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ON CONFORMALLY BERWALD M-TH ROOT (α, β)-METRICS

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Abstract. In this paper, we study the class of mth root (α, β)-metrics which is a significant class mixed of two classes of metrics: m-th root metrics and (α, β)-metrics. First, we find the necessary and sufficient condition under which the quartic (α, β)-metrics are conformally Berwald. Then, we find the necessary and sufficient condition under which the cubic (α, β)-metrics are conformally Berwald. Finally, we construct some conformal Finslerian invariants.
Keywords: (α, β)-metrics; Finslerian invariants; conformally Berwald metrics; Riemannian metrics.

1. Introduction

The conformal transformations of the class of Riemannian metrics have been well investigated and developed. The class of Finsler metrics are a natural generalization of the class of Riemannian metrics. The conformal transformation of Finsler metrics was initiated by Knebelman in [10] and studied by Hashiguchi in [4]. Let F and $\bar{F}$ be two Finsler metrics on a manifold M. In [4], Hashiguchi proved that F is conformal to $\bar{F}$ if and only if there exists a scalar function $\kappa = \kappa(x)$ such that $\bar{F} = e^{\kappa} F$. The scalar function κ is called the conformal factor. A Finsler metric is called a conformally flat metric if it is locally conformal to a locally Minkowski metric [26]. There are many efforts to find a conformally invariant curvature tensor similar to the Weyl conformal curvature of a Riemannian metric and to establish the condition for a Finsler metric to be conformally flat. In [20], Szilasi-Vincze gave an intrinsic proof of the Weyl theorem, which states that the projective and conformal properties of a Finsler metric determine its metric properties uniquely. Therefore the conformal properties of Finsler metrics deserve extra attention.

A Berwald metric is much closer to a Riemannian metric than the other class of Finsler metrics because any geodesic of a Berwald metric must be that of a Riemannian metric [17]. A Finsler metric F on a manifold M is said to be a Berwald metric

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if there exists a torsion-free affine connection ∇ on M whose parallel transport preserves F, namely, if $c = c(t)$ is a smooth path in M with the endpoints x_1 and x_2, and $P_c : T_{x_1}M \to T_{x_2}M$ is the ∇-parallel transport along c, then for all $y \in T_xM$, $F_{x_2}\big(P_c(y)\big) = F_{x_1}(y)$ holds. Thus a Riemannian metric viewed as a special Berwald metric, with the associated connection ∇ the Levi-Civita connection.

A Finsler metric conformally related to a Berwald metric is called conformally Berwald metric. In [6], Hashiguchi-Ichijyō proved that a Finsler metric $F = F(x, y)$ on a manifold M is conformal to a Berwald metric if and only if it is a Wagner metric (see also [28]). The Wagner metrics form an important class of the so-called generalized Berwald metrics admitting Finsler connections whose horizontal part depends only on the position - more precisely there exists a linear connection on M such that the indicatrix hypersurfaces are invariant under the parallel transport. Also, Berwald metrics in the classical sense are characterized by a similar property of the canonical Berwald connection. If a Berwald metric has vanishing Riemannian curvature, then it is called a locally Minkowski metric. In [8], Hashiguchi-Ichijyō determined all conformally flat Randers surfaces. Then, Hashiguchi proved that a conformally flat Randers metric is conformally Berwald metric and the associated Riemannian metric is also conformally flat [5]. He also studied the converse problem. In [1], Aikou obtained the conditions for a Finsler metric to be locally or globally conformal to a Berwald metric. In [7], Hōjō-Matsumoto-Okubo found the necessary and sufficient conditions under which a Randers metric and Kropina metric be a conformally Berwald metric. In [27], Vincze discussed the problem whether how we can check the conformality of a Finsler metric to a Berwald metric. His method is based on a differential 1-form constructing on the underlying manifold by the help of integral formulas such that its exterior derivative is conformally invariant. If the Finsler metric is conformal to a Berwald metric, then the exterior derivative vanishes [27]. In [15], Matveev-Nikolayevsky obtained some results regarding locally conformally Berwald closed metrics that are not globally conformally Berwald. In [30], Xia-Zhong found some explicit examples of complex Berwald metrics which are neither Hermitian metrics nor conformal changes of complex Minkowski metrics.

In order to find explicit examples of conformally Berwald metrics, one can investigate the class of m-th root Finsler metrics. Let M be an n-dimensional manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F : TM \to \mathbb{R}$ be a scalar function defined by $F = \sqrt[m]{A}$, where $A := \mathfrak{a}_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ and $\mathfrak{a}_{i_1\dots i_m}$ is symmetric in all its indices. Then F is called an m-th root Finsler metric on M [19]. For more progress, see [21], [24] and [25]. The fourth root metric is called a quartic metric [22][23]. The significant quartic metric $F = \sqrt[4]{y^iy^jy^ky^l}$ is called Berwald-Moór metric which has important role in the theory of space-time structure and gravitation as well as in unified gauge field theories [2][3][16].

We show that every 4-th root metric $F = \sqrt[4]{\mathfrak{a}_{ijkl}(x)y^iy^jy^ky^l}$ on a manifold M of dimension $n \geq 3$ can be written in the following form

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta=b_i(x)y^i$ is a 1-form on M. For $n=2$, F can be written as $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2}$. Then, we characterize conformally Berwald 4-th root (α,β)-metric as follows.

