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APPROXIMATION PROPERTIES OF MODIFIED BASKAKOV GAMMA OPERATORS

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Abstract. In this paper, we have studied an approximation properties of modified Baskakov-Gamma operator. Using Korovkin type theorem, firste we gave the approximation properties of this operator. Secondly, we computed the rate of convergence of this operator by means of the modulus of continuity and we gave an approximation properties of weighted spaces. Finally, we studied the Voronovskaya type theorem of this operator.

1. Introduction

The Baskakov operators and their connections with different branches of analysis such as convex and numerical analysis have been studied intensively.

In 1957, V.A. Baskakov defined the well _known Baskakov operators as follows[22];

$$B_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \qquad x\geq 0, n\in\mathbb{N}.$$

Later, many authors studied the approximation properties and gave many generalizations of these operators [1] ,[5],[10],[11],[15],[16],[17],[25],[26].Recently İnce İlarslan et al.[12]discussed some approximation properties of (p,q)-Baskakov-Kantoro -vich operators. Some authors studied the approximation properties Szasz type generalization[21].

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In 1998, V. Miheşan cosnructed and studied the convergence properties a generalization of the Baskakov operators as follows[24]:

$$B_n^a(f;x)=\sum_{k=0}^{\infty}e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right)\qquad x\geq 0,n\in\mathbb{N}, \tag{1.1}$$

where $P_k(x;a)=\sum_{i=0}^{k}\binom{k}{i}(x)_i\,a^{k-i}$.

In [6], Wafi and Khatoon examined the convergence features of the integral type modification of the operators (1.1)

$$V_n^a(f;x)=n\sum_{k=0}^{\infty}e^{\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt\qquad x\geq 0,n\in\mathbb{N}. \tag{1.2}$$

In 2010, Erençin and Başcanbaz-Tunca [2] identified a more general version of these operators, with the help of sequence, and examined the convergence features.

In 2011, Erençin constructed a Durrmeyer type modification of generalized Baskakov operators (1.1) as follows

$$L_n^{\alpha}(f;x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}f(t)\,dt\quad;\quad x\geq 0 \tag{1.3}$$

and studied some approximations properties[3].In (2012), Krech and Malejki investigated a modified type this operators[13].

In 2014, Erençin and Büyükdurakoğlu extended the operator (1.2) as

$$K_n(f;x)=e^{-\frac{a_nx}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{b_n}{d_n-c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}}f(t)\,dt\ \ ;\ x\geq 0,n\in\mathbb{N},$$

which is a more general version of the operators and examined the convergence features in weighted spaces[1].

In 2017, N. Rao and A. Wafi [8] defined as follows

$$L_{n,a}^{\alpha,\beta}(f,x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k+\alpha}{n+\beta}\right)$$

and examined the convergence features of Stancu variant the operator of (1.2) .

In 2015, Goyal and Agrawal examined the convergence features of bivariate generalization of operators L_n^{α} given by (1.3)[15].

Gamma operator is identified as[7]

$$G_n(f,x)=\int\limits_0^\infty \frac{x^{n+1}}{n!}e^{-xy}y^n f\left(\frac{n}{y}\right)dy \qquad x\in(0,\infty), n\in\mathbb{N}.$$

In 2011, L. Rempulska and M. Skorupka extended the modified version of Gamma operator as follows

$$G_{n,p}(f,x)=\int\limits_0^\infty \frac{x^{n+1}}{n!}e^{-xy}y^n F_p\left(x,\frac{n}{y}\right)dy$$

and investigated the approximation properties for differentiable functions in polynomial weighted spaces[14].

Different modification of this operator were examined[19],[20],[21].

In 2014, R. Malejki and E. Wachnicki[18] constructed integral type modification the operators B_n^α given by(1.1)as follows:

$$M_n^{\alpha,a}(f,x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^\infty \frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{1}{\Gamma(\alpha+k+1)}\int\limits_0^\infty e^{-ns}(ns)^{\alpha+k}f(s)\,ds$$

and studied approximation properties of the operator. In (2015), E. Pandey and S.P. Mishra investigated a differet type this operators[9].

In 2016, I. Krech and R. Malejki[13] defined a multivariate version of the operators $M_n^{\alpha,a}$.

In this paper, we give a new generalization consisting of the linear combination of Baskakov- Gamma operators.

