh\left(\frac{1}{2}\right)-x}{2bh\left(\frac{1}{2}\right)-a}\right)\cdot a\right)\\
&\leq h\left(\frac{(x-a)}{2bh\left(\frac{1}{2}\right)-a}\right)F\left(2bh\left(\frac{1}{2}\right)\right)+h\left(\frac{2bh\left(\frac{1}{2}\right)-x}{2bh\left(\frac{1}{2}\right)-a}\right)F(a)\\
&\leq h\left(\frac{x-a}{2bh\left(\frac{1}{2}\right)-a}\right)F\left(2bh\left(\frac{1}{2}\right)\right)+\left(1-h\left(\frac{x-a}{2bh\left(\frac{1}{2}\right)-a}\right)\right)F(a)\\
&= F(a)=F\left(2bh\left(\frac{1}{2}\right)\right).
\end{aligned}
$$

□

Remark 2.1. In the above lemma we use $\lambda = \frac{1}{2}$ to overcome ambiguity in a case of using variable from interval $(0, 1)$.

Lemma 2.2. *Let $h : [0, b] \to \mathbb{R}$, $(0, 1) \subseteq [0, b]$ be non negative function such that $h(\lambda) + h(1 - \lambda) \leq 1$ for all $\lambda \in (0, 1)$. Also let $0 \leq a < 2bh\left(\frac{1}{2}\right)$ and $f : [a, 2bh\left(\frac{1}{2}\right)] \to \mathbb{R}$ be h-convex function. Then F which is defined in (2.1) is decreasing on $\left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and increasing on $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2}, 2bh\left(\frac{1}{2}\right)\right]$.*

Proof. Suppose that f is the h-convex function and $x, y \in \left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ such that $x \leq y$, then there exists $\lambda \in (0, 1)$ such that $y = \lambda x + (1 - \lambda)\frac{a+2bh\left(\frac{1}{2}\right)}{2}$. By Lemma 2.1, F is h-convex function, so

$$
\begin{aligned}
F(y) &= F\left(\lambda x+(1-\lambda)\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right)\\
&\leq h(\lambda)F(x)+h(1-\lambda)F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right)\\
&= F(x)+h(1-\lambda)F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right)-(1-h(\lambda))F(x).
\end{aligned}
$$

As $h(\lambda) + h(1 - \lambda) \leq 1$, so

$$
F(y) \leq F(x)+h(1-\lambda)\left(F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right)-F(x)\right).
$$

Now, by using the expression (2.2) of Lemma 2.1(ii), one has $F(y) \leq F(x)$. Then F is decreasing on the interval $\left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$. Now let $x, y \in \left[\frac{a+2bh\left(\frac{1}{2}\right)}{2}, 2bh\left(\frac{1}{2}\right)\right]$ such that $x \leq y$, then there exists $\lambda \in (0,1)$ such that $x = \lambda y + (1-\lambda)\frac{a+2bh\left(\frac{1}{2}\right)}{2}$. By Lemma 2.1, F is h-convex function, so

$$\begin{aligned} F(x) &= F\left(\lambda y + (1-\lambda)\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right) \\ &\leq h(\lambda)F(y) + h(1-\lambda)F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right) \\ &= F(y) + h(1-\lambda)F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right) - (1-h(\lambda))F(y). \end{aligned}$$

As $h(\lambda) + h(1-\lambda) \leq 1$, so

$$F(x) \leq F(y) + h(1-\lambda)\left(F\left(\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right) - F(y)\right).$$

Again by using Lemma 2.1(ii), one can deduce that $F(x) \leq F(y)$. Then F is increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2}, 2bh\left(\frac{1}{2}\right)\right]$. □

Theorem 2.1. *Let $h : [0,b] \to \mathbb{R}$, $(0,1) \subseteq [0,b]$ be non negative function. Also let $0 \leq a < 2bh\left(\frac{1}{2}\right)$ and $f : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}$ be integrable function such that $f(x)$ is decreasing for $x \in \left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and f is symmetric about $\frac{a+2bh\left(\frac{1}{2}\right)}{2}$. Then for every h-convex function φ, one has*

$$\int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx \geq \frac{1}{2bh\left(\frac{1}{2}\right) - a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx.$$

Proof. Let us consider

$$\phi(x) := \varphi(x) + \varphi\left(a + 2bh\left(\frac{1}{2}\right) - x\right). \tag{2.3}$$