Theorem 1.1. *Let $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2+c_3\beta^4}$ be a non-Riemanian quartic (α,β)-metric on an n-dimensional manifold M, where c_i are nonzero constants. Then F is a conformally Berwald metric if and only if β satisfies following*

$$r_{ij}=\frac{r^s_s}{n-1}\Big(a_{ij}-\frac{1}{b^2}b_ib_j\Big)-\frac{1}{b^2}\Big(b_is_j+b_js_i\Big), \tag{1.1}$$

$$s_{ij}=\frac{1}{b^2}\Big(b_is_j-b_js_i\Big) \tag{1.2}$$

and the conformal factor $\kappa=\kappa(x)$ satisfies

$$\kappa_i=-\frac{1}{b^2}\Big(2s_i+\frac{1}{n-1}r^s_sb_i\Big), \tag{1.3}$$

where $\kappa_i:=\partial\kappa/\partial x^i$ and $b:=||\beta||_\alpha=\sqrt{a^{ij}b_ib_j}$.

Suppose that the quartic (α,β)-metric $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2+c_3\beta^4}$ is a Berwald metric. Then by Lemma 2.3, β is parallel with respect to α. Therefore $r_{ij}=s_{ij}=0$ and F satisfies (1.1) and (1.2). In this case, (1.3) implies that $\kappa=constant$. Thus, we conclude the following.

Corollary 1.1. *Let $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2+c_3\beta^4}$ be a non-Riemannian Berwald quartic (α,β)-metric. Then F is a conformally Berwald metric if and only if the conformal transformation is homothetic.*

It is remarkable that, the Corollary 1.1 confirms the Vincze's theorem in [27] that say a conformal transformation between two non-Riemannian Berwald metrics must be a homothety.

By the same argument used in proof of Theorem 1.1, one can get the following result.

Corollary 1.2. *Let $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2}$ be a non-Riemanian quartic (α,β)-metric on an n-dimensional manifold M, where c_i are nonzero constants. Then F is a conformally Berwald metric if and only if β satisfies (1.1) and (1.2) and the conformal factor $\kappa=\kappa(x)$ satisfies (1.3).*

The third root metric $F=\sqrt[3]{\mathfrak{a}_{ijk}(x)y^iy^jy^k}$ is called the cubic metric. In [29], Wegener studied cubic Finsler metrics of dimensions two and three. Wegener's paper is only an abstract of his PhD thesis without all details and calculations. In [12], Matsumoto wrote an improved version of Wegener's results. In [13], Matsumoto-Numata proved that every cubic (α,β)-metric on a manifold M of dimension $n \geq 3$ can be written in the following form

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}.$$

For $n = 2$, they showed that F is given by $F = \sqrt[3]{\alpha^2\beta}$. In this paper, we prove the following.

Theorem 1.2. *Let (M, F) be an n-dimensional Finsler manifold. Then the following hold:*

(i) The cubic (α,β)-metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ is a conformally Berwald metric if and only if β satisfies

$$r_{ij} = \frac{1}{b^2}(b_j r_i + b_i r_j) - b^r \bar{f}_r(c_1 a_{ij} + 3c_2 b_i b_j) - a_{ij} b^r k_r, \tag{1.4}$$

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{1.5}$$

and the conformal factor $\kappa = \kappa(x)$ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - u b_j) - 2(2c_1 + 3c_2 b^2)\bar{f}_j, \tag{1.6}$$

where c_1 and c_2 are nonzero constants, $\kappa_r := \partial\kappa/\partial x^r$, $f := b_i\kappa_j a^{ij}$, $f_i := \partial f/\partial x^i$, and

$$u := \frac{1}{2}(2c_1\bar{f}_r - \kappa_r)b^r, \quad \bar{f}_j := \frac{1}{3b^2(c_1 + c_2 b^2)}(s_j + r_j).$$

(ii) The cubic (α,β)-metric $F = \sqrt[3]{\alpha^2\beta}$ is a conformally Berwald metric if and only if β satisfies

$$r_{ij} = \frac{1}{b^2}(b_j r_i + b_i r_j) - b^r(\kappa_r + \frac{1}{3}\bar{f}_r)a_{ij} - \frac{2h}{b^2} b_i b_j, \tag{1.7}$$

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{1.8}$$

and the conformal factor $\kappa = \kappa(x)$ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - h b_j) - \frac{4}{3}\bar{f}_j, \tag{1.9}$$

where

$$h := \frac{1}{6}(2\bar{f}_r - 3\kappa_r)b^r, \qquad \bar{f}_j = \frac{1}{b^2}(s_j + r_j).$$

2. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark some notions about an (α,β)-metric. An (α,β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a scalar function on an open interval $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The metric α is called the associated Riemannian metric of the (α,β)-metric F. Throughout this paper, we assume that the associated Riemannian metric of an (α,β)-metric is positive-definite.

