ON THE AUTOMORPHISMS GROUP OF FINITE POWER GRAPHS

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Abstract. The power graph of a group G is the graph with vertex set G, having an edge joining x and y whenever one is a power of the other. The purpose of this paper is to study the automorphism groups of the power graphs of infinite groups.
Keywords: Power graphs, infinite groups, automorphism group.

1. Introduction

In this paper, we consider only simple undirected graphs. Let Γ and Δ be two graphs. These graphs are said to be *isomorphic* and write $\Gamma \cong \Delta$, if there exists a bijection Φ from $V(\Gamma)$ to $V(\Delta)$ such that $uv \in E(\Gamma)$ if and only if $\Phi(u)\Phi(v) \in E(\Delta)$. An isomorphism from Γ to itself is called an *automorphism* of Γ. We denote the set of all automorphisms on Γ as $\mathrm{Aut}(\Gamma)$. A (vertex) *coloring* of Γ is a mapping $c : V(\Gamma) \to S$, where S is the set of colors. The vertices assigned to a given color form a color class. If $|S| = k$, we say that c is a k-coloring (often we use $S = \{1, \ldots, k\}$). A coloring for Γ is proper if adjacent vertices have different colors and it is k-colorable if Γ has a proper k-coloring. The *chromatic number* $\chi(\Gamma)$ is the least number k such that Γ is k-colorable. Obviously, $\chi(\Gamma)$ exists as assigning distinct colors to vertices yield a proper $|V(\Gamma)|$-coloring. An optimal coloring of Γ is a $\chi(\Gamma)$-coloring and Γ is called k-chromatic if $\chi(\Gamma) = k$. Finally, Γ is *planar* if it has no subdivision of the $K_{3,3}$ and K_5.

Suppose G and H are two groups. The group G is called a *torsion group* if the order of elements of G is finite. The *free product* $G * H$ of groups G and H is the set of elements of the form $g_1 h_1 g_2 h_2 \ldots g_r h_r$, where $g_i \in G$ and $h_i \in H$, with g_1 and

h_r possibly equal to the identity elements of G and H. The *semidirect product* of a group G by a group H is a group T containing G and H, where H is normal in T and $G \cap H = \{1\}$ and is denoted $G \rtimes H$. The *symmetric group* S_n of degree n is the group of all permutations on n symbols. The *cyclic group* of order n is denoted by Z_n and for prime number p, the *Prüfer p-group* is denoted by $Z(p^\infty)$.

Graphs associated to algebraic constructions are significant, because they have valuable applications in mathematics and computer science (see, for example, the survey [12] and the monographs [13, 14]). The *power graph* of a group G is the graph with vertex set G, having an edge joining x and y whenever one is a power of the other. The concept of a power graph was first introduced and considered in [11] in the case of groups. Also, they in this paper described the structure of the power graphs of all finite abelian groups. For semigroups, it was first investigated in [10], and then in [8, 9]. Chakrabarty, Ghosh, and Sen in [4] characterized the class of semigroups S for which $\mathcal{P}(S)$ is connected or complete. As a consequence, they proved that $\mathcal{P}(G)$ is connected for any finite group G and $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m. In [3] Cameron and Ghosh proved that non-isomorphic finite groups may have isomorphic power graphs, but about abelian groups with isomorphic power graphs, the groups are also isomorphic. In [5] Doostabadi, Erfanian, and Jafarzadeh obtained some results on the power graph of infinite groups. The power graphs were also investigated in [2, 17, 15]. In [7], the automorphism groups of the power graph in general for finite groups are computed. Also, Feng et al. [6], computed the full automorphism group of the power graph of a finite group. In [1] Abawajy, Kelarev, and Chowdhury gave a survey of all results on the power graphs of groups and semigroups obtained in the literature.

The purpose of this paper is to study the power graphs of infinite groups and their automorphism groups.

2.   Main Results

Let p be a fixed prime number. An infinite group T is called a *Tarski Monster group* for p if every nontrivial subgroup (i.e. Every subgroup other than 1 and G itself) has p elements. The group G is necessarily finitely generated. It is generated by every two non-commuting elements. It is simple. The Tarski groups were first constructed by Olshanskii in 1979. Olshanskii showed in fact that there are continuum-many non-isomorphic Tarski Monster groups for each prime $p > 10^{75}$ [18, 19].

Let A be an abelian group. The *generalized dihedral group* $Dih(A)$ is the semidirect product $A \rtimes Z_2$, where Z_2 is the cyclic group of order 2, and the generator of Z_2 maps elements of A to their inverses. If A is cyclic, then $Dih(A)$ is called a *dihedral group*. The *finite dihedral group* $Dih(Z_n)$ is commonly denoted by D_n or D_{2n}. The *infinite dihedral group* $Dih(Z)$ is denoted by D_∞ and is isomorphic to the free product $Z_2 * Z_2$ of two cyclic groups of order 2.

First, we state two theorems for the automorphism group of graphs.

Theorem 2.1.   *Let Γ be a graph, then* $\mathrm{Aut}(n\Gamma) = \mathrm{Aut}(\Gamma) \wr S_n$.

Theorem 2.2. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be distinct graphs and*

$$\Gamma = n_1 \Gamma_1 \bigcup n_2 \Gamma_2 \bigcup \ldots \bigcup n_r \Gamma_r.$$

Then

$$\mathrm{Aut}(\Gamma) = (\mathrm{Aut}(\Gamma_1) \wr S_{n_1}) \times \ldots \times (\mathrm{Aut}(\Gamma_r) \wr S_{n_r}).$$

In the next theorem, we obtain the automorphism group of the power graph of $(Z \times Z \times \ldots \times Z, +)$, where Z is a set of integer numbers.