Since Lemma 2.1(i) leads us to the fact that ϕ is h-convex on $[a, 2bh(\frac{1}{2})]$, so ϕ is decreasing on $\left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ by Lemma 2.2. Now using this fact along with given condition that f is decreasing on $\left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$, one has the following result due to Chebyshev's inequality given in (1.1):

$$\frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2} - a}\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)\left(\varphi(x) + \varphi\left(a + 2bh\left(\frac{1}{2}\right) - x\right)\right)dx$$

$$\geq \frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2}-a}\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)dx\right)$$

$$\times \frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2}-a}\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi(x)dx + \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi\left((a+2bh\left(\frac{1}{2}\right)-x\right)dx\right).$$

Now by using the symmetry of f about $\frac{a+2bh(\frac{1}{2})}{2}$, one has

$$\frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2}-a}\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\left(\varphi(x)+\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)\right)dx$$

$$\geq \frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2}-a}\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx\right)$$

$$\times \frac{1}{\frac{a+2bh\left(\frac{1}{2}\right)}{2}-a}\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi(x)dx + \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi\left((a+2bh\left(\frac{1}{2}\right)-x\right)dx\right).$$

This implies

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi(x)dx$$

$$+ \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx$$

$$\geq \frac{2}{2bh\left(\frac{1}{2}\right)-a}\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx$$

$$\times \left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi(x)dx + \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx\right). \tag{2.4}$$

By using the identities

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)dx = \frac{1}{2}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx, \tag{2.5}$$

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi(x)dx = \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} \varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx, \tag{2.6}$$

and

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx$$

$$= \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx, \tag{2.7}$$

in inequality (2.4), one can deduce that

$$\begin{aligned}&\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)\varphi(x)dx + \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx\\ \geq\ & \frac{2}{2bh\left(\frac{1}{2}\right)-a}\left(\frac{1}{2}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx\right)\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} \varphi(x)dx + \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx\right).\end{aligned}$$

After simplification, our required result is obtained. □

The following theorem presents the weighted version of Chebyshev's type inequality for h-convex functions.

Theorem 2.2. *Under the assumptions of Theorem 2.1 and in addition if* $p : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}_+$ *be an integrable symmetric function about* $\frac{a+2bh\left(\frac{1}{2}\right)}{2}$, *then one has the inequality*

$$\begin{aligned}&\int_a^{2bh\left(\frac{1}{2}\right)} p(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} p(x)f(x)\varphi(x)dx\\ \geq\ & \int_a^{2bh\left(\frac{1}{2}\right)} p(x)f(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} p(x)\varphi(x)dx.\end{aligned} \tag{2.8}$$

Proof. Its proof is similar to the proof of the previous one by using weighted Chebyshev's inequality given in (1.2). □

Theorem 2.3. *Let* $h : [0, b] \to \mathbb{R}$, $(0, 1) \subseteq [0, b]$ *be a non negative function. Also let* $0 \leq a < 2bh\left(\frac{1}{2}\right)$ *and* $f, \varphi : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}$ *be* h*-convex functions and* $p : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}_+$ *be an integrable symmetric function about* $\frac{a+2bh\left(\frac{1}{2}\right)}{2}$. *Then*

$$\begin{aligned}&\int_a^{2bh\left(\frac{1}{2}\right)} p(x)f(x)\varphi(x)dx + \int_a^{2bh\left(\frac{1}{2}\right)} p(x)f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx\\ \geq\ & \frac{2}{\int_a^{2bh\left(\frac{1}{2}\right)} p(x)dx}\int_a^{2bh\left(\frac{1}{2}\right)} p(x)f(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} p(x)\varphi(x)dx.\end{aligned} \tag{2.9}$$

Proof. Let us assume F and ϕ which are define in (2.1) and (2.3). Since f and φ are h-convex functions, by using Lemma 2.2 one can deduce that F and ϕ have same monotonicity on interval $\left[a, \frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$. By applying Chebyshev's inequality,

$$\begin{aligned}
&\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)F(x)\phi(x)dx \\
\geq\ & \frac{1}{\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)dx} \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)F(x)dx \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)\phi(x)dx.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f(x)\varphi(x)dx \\
+\ &\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\
+\ &\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\
+\ &\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi(x)dx \\
\geq\ & \frac{1}{\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)dx} \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)\left(f(x)+f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\right)dx \\
&\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)\left(\varphi(x)+\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)\right)dx\right). \qquad (2.10)
\end{aligned}$$

Since

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)dx = \frac{1}{2}\int_a^{2bh\left(\frac{1}{2}\right)} p(x)dx. \qquad (2.11)$$