For an (α,β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, one can define $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\{\theta_i^j := \gamma^j_{ik}(x)dx^k\}$ denote the Levi-Civita connection forms of the Riemannian metric α. Let us put

$$\begin{gathered}
r_{ij} := \tfrac{1}{2}\big(b_{i|j} + b_{j|i}\big), \quad s_{ij} := \tfrac{1}{2}\big(b_{i|j} - b_{j|i}\big),\\
r_j := b^i r_{ij}, \quad r := b^ib^j r_{ij}, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j,\\
r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j, \quad s^i{}_j := a^{im}s_{mj}, \quad r^i{}_j := a^{im}r_{mj}.
\end{gathered}$$

Then β is parallel with respect to α if and only if $b_{i|j} = 0$ or equivalently $r_{ij} = s_{ij} = 0$.

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. Assume that F is conformally related to a Finsler metric $\bar{F}$ on M, that is, there is a scalar function $\kappa = \kappa(x)$ on M such that $\bar{F} = e^{\kappa(x)}F$. It is easy to see that $\bar{F} = \bar{\alpha}\phi(\bar{\beta}/\bar{\alpha})$ is also an (α,β)-metric, where $\bar{\alpha} = e^{\kappa(x)}\alpha$ and $\bar{\beta} = e^{\kappa(x)}\beta$. Put $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta} = \bar{b}_i(x)y^i$. Let us define

$$b := \|\beta_x\|_\alpha = \sqrt{a^{ij}b_ib_j}, \quad \bar{b} := \|\bar{\beta}_x\|_{\bar{\alpha}} = \sqrt{\bar{a}^{ij}\bar{b}_i\bar{b}_j}.$$

Thus

$$b = \bar{b}. \tag{2.1}$$

Let (M,F) be a Finsler manifold. A global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$, where $G^i = G^i(x,y)$ are given by

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

The vector field $\mathbf{G}$ is called the associated spray to (M,F). F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ is quadratic in $y \in T_xM$ for any $x \in M$. Then (M,F) is called a Berwald manifold. The important described characteristic of a Berwald manifold is that all its tangent spaces are linearly isometric to a common Minkowski space [18].

In order to prove Theorem 1.1, we need the following.

Lemma 2.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F is conformally related to a Finsler metric $\bar{F}$ on M, i.e., $\bar{F} = e^{\kappa(x)}F$, where $\kappa = \kappa(x)$ is scalar function on M. Then the following hold*

$$\bar{r}_{ij} = \frac{e^\kappa}{2}\left(2r_{ij} + 2fa_{ij} - b_j\kappa_i - b_i\kappa_j\right), \tag{2.2}$$

$$\bar{s}_{ij} = \frac{e^\kappa}{2}\left(2s_{ij} - b_j\kappa_i + b_i\kappa_j\right), \tag{2.3}$$

where $\kappa_i := \partial\kappa/\partial x^i$ and $f := \kappa_t b^t$.

Proof. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric which is conformally related to a Finsler metric $\bar{F}$ on M, that is, there is a scalar function $\kappa = \kappa(x)$ on M such that $\bar{F} = e^{\kappa(x)}F$. If we write $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta} = \bar{b}_i(x)y^i$, then the following hold

$$\bar{a}_{ij} = e^{2\kappa}a_{ij}, \qquad \bar{b}_i = e^\kappa b_i. \tag{2.4}$$

Therefore, we get

$$\bar{a}^{ij} = e^{-2\kappa}a^{ij}, \quad \bar{b}^i = e^{-\kappa}b^i.$$

Let G^i and $\bar{G}^i$ be the spray coefficients of F and $\bar{F}$, respectively. By using the Rapcsák's identity, the following relationship between G^i and $\bar{G}^i$ holds

$$\bar{G}^i = G^i + \frac{\bar{F}_{;m}y^m}{2\bar{F}}y^i + \frac{\bar{F}}{2}\bar{g}^{il}\Big\{\bar{F}_{;k,l}y^k - \bar{F}_{;l}\Big\}, \tag{2.5}$$

where "$;$" and "$,$" denote the horizontal and vertical derivation with respect to the Berwald connection of F. Since $F_{;m} = 0$, then the following hold

$$\bar{F}_{;m} = \kappa_m e^\kappa F, \quad \bar{F}_{;m,l} = \kappa_m e^\kappa F_{,l}, \quad \bar{g}_{ij} = e^{2\kappa}g_{ij}, \quad \bar{g}^{ij} = e^{-2\kappa}g^{ij}. \tag{2.6}$$

By putting (2.6) in (2.5), we get

$$\bar{G}^i = G^i + \kappa_0 y^i - \frac{1}{2}F^2\kappa^i, \tag{2.7}$$

where $\kappa_0 := \kappa_i y^i$ and $\kappa^i := g^{im}\kappa_m$. Let us put

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}.$$

Then taking twice vertical derivation of (2.7) yields

$$\bar{G}^i{}_{jk} = G^i{}_{jk} + \kappa_j\delta^i_k + \kappa_k\delta^i_j - g_{jk}\kappa^i. \tag{2.8}$$

By (2.4) and (2.8), we get the following

$$\bar{b}_{i||j} = e^\kappa(b_{i|j} - b_j\kappa_i + fa_{ij}), \tag{2.9}$$

where "$|$" and "$||$" denote the covariant derivatives with respect to α and $\bar{\alpha}$, respectively. By (2.9), we get (2.2) and (2.3). $\square$

In order to prove Theorem 1.1, we need to the following.