Theorem 2.3. *The automorphism group of the power graph of $(Z \times Z \times \ldots \times Z, +)$ is isomorphic to*

$$\mathrm{Aut}(\mathcal{P}(Z \times Z \times \ldots \times Z, +)) = (S_P \times S_P \times \ldots \times S_P) \wr S_2.$$

Where P denotes the set of all prime numbers.

Proof. Consider the group Z under addition. Two integers a and b of Z are adjacent in the power graph if and only if $a \mid b$ or $b \mid a$. This graph has three components containing $\{0\}$, Z^+, and Z^-, where Z^+ and Z^- are the set of all positive and negative integers, respectively. On the other hand, any integer n is adjacent to all positive multiples of n and there is no edge connecting a positive and a negative number. The components containing all positive and all negative integers is denoted by H_1 and H_2 and it can be easily seen that $H_1 \cong H_2$. We now calculate the automorphism group of subgraph H_1. By power graph structure of Z, $\mathrm{Aut}(H_1)$ is isomorphic to the automorphism group of the partially ordered set $L = (N, \mid)$, that N is set of the natural numbers. This group is isomorphic to the symmetric group S_P. Therefore, $\mathrm{Aut}(\mathcal{P}(Z)) = S_P \wr S_2$.

We now apply induction and Theorems 2.1 and 2.2. Since $\mathrm{Aut}(L \times L \times \ldots \times L) \cong S_P \times S_P \times \ldots \times S_P$, so,

$$\mathrm{Aut}(\mathcal{P}(Z \times Z \times \ldots \times Z, +)) = (S_P \times S_P \times \ldots \times S_P) \wr S_2.$$

This completes the proof. $\square$

By [16, Proposition 7], $\mathcal{P}(D_{2n})$ is a union of $\mathcal{P}(Z_n)$ and n copies of K_2 that share the identity element of D_{2n} and by [15, Corollary 2.4], the automorphism group of the power graph D_{2n}, if n is a prime power, then is equal to $S_{n-1} \times S_n$ and otherwise, is equal to $S_n \times \prod_{d \mid \varphi(n)} S_{\varphi(d)}$ that φ is Euler's totient function. In the next two theorems, we compute power graphs and automorphism groups of these graphs for generalized dihedral group $Dih(A)$ and D_∞.

Theorem 2.4. *For the dihedral groups $Dih(A)$ and D_∞,*

$$\begin{aligned} \mathrm{Aut}(\mathcal{P}(Dih(A))) &= \mathrm{Aut}(\mathcal{P}(A)) \times S_{|A|}, \\ \mathrm{Aut}(\mathcal{P}(D_\infty)) &= (S_P \wr S_2) \times S_L. \end{aligned}$$

where $L = (N, \mid)$.

Proof. In the generalized dihedral group $Dih(A)$, for an abelian group A, all elements outside A have order two. So, according to the power graph structure, the power graph of $Dih(A)$, is the union of a copy of $\mathcal{P}(A)$ and $|A|$ copies of K_2 that share in identity. So $\text{Aut}(\mathcal{P}(Dih(A))) = \text{Aut}(\mathcal{P}(A)) \times S_{|A|}$. In this case, if A is a finite abelian group, then the structure of the power graph of it described in [11].

Consider the infinite dihedral group $D_\infty = <r, s|s^2 = 1, srs = r^{-1}>$. The power graph of this group is a union of a copy of $\mathcal{P}(Z)$ and infinite copies of K_2 that share in identity. Thus, by Theorem 2.3, $\text{Aut}(\mathcal{P}(D_\infty)) = (S_P \wr S_2) \times S_L$. $\square$

For the sake of completeness, we mention here two important results which are crucial in our investigation of the power graphs of the infinite groups.

Theorem 2.5. *[5] Let G be an infinite group. Then $\mathcal{P}(G)$ is complete if and only if $G \cong Z(p^\infty)$ for some prime p.*

By previous theorem $\text{Aut}(Z(p^\infty)) \cong S_N$.

Theorem 2.6. *[5] Let G be a group. Then $\mathcal{P}(G)$ is planar if and only if G is a torsion group and $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.*

By previous theorem for the Tarski Monster group T, $\mathcal{P}(T)$ is planar.

Theorem 2.7. *For the Tarski Monster group T, $\text{Aut}(\mathcal{P}(T)) = S_p \wr S_N$. Also $\chi(\mathcal{P}((T)) = p$.*

Proof. The power graph of the Tarski Monster group T for the prime number p is a union infinite copy of K_p that share in identity. So by Theorem 2.1, the automorphism group of this graph is $\text{Aut}(\mathcal{P}(T)) = S_p \wr S_N$. Also, according to power graph structure of the Tarski Monster group, $\chi(\mathcal{P}((T)) = p$. $\square$

Direct product of graphs $\Gamma_1(V_1, E_1), \ldots, \Gamma_n(V_n, E_n)$ is graph $\Gamma(V, E)$, where $V = V_1 \times \ldots \times V_n$ and E is the set of all pairs $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ such that $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$ and $(a_i, b_i) \in E_i \cup \triangle(V_i)$ for all $1 \leq i \leq n$[11].

Theorem 2.8. *[11] If group G is a direct product of the p_i-primary components of itself, that p_i's are pairwise distinct primes, then the power graph of G is the direct product of the power graphs of the p_i-primary component.*

Corollary 2.1. *We know that $Q/Z = \bigoplus_{p \in P} Z(p^\infty)$ that $Z(p^\infty)$'s are p-primary components of Q/Z. By theorem previous and Theorem 2.5, the power graph of group Q/Z is a direct product of the power graphs of complete graphs.*

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