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)dx = \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx. \qquad (2.12)$$

$$\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx$$

$$= \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi(x)dx. \tag{2.13}$$

By using the identities (2.11), (2.12) and (2.13) in inequality (2.10), one has

$$\begin{aligned}
&\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f(x)\varphi(x)dx + \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} p(x)f(x)\varphi(x)dx \\
&+ \int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\
&+ \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} p(x)f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\
&\geq \frac{1}{\int_a^{2bh\left(\frac{1}{2}\right)} p(x)dx}\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)f(x)dx + \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} p(x)f(x)dx\right) \\
&\left(\int_a^{\frac{a+2bh\left(\frac{1}{2}\right)}{2}} p(x)\varphi(x)dx + \int_{\frac{a+2bh\left(\frac{1}{2}\right)}{2}}^{2bh\left(\frac{1}{2}\right)} p(x)\varphi(x)dx\right).
\end{aligned}$$

It follows that (2.9) holds and the proof of Theorem 2.3 is completed. □

The following result is given in [Theorem 1.1, [11]].

Corollary 2.1. *Let $f, \varphi : [a, b] \to \mathbb{R}$ be convex functions and $p : [a, b] \to \mathbb{R}_+$ be integrable symmetric function about $\frac{a+b}{2}$. Then*

$$\begin{aligned}
&\int_a^b p(x)f(x)\varphi(x)dx + \int_a^b p(x)f(x)\varphi(a+b-x)dx \\
&\geq \frac{2}{\int_a^b p(x)dx}\int_a^b p(x)f(x)dx \int_a^b p(x)\varphi(x)dx. \qquad (2.14)
\end{aligned}$$

Proof. By putting $h\left(\frac{1}{2}\right) = \frac{1}{2}$ in (2.9), above result can be obtained. □

Remark 2.2. One can note that inequality (2.8) is valid, when the function f is decreasing, symmetric and φ is h-convex. Imposing the condition of symmetry on φ along with that f and φ are h-convex functions, one can get inequality (2.8) of Theorem 2.3.

Generalizations of Chebyshev's type inequality are valid for similar ordered functions. The first result of this kind was given by Hardy, Littlwood and Polya (see [8], p. 168).
For h-convex function, the following lemma is very helpful to establish a refinement of Chebyshev's type inequality for similar ordered functions as well as oppositely ordered functions.

Lemma 2.3. *Let $f, \varphi : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}$ be two integrable functions. If f and φ are similarly ordered, then*

$$\int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx \geq \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx. \tag{2.15}$$

If f and φ are oppositely ordered, then above inequality is reversed.

Proof. Since f and φ are similarly ordered, then for all $x \in \left[a, 2bh\left(\frac{1}{2}\right)\right]$

$$\left(f(x) - f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\right)\left(\varphi(x) - \varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)\right) \geq 0,$$

which implies that

$$\begin{aligned} & f(x)\varphi(x) + f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right) \\ \geq\; & f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right) + f\left(a+2bh\left(\frac{1}{2}\right)-x\right)\varphi(x). \end{aligned}$$

By integrating both sides over $\left[a, 2bh\left(\frac{1}{2}\right)\right]$, the above inequality (2.15) can be obtained which is our required result. □

Theorem 2.4. *Let $h : [0, b] \to \mathbb{R}$, $(0, 1) \subseteq [0, b]$ be a non negative function. Also let $0 \leq a < 2bh\left(\frac{1}{2}\right)$ and $f, \varphi : \left[a, 2bh\left(\frac{1}{2}\right)\right] \to \mathbb{R}$ be h-convex functions.*

(i) If f and φ are similarly ordered, then

$$\begin{aligned} & \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx \\ \geq\; & \frac{1}{2}\left(\int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx + \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx\right) \\ \geq\; & \frac{1}{2bh\left(\frac{1}{2}\right)-a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx. \end{aligned} \tag{2.16}$$

(ii) If f and φ are oppositely ordered, then

$$\begin{aligned} & \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\ \geq\; & \frac{1}{2bh\left(\frac{1}{2}\right)-a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx \int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx \\ \geq\; & \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx. \end{aligned} \tag{2.17}$$

Proof. (i) Since f and φ are h-convex functions and similarly ordered, then by using Lemma 2.3 one has

$$\int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx \geq \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx,$$

which can be written as

$$\begin{aligned} & 2\int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx \\ \geq & \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx + \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx. \end{aligned} \tag{2.18}$$