Lemma 2.2. *Let $F = \sqrt[4]{\mathfrak{a}_{ijkl}(x)y^iy^jy^ky^l}$ be a quartic metric on an n-dimensional manifold M. Then the following hold:*

(1) *If $n = 2$, then by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$, F is always written in the form*

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2},$$

where c_1 and c_2 are real constants and α^2 may be degenerate.

(2) *If $n \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta = \beta_i(x)y^i$ which is homogeneous in α and β of degree one, then it is written in the following form*

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},$$

where c_1, c_2 and c_3 are real constants.

Proof. The proof is very tedious, computational and straightforward. By the same argument used by Matsumoto-Numata for the cubic Finsler metrics in [13], one can get the proof. Here, we omit the process of proof. □

In [9], Kim-Park claimed that using the homogeneousness of a Finsler metric, one can consider the general form of m-th root metric ($m \geq 3$) admitting (α,β)-metric and obtain the following

$$\begin{aligned}
F &= \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3},\\
F &= \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},\\
&\vdots\\
F &= \sqrt[m]{\Sigma_0^s c_{m-2r}\alpha^{2r}\beta^{m-2r}}, \quad s \leq \frac{m}{2},
\end{aligned}$$

where c_i are constants. They studied quartic metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ and proved the following.

Lemma 2.3. ([9]) *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemannian quartic metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a non-zero 1-form on M and c_i $(1 \leq i \leq 3)$ are non-zero constants. Then F is a Berwald metric if and only if β is parallel with respect to α.*

Proof of Theorem 1.1: By Lemma 2.1, we have

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - \kappa_i b_j + a_{ij}\kappa_m b^m), \tag{2.10}$$

where "|" and "||" denote the covariant derivatives with respect to α and $\bar{\alpha}$, respectively. By assumption, $\bar{F}$ is a Berwald metric. Then by Lemma 2.3, (2.10) reduces to following

$$b_{i|j} - \kappa_i b_j + b^r \kappa_r a_{ij} = 0. \tag{2.11}$$

Multiplying (2.11) with b^i and a^{ij} yield, respectively

$$b^j b_{i|j} = b^2 \kappa_i - b^r \kappa_r b_i, \tag{2.12}$$

$$b^r \kappa_r = -\frac{1}{n-1} a^{ij} b_{i|j}. \tag{2.13}$$

Putting (2.13) in (2.12) yields

$$\kappa_i = \frac{1}{b^2}\Big[b^r b_{i|r} - \frac{1}{n-1} a^{rs} b_{r|s} b_i\Big]. \tag{2.14}$$

It is remarkable that since κ_i is a gradient vector, then

$$\kappa_{i|j} - \kappa_{j|i} = 0.$$

(2.11) can be written as

$$r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) - b^r \kappa_r a_{ij}, \tag{2.15}$$

$$s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i). \tag{2.16}$$

(2.15) and (2.16) give respectively

$$b^r \kappa_r = -\frac{1}{n-1} a^{rs} r_{rs}, \tag{2.17}$$

$$s_j = \frac{1}{2}\Big(\kappa_r b^r b_j - b^2 \kappa_j\Big). \tag{2.18}$$

Putting (2.17) and (2.18) in (2.15) and (2.16) yield, respectively

$$r_{ij} = \frac{r^s_s}{n-1}\Big(a_{ij} - \frac{1}{b^2} b_i b_j\Big) - \frac{1}{b^2}\Big(b_i s_j + b_j s_i\Big), \tag{2.19}$$

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{2.20}$$

Now (2.14) can be written as

$$\kappa_i = \frac{1}{b^2}\Big(b^r r_{ir} - s_i - \frac{1}{n-1} a^{rs} r_{rs} b_i\Big). \tag{2.21}$$

and (2.19) gives

$$b^r r_{ir} = -s_i. \tag{2.22}$$

By putting (2.22) in (2.21), we get

$$\kappa_i = -\frac{1}{b^2}\Big(2s_i + \frac{1}{n-1} r^s_s b_i\Big). \tag{2.23}$$

This completes the proof. □

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where open $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Then β is called Killing with respect to α if and only if $r_{ij} = 0$.

Corollary 2.1. *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemanian quartic (α,β)-metric on an n-dimensional manifold M, where c_i are nonzero constants and β is a Killing 1-form. Then F is a conformally Berwald metric if and only if it is a Berwald metric.*

Proof. By Theorem 1.1, β satisfies (1.1) and (1.2). Contracting (1.1) with b^i implies that

$$r_i + s_i = 0. \tag{2.24}$$

Let β be a Killing 1-form with respect to α, i.e., $r_{ij} = 0$. Then (2.24) yields $s_i = 0$. Putting it in (1.2) implies that $s_{ij} = 0$. Thus β is parallel with respect to α. By Lemma 2.3, F reduces to a Berwald metric. In this case, by (1.3) one can verify that the conformal change reduces to a homothetic change. □

3. Proof of Theorem 1.2

In this section, we are going to find the necessary and sufficient condition under which a cubic (α,β)-metric is conformally Berwald. For this aim, we remark that the (α,β)-metric $F = \alpha^{m+1}\beta^{-m}$ is called m-Kropina metric. In [13], Matsumoto-Numata studied the class of cubic metrics and proved the following.

Lemma 3.1. (Matsumoto-Numata [13]) *Let $F = \sqrt[3]{\mathfrak{a}_{ijk}(x)y^iy^jy^k}$ be a cubic metric on an n-dimensional manifold M. Then the following hold:*

(i) *If $n = 2$, then by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$, F is a $(-\frac{1}{3})$-Kropina metric*

$$F = \sqrt[3]{\alpha^2\beta}, \tag{3.1}$$

where α^2 may be degenerate.

(ii) *If $n \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta = b_i(x)y^i$ and it is homogeneous in α and β of degree one, then it is written in the following form*

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}, \tag{3.2}$$

where c_1 and c_2 are constants.

Also, in [9], Kim-Park studied cubic (α,β)-metrics and proved the following.

Lemma 3.2. (Kim-Park [9]) *Let $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ be a cubic (α,β)-metric on a manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. Then F is a Berwald metric if and only if there exists functions $f_i=f_i(x)$ on M satisfy following*

$$b_{i|j}=3(c_1+c_2b^2)b_if_j+(c_1+3c_2b^2)b_jf_i-b_kf^k(c_1a_{ij}+3c_2b_ib_j), \tag{3.3}$$

where c_1, c_2 and c_3 are constants and $b^2=b_ib^i$. In this case, f_i are given by following

$$f_j=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(b^2)}{c_1+c_2b^2}\right]. \tag{3.4}$$

Now, we can consider the case (i) in Theorem 1.2 and prove the following.

Lemma 3.3. *Let (M,F) be an n-dimensional Finsler manifold. Then the cubic (α,β)-metric $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ is conformally Berwald if and only if β satisfies following*

$$s_{ij}=\frac{1}{b^2}\left(b_is_j-b_js_i\right), \tag{3.5}$$

$$r_{ij}=\frac{1}{b^2}\left(b_jr_i+b_ir_j\right)-(c_1a_{ij}+3c_2b_ib_j)\bar{f}_rb^r-a_{ij}k_rb^r, \tag{3.6}$$

and the conformal factor $\kappa=\kappa(x)$ satisfies

$$\kappa_j=\frac{2}{b^2}(r_j-ub_j)-2(2c_1+3c_2b^2)\bar{f}_j, \tag{3.7}$$

where

$$\bar{f}_j=\frac{1}{3b^2(c_1+c_2b^2)}(s_j+r_j),\qquad u:=\frac{1}{2}(2c_1\bar{f}_r-\kappa_r)b^r.$$

Proof. Let $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ be a cubic metric on a manifold M which is conformally related to the Berwald metric $\bar{F}$, namely, $\bar{F}=e^{\kappa}F$, where $\kappa=\kappa(x)$ is a scalar function on M. Thus $\bar{F}=\sqrt[3]{c_1\bar{\alpha}^2\bar{\beta}+c_2\bar{\beta}^3}$ is also a cubic (α,β)-metric, where $\bar{\alpha}=e^{\kappa(x)}\alpha$ and $\bar{\beta}=e^{\kappa(x)}\beta$. Put $\bar{\alpha}=\sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta}=\bar{b}_i(x)y^i$. Then by Lemma 3.2, there exist functions $\bar{f}_i=\bar{f}_i(x)$ on M such that $\bar{\beta}$ satisfies following

$$\bar{b}_{i||j}=3(c_1+c_2\bar{b}^2)\bar{b}_i\bar{f}_j+(c_1+3c_2\bar{b}^2)\bar{b}_j\bar{f}_i-\bar{b}_m\bar{f}^m(c_1\bar{a}_{ij}+3c_2\bar{b}_i\bar{b}_j), \tag{3.8}$$

where "||" denotes the covariant derivatives with respect to $\bar{\alpha}$ and $\bar{f}_i$ are given by following

$$\bar{f}_i=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(\bar{b}^2)}{c_1+c_2\bar{b}^2}\right]=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(b^2)}{c_1+c_2b^2}\right].$$

Here, $\bar{f}^m := \bar{a}^{mk}\bar{f}_k$. On the other hand, by Lemma 2.1 the following holds

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - \kappa_i b_j + b^m \kappa_m a_{ij}), \tag{3.9}$$

where "|" denotes the covariant derivatives with respect to α. By (2.1), (2.4), (3.8) and (3.9), we get

$$\begin{aligned} b_{i|j} - \kappa_i b_j + b^m\kappa_m a_{ij} = 3(c_1 + c_2 b^2) b_i \bar{f}_j + (c_1 + 3c_2 b^2) b_j \bar{f}_i \\ - b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j). \end{aligned} \tag{3.10}$$

(3.10) implies that

$$\begin{aligned} r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) + (2c_1 + 3c_2 b^2)(b_i \bar{f}_j + b_j \bar{f}_i) - b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j) \\ - b^m \kappa_m a_{ij} \end{aligned} \tag{3.11}$$

and

$$s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i) + c_1 (b_i \bar{f}_j - b_j \bar{f}_i). \tag{3.12}$$