By using Theorem 2.3 and (2.18), one can obtain (2.16).
(ii) Since f and φ are h-convex functions, then by Theorem 2.3

$$\begin{aligned} & \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi\left(a+2bh\left(\frac{1}{2}\right)-x\right)dx \\ - & \frac{1}{2bh\left(\frac{1}{2}\right)-a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx\int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx \\ \geq & \frac{1}{2bh\left(\frac{1}{2}\right)-a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx\int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx - \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx. \end{aligned} \tag{2.19}$$

On the other hand, one has

$$\frac{1}{2bh\left(\frac{1}{2}\right)-a}\int_a^{2bh\left(\frac{1}{2}\right)} f(x)dx\int_a^{2bh\left(\frac{1}{2}\right)} \varphi(x)dx \geq \int_a^{2bh\left(\frac{1}{2}\right)} f(x)\varphi(x)dx, \tag{2.20}$$

because f and φ are oppositely ordered. By (2.19) and (2.20), one can obtain (2.17). □

The following result is given in [Theorem 1.2, [11]].

Corollary 2.2. *Let $f, \varphi : [a,b] \to \mathbb{R}$ be convex functions.*

(i) If f and φ are similarly ordered, then

$$\begin{aligned} \int_a^b f(x)\varphi(x)dx & \geq \frac{1}{2}\left(\int_a^b f(x)\varphi(x)dx + \int_a^b f(x)\varphi(a+b-x)dx\right) \\ & \geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b \varphi(x)dx. \end{aligned}$$

(ii) If f and φ are oppositely ordered, then

$$\begin{aligned}\int_a^b f(x)\varphi(a+b-x)dx &\geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b \varphi(x)dx\\ &\geq \int_a^b f(x)\varphi(x)dx.\end{aligned}$$

Proof. By putting $h\left(\frac{1}{2}\right)=\frac{1}{2}$ in (2.16) and (2.17), the above results can be obtained. □

Theorem 2.5. *Let $h:[0,b]\to\mathbb{R}$, $(0,1)\subseteq[0,b]$ be a non negative function. Also let $0\leq a<2bh\left(\frac{1}{2}\right)$ and $f,\varphi:\left[a,2bh\left(\frac{1}{2}\right)\right]\to\mathbb{R}$, where f is h-convex function and φ is decreasing on the interval $\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$. Then the inequality (2.9) holds.*

Proof. Let us assume F and ϕ which are define in (2.1) and (2.3). Since f is h-convex function then by Lemma 2.2, F is decreasing on the interval $\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$. In order to prove that inequality (2.9), we need to prove ϕ is decreasing on the interval $\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$.

Let $x,y\in\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and set $x^*=a+2bh\left(\frac{1}{2}\right)-x$ and $y^*=a+2bh\left(\frac{1}{2}\right)-y$, where $x^*,y^*\in\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$. It is clear that if $x\leq y$, then $x^*\geq y^*$. Since φ is decreasing on the interval $\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$ and increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$, then one has

$$\varphi(x)\geq\varphi(y)\ \ \textit{and}\ \ \varphi(x^*)\geq\varphi(y^*).$$

Then

$$\phi(x)=\varphi(x)+\varphi(x^*)\geq\varphi(y)+\varphi(y^*)=\phi(y)$$

which implies that ϕ is decreasing on the interval $\left[a,\frac{a+2bh\left(\frac{1}{2}\right)}{2}\right]$. By the same method one can prove that ϕ is increasing on the interval $\left[\frac{a+2bh\left(\frac{1}{2}\right)}{2},2bh\left(\frac{1}{2}\right)\right]$. Then F and ϕ have same monotonicity and by applying Chebyshev's inequality with

$p : [a, 2bh\left(\frac{1}{2}\right) \to \mathbb{R}_+]$ is integrable symmetric function about $\frac{a+2bh\left(\frac{1}{2}\right)}{2}$ inequality (2.9) can be obtained. □

The following result is given in [[11], Theorem 1.3].

Corollary 2.3. *Let $f, \varphi : [a, b] \to \mathbb{R}$ where f is convex function and φ is decreasing on $[a, \frac{a+b}{2}]$ and increasing on $[\frac{a+b}{2}, b]$. Then (2.14) hold.*

Proof. By putting $h\left(\frac{1}{2}\right) = \frac{1}{2}$ in (2.9), the required inequality (2.14) can be obtained. □

3. Mean value theorems

Lemma 3.1. *[22] Let $h : [0, b] \to \mathbb{R}^+$ be supermultiplicative such that $h(\lambda) + h(1-\lambda) \leq 1$ for all $\lambda \in (0, 1)$. If $f : [0, b] \to \mathbb{R}$ is h-convex, then $\frac{f(x)-f(a)}{h(x-a)}$ is increasing for $x > a$.*