Multiplying (3.12) with b^i yields

$$s_j = \left(c_1 \bar{f}_j - \frac{\kappa_j}{2}\right) b^2 - b_j \left(c_1 \bar{f}_i - \frac{\kappa_i}{2}\right) b^i. \tag{3.13}$$

By (3.12) and (3.13), we get

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{3.14}$$

Let us put

$$u := \frac{b^r}{2}\Big(2c_1 \bar{f}_r - \kappa_r\Big).$$

Then contracting (3.11) with b^i gives

$$r_j = u b_j + \Big((2c_1 + 3c_2 b^2)\bar{f}_j + \frac{\kappa_j}{2}\Big) b^2. \tag{3.15}$$

By (3.15), we obtain

$$\kappa_j = 2\left[\frac{r_j - u b_j}{b^2} - (2c_1 + 3c_2 b^2)\bar{f}_j\right]. \tag{3.16}$$

Considering (3.15), the relation (3.11) can be written as follows

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - b^r \bar{f}_r (c_1 a_{ij} + 3c_2 b_i b_j) - a_{ij} b^r k_r. \tag{3.17}$$

Comparing (3.13) and (3.15) yield

$$\bar{f}_j = \frac{1}{3b^2(c_1 + c_2 b^2)}\Big(s_j + r_j\Big). \tag{3.18}$$

Conversely, we make the conformally changed $\bar{F}$ from F by the conformal change $\bar{F} = e^{\kappa(x)}F$. Suppose that the metric F satisfies (3.5) and (3.6), and the conformal factor κ satisfies (3.7). Then (3.5), (3.6) and (3.7) lead to

$$\begin{aligned} b_{i|j} - \kappa_i b_j + b^m \kappa_m a_{ij} &= r_{ij} + s_{ij} - \kappa_i b_j + \kappa_m b^m a_{ij} \\ &= 3db_i \bar{f}_j + (c_1 + 3c_2 b^2) b_j \bar{f}_i - b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j), \end{aligned} \tag{3.19}$$

where $d := c_1 + c_2 b^2$. By (3.10) and (3.19), $\bar{F}$ is a Berwald metric. It follows that F is a conformally Berwald metric. □

In [11], Matsumoto studied Kropina metrics and characterized m-Kropina metrics of Berwald-type as follows.

Lemma 3.4. (Matsumoto [11]) *Let $F = \alpha^{m+1}\beta^{-m}$ be the m-Kropina metric on a manifold M. Then F is a Berwald metric if and only if there exists a covariant vector field $f_i = f_i(x)$ such that the following holds*

$$b_{i|j} = m(a_{ij} b_k f^k - b_j f_i) + b_i f_j,$$

where $f^k = a^{lk} f_l$.

Using Lemma 3.4, we prove the following.

Lemma 3.5. *Let (M, F) be an n-dimensional Finsler manifold M. Then the cubic (α, β)-metric $F = \sqrt[3]{\alpha^2\beta}$ is conformally Berwald if and only if β satisfies following*

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big), \tag{3.20}$$

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - \Big(b^r \kappa_r + \frac{1}{3} b^r \bar{f}_r\Big) a_{ij} - \frac{2h}{b^2} b_i b_j, \tag{3.21}$$

and the conformal factor κ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - h b_j) - \frac{4}{3}\bar{f}_j, \tag{3.22}$$

where

$$h := \frac{1}{6}(2\bar{f}_r - 3\kappa_r) b^r, \qquad \bar{f}_j = \frac{1}{b^2}(s_j + r_j).$$

Proof. Let $F = \sqrt[3]{\alpha^2\beta}$ be a cubic metric on a manifold M which is conformally related to the Berwald metric $\bar{F}$, namely, $\bar{F} = e^{\kappa}F$, where $\kappa = \kappa(x)$ is a scalar function on M. Thus $\bar{F} = \sqrt[3]{\bar{\alpha}^2\bar{\beta}}$ is also a cubic (α, β)-metric, where $\bar{\alpha} = e^{\kappa(x)}\alpha$ and $\bar{\beta} = e^{\kappa(x)}\beta$. Put $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x) y^i y^j}$ and $\bar{\beta} = \bar{b}_i(x) y^i$. By Lemma 3.4, $F = \sqrt[3]{\alpha^2\beta}$ is a Berwald metric if and only if there exists f_i satisfying

$$\bar{b}_{i||j} = -\frac{1}{3}\bar{a}_{ij}\bar{b}_r \bar{f}^r + \frac{1}{3}\bar{b}_j \bar{f}_i + \bar{b}_i \bar{f}_j, \tag{3.23}$$

where "$||$" denotes the covariant derivatives with respect to $\bar{\alpha}$ and $\bar{f}^k := \bar{a}^{lk}\bar{f}_l$. By Lemma 2.1, the following hold

$$\bar{b}_{i||j} = e^\kappa(b_{i|j} - \kappa_i b_j + b^r\kappa_r a_{ij}), \quad \bar{a}_{ij} = e^{2\kappa}a_{ij}, \quad \bar{b}_i = e^\kappa b_i. \tag{3.24}$$

where "$|$" denotes the covariant derivatives with respect to α. By (3.23) and (3.24), we get

$$b_{i|j} - \kappa_i b_j + b^r\kappa_r a_{ij} = -\frac{1}{3}a_{ij}b^r\bar{f}_r + \frac{1}{3}b_j\bar{f}_i + b_i\bar{f}_j \tag{3.25}$$

which is equivalent to

$$r_{ij} = \frac{1}{2}\Big(\kappa_i b_j + \kappa_j b_i\Big) - \frac{1}{3}\Big(a_{ij}\bar{f}_r b^r - 2(b_j\bar{f}_i + b_i\bar{f}_j)\Big) - a_{ij}\kappa_r b^r, \tag{3.26}$$

$$s_{ij} = \frac{1}{2}\Big(\kappa_i b_j - \kappa_j b_i\Big) + \frac{1}{3}\Big(b_i\bar{f}_j - b_j\bar{f}_i\Big). \tag{3.27}$$

Multiplying (3.27) with b^i yields

$$s_j = b^2\Big(\frac{\bar{f}_j}{3} - \frac{\kappa_j}{2}\Big) - \Big(\frac{\bar{f}_i}{3} - \frac{\kappa_i}{2}\Big)b^i b_j. \tag{3.28}$$

Consequently, eliminating f_i from (3.27) we obtain

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{3.29}$$

Let us put

$$h := \frac{1}{6}\Big(2\bar{f}_r - 3\kappa_r\Big)b^r.$$

Then multiplying (3.26) with b^i yields

$$r_j = hb_j + \frac{b^2}{6}\Big(4\bar{f}_j + 3\kappa_j\Big). \tag{3.30}$$

(3.30) implies that

$$\kappa_j = \frac{2}{b^2}(r_j - hb_j) - \frac{4}{3}\bar{f}_j. \tag{3.31}$$

By (3.30) and (3.28), we get

$$\bar{f}_j = \frac{1}{b^2}\Big(s_j + r_j\Big). \tag{3.32}$$

Multiply (3.30) with b_i and construct $(b_j r_i + b_i r_j)/b^2$. By considering (3.26), we get the following

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - \Big(b^r\kappa_r + \frac{1}{3}b^r\bar{f}_r\Big)a_{ij} - \frac{2h}{b^2}b_i b_j. \tag{3.33}$$

Conversely, we make the conformally changed $\bar{F}$ from F by the conformal change $\bar{F} = e^{\kappa(x)}F$. Suppose that the metric F satisfies (3.20) and (3.21), and the conformal factor κ satisfies (3.22). Then (3.20), (3.21) and (3.22) lead to

$$\begin{aligned} b_{i|j} - \kappa_i b_j \quad &+ b^m\kappa_m a_{ij} = r_{ij} + s_{ij} - \kappa_i b_j + b^m\kappa_m a_{ij} \\ &= b_i\Big(\frac{s_j}{b^2} + \frac{r_j}{b^2}\Big) - b_j\Big(\frac{s_i}{b^2} + \frac{r_i}{b^2}\Big) - 2\frac{r_i}{b^2}b_j - \frac{2h}{b^2}b_ib_j - \frac{1}{3}b^r\bar{f}_r a_{ij} - \kappa_i b_j \\ &= -\frac{1}{3}a_{ij}b^r\bar{f}_r + \frac{1}{3}b_j\bar{f}_i + b_i\bar{f}_j. \end{aligned} \tag{3.34}$$

By (3.25) and (3.34), $\bar{F}$ is a Berwald metric and then F is a conformally Berwald metric. □

Proof of Theorem 1.2: By Lemmas 3.3 and 3.5, we get the proof. □

4. Some Conformal Invariants

In the theory of conformal changes of Riemannian metrics, the Weyl invariant tensor plays important roles. Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension $n \geq 4$. In local coordinate system, the Weyl tensor is written as follows

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\Big\{g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik}\Big\} - \frac{\mathbf{S}}{(n-1)(n-2)}\Big\{g_{ik}g_{jl} - g_{il}g_{jk}\Big\}.$$

where R_{ijkl} is the Riemann tensor of Riemannian metric $\mathbf{g}$, $R_{ij} = R^k_{\ ikj}$ is the Ricci tensor and $\mathbf{S} = g^{ij}R_{ij} = R^j_j$ is the scalar curvature of $\mathbf{g}$. In dimensions 2 and 3, the Weyl curvature tensor vanishes identically. If the Weyl tensor vanishes in dimension 4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity. The Weyl tensor is invariant under conformal changes: if $\tilde{\mathbf{g}} = e^{f(x)}\mathbf{g}$ for some positive scalar function $f = f(x)$ then $\tilde{W} = W$. For this reason, the Weyl tensor is also called the *conformal tensor*. It follows that a necessary condition for a Riemannian manifold to be conformally flat is that the Weyl tensor vanish. The existence of this conformal invariant is quite remarkable since there is no known generalization of the Weyl conformal curvature tensor to Finsler geometry [7]. Then the following natural question arises:

Is there any conformal invariant in Finsler Geometry?

Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]\Big|_{s=t=0}, \quad u, v \in T_xM.$$

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form $dV_{BH} = \sigma_F(x)\omega^1 \wedge \cdots \wedge \omega^n$ is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det\big(g_{ij}(x, y)\big)}}{\sigma_F(x)}.$$

Now, let $\bar{F} = e^{\kappa}F$ be two conformal Finsler metrics on an n-dimensional manifold M, where $\kappa = \kappa(x)$ is a scalar function on M. It is easy to verify that

$$\bar{g}_{ij}(x, y) = e^{2\kappa} g_{ij}(x, y), \quad \det(\bar{g}_{ij}) = e^{2n\kappa} \det(g_{ij}), \quad \sigma_{\bar{F}} = e^{n\kappa}\sigma_F.$$

Thus, we conclude the following.

Lemma 4.1. *Let $\bar{F} = e^{\kappa}F$ be two conformal Finsler metrics on a manifold M. Then $\bar{\tau} = \tau$.*

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0},$$

where $u, v, w \in T_xM$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. Thus $\mathbf{C} = 0$ if and only if F is Riemannian. Using the notion of Cartan torsion, one can define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

At any point $x \in M$, Shen defined the norms of $\mathbf{C}$ and $\mathbf{I}$ in [18] as follows

$$||\mathbf{C}|| = \sup_{y, u \in T_xM_0} \frac{F(y)|\mathbf{C}_y(u, u, u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}} = \sup_{y, u \in I_xM} \frac{|\mathbf{C}_y(u, u, u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}}, \tag{4.1}$$

$$||\mathbf{I}|| = \sup_{y, u \in T_xM_0} \frac{F(y)|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}} = \sup_{y, u \in I_xM} \frac{|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}}, \tag{4.2}$$

where I_xM is the indicatrix of F at $x \in M$.

For a vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}.$$

Then F is said to be C-reducible if $\mathbf{M}_y = 0$.

Lemma 4.2. (Matsumoto-Hōjō Lemma) *A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if its Matsumoto torsion vanish.*

For a non-zero vector $y \in T_xM_0$, define the torsion $\mathbf{A}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{A}_y(u,v,w) := \mathbf{C}_y(u,v,w) - \frac{P}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w) + \mathbf{I}_y(v)\mathbf{h}_y(u,w) + \mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\}$$
$$- \frac{Q}{||\mathbf{I}||^2}\mathbf{I}_y(u)\mathbf{I}_y(v)\mathbf{I}_y(w), \tag{4.3}$$

where $P = P(x,y)$ and $Q = Q(x,y)$ are scalar functions on TM and $||\mathbf{I}||^2 = I^i I_i$. A Finsler metric F on an n-dimensional manifold M is called semi-C-reducible if $\mathbf{A}_y = 0$. In [14], Matsumoto-Shibata proved that every (α,β)-metric is semi-C-reducible.

Theorem 4.1. ([14]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

Let us define

$$||\mathbf{M}|| = \sup_{y,u \in T_xM_0} \frac{F(y)|\mathbf{M}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}} = \sup_{y,u \in I_xM} \frac{|\mathbf{M}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}}, \tag{4.4}$$

$$||\mathbf{A}|| = \sup_{y,u \in T_xM_0} \frac{F(y)|\mathbf{A}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}} = \sup_{y,u \in I_xM} \frac{|\mathbf{A}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}}. \tag{4.5}$$

Then, we get the following.

Theorem 4.2. *Let (M,F) be an n-dimensional Finsler manifold. Then the following are conformally invariant:*

(i) $\mathcal{C} := F^2||\mathbf{C}||^2$;

(ii) $\mathcal{M} := F^2||\mathbf{M}||^2$;

(iii) $\mathcal{A} := F^2||\mathbf{A}||^2$.

Proof. We have $\bar{C}_{ijk} = e^{2\kappa}C_{ijk}$. Then $\bar{C}^{ijk} = e^{-4\kappa}C^{ijk}$ which yields

$$||\bar{\mathbf{C}}||^2 = e^{2\kappa}||\mathbf{C}||^2. \tag{4.6}$$

Then $\mathcal{C} = \mathcal{C}(x,y)$ is a conformally invariant.

In local coordinates, the Matsumoto torsion is given by following

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

where $h_{ij} := FF_{y^iy^j}$ is the angular metric. Since

$$h_{ij} = e^{2\kappa}\bar{h}_{ij}, \quad I_i = \bar{I}_i,$$

then

$$\bar{M}_{ijk} = e^{2\kappa} M_{ijk}$$

which implies that

$$\bar{M}^{ijk} = e^{-4\kappa} M^{ijk}.$$

Then

$$||\bar{\mathbf{M}}||^2 = e^{2\kappa} ||\mathbf{M}||^2.$$

Thus $\mathcal{M} = \mathcal{M}(x, y)$ is a conformally invariant.

Finally, in local coordinates $\mathbf{A}_y$ is written as follows

$$A_{ijk} := C_{ijk} - \frac{P}{1+n}\Big\{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \Big\} - \frac{Q}{||\mathbf{I}||^2} I_i I_j I_k.$$

We get $\bar{A}_{ijk} := e^{2\kappa} A_{ijk}$. Then $||\bar{\mathbf{A}}||^2 = e^{2\kappa} ||\mathbf{A}||^2$. Then, $\mathcal{A} = \mathcal{A}(x, y)$ is a conformally invariant. $\square$

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