Proof. Suppose f is an h-convex function and

$$P_h(x) = \frac{f(x) - f(a)}{h(x - a)}.$$

We take $y > x > a$ and $x = \lambda y + (1 - \lambda)a$, then

$$\begin{aligned} P_h(x) &= \frac{f(\lambda y + (1 - \lambda)a) - f(a)}{h(\lambda y + (1 - \lambda)a - a)} \\ &\leq \frac{h(\lambda)f(y) + [h(1 - \lambda) - 1]f(a)}{h(\lambda(y - a))}. \end{aligned}$$

Using the fact that h is supermultiplicative, one has

$$P_h(x) \leq \frac{h(\lambda)f(y) + [h(1 - \lambda) - 1]f(a)}{h(\lambda)h(y - a)}.$$

Since $h(1 - \lambda) - 1 \leq - \leq h(\lambda)$, this implies

$$P_h(x) \leq \frac{f(y)}{h(y - a)} - \frac{f(a)}{h(y - a)} = P_h(y).$$

Hence we have have proved that if f is h-convex then $\frac{f(x)-f(a)}{h(x-a)}$ is increasing for $x > a$. □

The following Lemma is very helpful in proving mean value theorem related to the non negative functional of Chebyshev's type inequality for h-convex functions.

Lemma 3.2. *Let $h : [0,b] \to \mathbb{R}$, $(0,1) \subseteq [0,b]$ be a non negative differentiable function. Also let $\phi : [a,b] \to \mathbb{R}$ be a differentiable function, where $0 \le a < 2bh\left(\frac{1}{2}\right)$ such that*

$$m_1 \le \frac{h(x-a)\phi'(x) - (\phi(x)-\phi(a))h'(x-a)}{h^2(x-a)} \le M_1,$$

for all $x, a \in [a,b]$. Then the functions

$$\psi_1(x) = M_1 x h(x-a) - \phi(x), \quad \psi_2(x) = \phi(x) - m_1 x h(x-a)$$

are h-convex in $[a,b]$.

Proof. Suppose

$$P_{h,\psi_1}(x) = \frac{\psi_1(x)-\psi_1(a)}{h(x-a)} = \frac{M_1 x h(x-a)}{h(x-a)} - \frac{\phi(x)-\phi(a)}{h(x-a)}.$$

So we have

$$P'_{h,\psi_1}(x) = M_1 - \frac{h(x-a)\phi'(x) - (\phi(x)-\phi(a))h'(x-a)}{h^2(x-a)}.$$

By the given condition, one has

$$P'_{h,\psi_1}(x) \ge 0 \quad \textit{for all} \quad x \in [a,b].$$

Similarly one can show that

$$P'_{h,\psi_2}(x) \ge 0 \quad \textit{for all} \quad x \in [a,b].$$

This gives us P_{h,ψ_1} and P_{h,ψ_2} are increasing on $x \in [a,b]$. Hence by Lemma 3.1 ψ_1 and ψ_2 are h-convex in $[a,b]$. □

Theorem 3.1. *Let $h : [0,b] \to \mathbb{R}$, $(0,1) \subseteq [0,b]$ be a non negative differentiable function. Also let $0 \le a < 2bh\left(\frac{1}{2}\right)$ and $f : [a, 2bh(\frac{1}{2})] \to \mathbb{R}$ be an integrable function such that the following two conditions hold:*

(i) f is decreasing on $[a, \frac{a+2bh(\frac{1}{2})}{2}]$.

(ii) f is symmetric about $\frac{a+2bh(\frac{1}{2})}{2}$.

If $\phi, h \in C^1[a, 2bh(\frac{1}{2})]$ then there exists $\xi \in (a, 2bh(\frac{1}{2}))$ such that

$$T_h(f,\phi) = \frac{h(\xi-a)\phi'(\xi) - (\phi(\xi)-\phi(a))h'(\xi-a)}{h^2(\xi-a)} T_h(f,\gamma),$$

provided that $T_h(f,\gamma)$ is non-zero, where $\gamma(x) = x^2$.

Proof. As $\phi, h \in C^1[a, 2bh(\frac{1}{2})]$, so there exist real numbers m_1 and M_1 such that

$$m_1 \leq \frac{h(x-a)\phi'(x) - (\phi(x) - \phi(a))h'(x-a)}{h^2(x-a)} \leq M_1,$$

for each $x \in [a, 2bh(\frac{1}{2})]$.
Now let us consider the function ψ_1 and ψ_2 defined in Lemma 3.2. As ψ_1 is h-convex in $[a, 2bh(\frac{1}{2})]$,

$$T_h(f, \psi_1) \geq 0,$$

that is

$$T_h(f, M_1 x h(x-a) - \phi(x)) \geq 0,$$

which gives

$$M_1 T_h(f, \gamma) \geq T_h(f, \phi). \tag{3.1}$$

Similarly ψ_2 is h-convex in $[a, 2bh(\frac{1}{2})]$, therefore one has

$$m_1 T_h(f, \gamma) \leq T_h(f, \phi). \tag{3.2}$$

By the assumption $T_h(f, \gamma) \neq 0$, combining (3.1) and (3.2) one has

$$m_1 \leq \frac{T_h(f, \phi)}{T_h(f, \gamma)} \leq M_1.$$

Hence, there exist $\xi \in [a, 2bh(\frac{1}{2})]$ such that

$$\frac{T_h(f, \phi)}{T_h(f, \gamma)} = \frac{h(\xi-a)\phi'(\xi) - (\phi(\xi) - \phi(a))h'(\xi-a)}{h^2(\xi-a)}. \tag{3.3}$$

Hence proved required result. □

Corollary 3.1. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function such that the following two conditions hold:*

(i) f is decreasing on $[a, \frac{a+b}{2}]$.

(ii) f is symmetric about $\frac{a+b}{2}$.

If $\phi \in C^1[a, b]$, then there exists $\xi \in (a, b)$ such that

$$T(f, \phi) = \frac{(\xi-a)\phi'(\xi) - \phi(\xi) + \phi(a)}{(\xi-a)^2} T(f, \gamma),$$

provided that $T(f, \gamma)$ is non zero, where $\gamma(x) = x^2$.

Proof. By putting $h(\xi - a) = \xi - a$ in (3.3), above result can be obtained. □

Theorem 3.2. *Let $h : [0, b] \to \mathbb{R}$, $(0, 1) \subseteq [0, b]$ be a non negative differentiable function. Also let $0 \le a < 2bh\left(\frac{1}{2}\right)$ and $f : [a, 2bh(\frac{1}{2})] \to \mathbb{R}$ be an integrable function such that the following two conditions hold:*

(i) f is decreasing on $[a, \frac{a+2bh(\frac{1}{2})}{2}]$.

(ii) f is symmetric about $\frac{a+2bh(\frac{1}{2})}{2}$.

If $\phi_1, \phi_2, h \in C^1[a, 2bh(\frac{1}{2})]$, then there exist $\xi \in (a, 2bh(\frac{1}{2}))$ such that

$$\frac{T_h(f, \phi_1)}{T_h(f, \phi_2)} = \frac{h(\xi - a)\phi_1'(\xi) - (\phi_1(\xi) - \phi_1(a))h'(\xi - a)}{h(\xi - a)\phi_2'(\xi) - (\phi_2(\xi) - \phi_2(a))h'(\xi - a)}, \tag{3.4}$$

provided that the denominators are non zero, where $\gamma(x) = x^2$.

Proof. Suppose that a function $p \in C^1[a, 2bh(\frac{1}{2})]$ be defined as:

$$p = c_1\phi_1 - c_2\phi_2,$$

where

$$c_1 = T_h(f, \phi_2), \quad c_2 = T_h(f, \phi_1).$$

Then using Theorem 3.1 with $\phi = p$, one has

$$h(\xi - a)(c_1\phi_1 - c_2\phi_2)'(\xi) - ((c_1\phi_1 - c_2\phi_2)(\xi) - (c_1\phi_1 - c_2\phi_2)(a))h'(\xi - a) = 0,$$

that is

$$h(\xi - a)(c_1\phi_1'(\xi) - c_2\phi_2'(\xi)) - (c_1\phi_1(\xi) - c_2\phi_2(\xi) - c_1\phi_1(a) + c_2\phi_2(a))h'(\xi - a) = 0,$$

which gives

$$c_1(h(\xi-a)\phi_1'(\xi)-(\phi_1(\xi)-\phi_1(a))h'(\xi-a)-c_2(h(\xi-a)\phi_2'(\xi)-(\phi_2(\xi)-\phi_2(a))h'(\xi-a) = 0,$$

which implies

$$c_1(h(\xi-a)\phi_1'(\xi)-(\phi_1(\xi)-\phi_1(a))h'(\xi-a) = c_2(h(\xi-a)\phi_2'(\xi)-(\phi_2(\xi)-\phi_2(a))h'(\xi-a)$$

and

$$\frac{c_2}{c_1} = \frac{h(\xi - a)\phi_1'(\xi) - (\phi_1(\xi) - \phi_1(a))h'(\xi - a)}{h(\xi - a)\phi_2'(\xi) - (\phi_2(\xi) - \phi_2(a))h'(\xi - a)}.$$

After putting value of c_1 and c_2, we get (3.4). □

Corollary 3.2. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function such that the following two conditions hold:*

(i) f is decreasing on $[a, \frac{a+b}{2}]$.

(ii) f is symmetric about $\frac{a+b}{2}$.

If $\phi_1, \phi_2, h \in C^1[a,b]$, then there exist $\xi \in (a,b)$ such that

$$\frac{T(f,\phi_1)}{T(f,\phi_2)} = \frac{(\xi-a)\phi_1'(\xi) - \phi_1(\xi) + \phi_1(a)}{(\xi-a)\phi_2'(\xi) - \phi_2(\xi) + \phi_2(a)},$$

provided that the denominators are non zero, where $\gamma(x) = x^2$.

Proof. Above result can be obtained by taking $h(\xi - a) = \xi - a$ in (3.4). □

4. Results for s-convex function

The following results hold for s-convex functions:

Theorem 4.1. *Let s be a real number, $s \in (0,1]$ and $\alpha, \beta \geq 0$. Also let $0 \leq a < b\alpha^{1-s}$ and $f : [a, b\alpha^{1-s}] \to \mathbb{R}$ be an integrable function such that f is decreasing for $x \in \left[a, \frac{a+b\alpha^{1-s}}{2}\right]$ and f is symmetric about $\frac{a+b\alpha^{1-s}}{2}$. Then for every s-convex function in first sense φ, one has*

$$\int_a^{b\alpha^{1-s}} f(x)\varphi(x)dx \geq \frac{1}{b\alpha^{1-s} - a} \int_a^{b\alpha^{1-s}} f(x)dx \int_a^{b\alpha^{1-s}} \varphi(x)dx.$$

Proof. The proof of above theorem is similar to the proof of Theorem 2.1 by taking $h(\alpha) = \alpha^s$. □

Theorem 4.2. *Under the assumptions of Theorem 4.1 and in addition if $p : \left[a, b\alpha^{1-s}\right] \to \mathbb{R}_+$ be integrable symmetric function about $\frac{a+b\alpha^{1-s}}{2}$, then one has the inequality*

$$\begin{aligned} &\int_a^{b\alpha^{1-s}} p(x)dx \int_a^{b\alpha^{1-s}} p(x)f(x)\varphi(x)dx \\ &\geq \int_a^{b\alpha^{1-s}} p(x)f(x)dx \int_a^{b\alpha^{1-s}} p(x)\varphi(x)dx. \end{aligned} \tag{4.1}$$

Proof. The proof of above theorem is similar to the proof of Theorem 2.2 by taking $h(\alpha) = \alpha^s$. □

Theorem 4.3. *Let s be a real number, $s \in (0,1]$. Also let $0 \leq a < b\alpha^{1-s}$ and $f, \varphi : \left[a, b\alpha^{1-s}\right] \to \mathbb{R}$ be s-convex functions in first sense and $p : \left[a, b\alpha^{1-s}\right] \to \mathbb{R}_+$ be an integrable symmetric function about $\frac{a+b\alpha^{1-s}}{2}$. Then*

$$\begin{aligned} &\int_a^{b\alpha^{1-s}} p(x)f(x)\varphi(x)dx + \int_a^{b\alpha^{1-s}} p(x)f(x)\varphi\left(a + b\alpha^{1-s} - x\right)dx \\ &\geq \frac{2}{\int_a^{b\alpha^{1-s}} p(x)dx} \int_a^{b\alpha^{1-s}} p(x)f(x)dx \int_a^{b\alpha^{1-s}} p(x)\varphi(x)dx. \end{aligned} \tag{4.2}$$

Proof. The proof of the above theorem is similar to the proof of Theorem 2.3 by taking $h(\alpha) = \alpha^s$. $\square$

Theorem 4.4. *Let s be a real number, $s \in (0,1]$. Also let $0 \leq a < b\alpha^{1-s}$ and $f, \varphi : [a, b\alpha^{1-s}] \to \mathbb{R}$ be s-convex functions in first sense.*

(i) If f and φ are similarly ordered, then

$$\begin{aligned}
& \int_a^{b\alpha^{1-s}} f(x)\varphi(x)dx \\
\geq\ & \frac{1}{2}\left(\int_a^{b\alpha^{1-s}} f(x)\varphi\left(a + b\alpha^{1-s} - x\right)dx + \int_a^{b\alpha^{1-s}} f(x)\varphi(x)dx\right) \\
\geq\ & \frac{1}{b\alpha^{1-s} - a}\int_a^{b\alpha^{1-s}} f(x)dx \int_a^{b\alpha^{1-s}} \varphi(x)dx.
\end{aligned}$$

(ii) If f and φ are oppositely ordered, then

$$\begin{aligned}
& \int_a^{b\alpha^{1-s}} f(x)\varphi\left(a + b\alpha^{1-s} - x\right)dx \\
\geq\ & \frac{1}{b\alpha^{1-s} - a}\int_a^{b\alpha^{1-s}} f(x)dx \int_a^{b\alpha^{1-s}} \varphi(x)dx \\
\geq\ & \int_a^{b\alpha^{1-s}} f(x)\varphi(x)dx.
\end{aligned}$$

Proof. The proof of the above theorem is similar to the proof of Theorem 2.4 by taking $h(\alpha) = \alpha^s$. $\square$

Theorem 4.5. *Let s be a real number, $s \in (0,1]$ and $\alpha, \beta \geq 0$. Also let $0 \leq a < b\alpha^{1-s}$ and $f, \varphi : [a, b\alpha^{1-s}] \to \mathbb{R}$, where f is s-convex function in first sense and φ is decreasing on the interval $\left[a, \frac{a+b\alpha^{1-s}}{2}\right]$ and increasing on the interval $\left[\frac{a+b\alpha^{1-s}}{2}, b\alpha^{1-s}\right]$. Then the inequality (4.2) holds.*

Proof. The proof of the above theorem is similar to the proof of Theorem 2.5 by taking $h(\alpha) = \alpha^s$. $\square$

Theorem 4.6. *Let s be a real number, $s \in (0,1]$ and $\alpha, \beta \geq 0$. Also let $0 \leq a < b\alpha^{1-s}$ and $f : [a, b\alpha^{1-s}] \to \mathbb{R}$ be integrable function such that the following two conditions hold:*

(i) f is decreasing for $[a, \frac{a+b\alpha^{1-s}}{2}]$.

(ii) f *is symmetric about* $\frac{a+b\alpha^{1-s}}{2}$.

If $\phi \in C^1[a, b\alpha^{1-s}]$ *then there exist* $\xi \in (a, b\alpha^{1-s})$ *such that*

$$T_s(f, \phi) = \frac{(\xi - a)^s \phi'(\xi) - s(\phi(\xi) - \phi(a))(\xi - a)^{s-1}}{(\xi - a)^{2s}} T_s(f, \gamma),$$

provided that $T_h(f, \gamma)$ *is non-zero, where* $\gamma(x) = x^2$.

Proof. The proof of the above theorem is similar to the proof of Theorem 3.1 by taking $h(\alpha) = \alpha^s$. □

Theorem 4.7. *Let* s *be a real number,* $s \in (0, 1]$ *and* $\alpha, \beta \geq 0$. *Also let* $0 \leq a < b\alpha^{1-s}$ *and* $f : [a, b\alpha^{1-s}] \to \mathbb{R}$ *be integrable function such that the following two conditions hold:*

(i) f *is decreasing on* $[a, \frac{a+b\alpha^{1-s}}{2}]$.

(ii) f *is symmetric about* $\frac{a+b\alpha^{1-s}}{2}$.

If $\phi_1, \phi_2 \in C^1[a, b\alpha^{1-s}]$, *then there exist* $\xi \in (a, b\alpha^{1-s})$ *such that*

$$\frac{T_s(f, \phi_1)}{T_s(f, \phi_2)} = \frac{(\xi - a)^s \phi_1'(\xi) - s(\phi_1(\xi) - \phi_1(a))(\xi - a)^{s-1}}{(\xi - a)^s \phi_2'(\xi) - s(\phi_2(\xi) - \phi_2(a))(\xi - a)^{s-1}},$$

provided that the denominators are non zero, where $\gamma(x) = x^2$.

Proof. The proof of the above theorem is similar to the proof of Theorem 3.2 by taking $h(\alpha) = \alpha^s$. □

5. Concluding remarks

This research article have been prepared to extrapolate Chebyshev's type inequalities. By using h-convex functions, Chebyshev's type inequality, weighted version of Chebyshev's type inequality and a refinement of Chebyshev's type inequality for similar ordered functions as well as oppositely ordered functions have been established. Furthermore, the associated Chebyshev's functional are estimated via mean value theorems. Also we discussed several results for s-convex functions which are special cases of proved results.

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