2. Constructions of the Operators

Let $x\in(0,\infty)$, $n\in\mathbb{N}$, $0<\alpha<\beta$ and f be defined on the space $C_B(0,\infty)$ of all continuous bounded functions. We define the operator as follows:

$$S_{n,a}^{\alpha,\beta}(f;x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^\infty \frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\int\limits_0^\infty \frac{x^{n+1}}{n!}y^n e^{-xy}f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right)dy \tag{2.1}$$

where $a>0$ is a constant and

$$P_k(n,a)=\sum_{i=0}^k \binom{k}{i}(n)_i\, a^{k-i}$$

with $(n)_0 = 1, (n)_i = n(n+1)(n+2)...(n+i-1); i \geq 1$denotes Pochammer Symbol. With the help of derivatives, $e^{-\frac{ax}{1+x}} \sum\limits_{k=0}^{\infty} \frac{P_{k+1}(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = n(1+x) + a$ and

$e^{-\frac{ax}{1+x}} \sum\limits_{k=0}^{\infty} \frac{P_{k+2}(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = n(n+1)(1+x)^2 + 2an(1+x) + a^2$ can be easily proved.

3. Auxiliary results

Lemma 3.1. *For the operators (2.1), we have*

$$\begin{aligned} S_{n,a}^{\alpha,\beta}(1;x) &= 1, \\ S_{n,a}^{\alpha,\beta}(t;x) &= \tfrac{nx+\alpha}{n+\beta} \tfrac{ax}{(1+x)(n+\beta)}, \\ S_{n,a}^{\alpha,\beta}\left(t^2;x\right) &= \tfrac{n^2(1+n)x^2}{(n+\beta)^2(n-1)} + \tfrac{\left[2an^2(1+x)+\alpha^2 n\right]}{(n+\beta)^2(n-1)} \tfrac{x^2}{(1+x)^2} \\ &\times \tfrac{\left\{\left[n^2+2\alpha n(n-1)\right](1+x)+\left[an+2a\alpha(n-1)\right]\right\}}{(n+\beta)^2(n-1)} \tfrac{x}{1+x} + \tfrac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Proof. Using the operator (2.1), , it follows

$$S_{n,a}^{\alpha,\beta}(1;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} dy$$

If we say $xy = t$ then it follows

$$S_{n,a}^{\alpha,\beta}(1;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = 1,$$

which proves the first result.

For $f(t) = t$ we have

$$\begin{aligned} &S_{n,a}^{\alpha,\beta}(t;x) \\ &= e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \tfrac{P_k(n,a)}{k!} \tfrac{x^k}{(1+x)^{n+k}} \tfrac{x^{n+1}}{n!} \tfrac{1}{n+\beta} \left[n \int_0^{\infty} y^n e^{-xy} \tfrac{k}{xy} dy + \alpha \int_0^{\infty} y^n e^{-xy} dy \right] \\ &= \tfrac{1}{n+\beta} \tfrac{x}{1+x} \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \tfrac{P_{k+1}(n,a)}{k!} \tfrac{x^k}{(1+x)^{n+k}} \right) + \tfrac{\alpha}{n+\beta} = \tfrac{nx+\alpha}{n+\beta} + \tfrac{ax}{(1+x)(n+\beta)}. \end{aligned}$$

For $f(t) = t^2$, it follows

$$S_{n,a}^{\alpha,\beta}\left(t^2;x\right) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left(\frac{\frac{kn}{xy} + \alpha}{n+\beta} \right)^2 dy$$

$$= \frac{e^{-\frac{ax}{1+x}}}{(n+\beta)^2}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^{k+n+1}}{(1+x)^{n+k}n!}$$
$$\times\left[n^2\int_0^{\infty}y^n e^{-xy}\frac{k^2}{x^2y^2}dy+2\alpha n\int_0^{\infty}y^n e^{-xy}\frac{k}{xy}dy+\alpha^2\int_0^{\infty}y^n e^{-xy}dy\right]$$
$$= \frac{n}{(n-1)(n+\beta)^2}\frac{x^2}{(1+x)^2}\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+2}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)$$
$$+\frac{n}{(n-1)(n+\beta)^2}\frac{x}{1+x}\times\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+1}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)$$
$$+\frac{2\alpha}{(n+\beta)^2}\frac{x}{1+x}\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+1}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)+\frac{\alpha^2}{(n+\beta)^2}$$
$$= \frac{n^2(1+n)x^2}{(n+\beta)^2(n-1)}+\frac{\left[2an^2(1+x)+\alpha^2n\right]}{(n+\beta)^2(n-1)}\frac{x^2}{(1+x)^2}$$
$$+\frac{\left\{\left[n^2+2\alpha n(n-1)\right](1+x)+[an+2a\alpha(n-1)]\right\}}{(n+\beta)^2(n-1)}\frac{x}{1+x}+\frac{\alpha^2}{(n+\beta)^2},$$

which completes the proof.

□

$S_{n,a}^{\alpha,\beta}\left(t^3;x\right)$ and $S_n^a\left(t^4;x\right)$ can be proved in a similarly way that of the proof of Lemma 3.1.

Theorem 3.1. *Let $f\in C_B(0,\infty)$, $x\in(0,\infty)$ and $n\in\mathbb{N}$. Then we have*

$$\lim_{n\to\infty}\left(S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right)=0.$$

Proof. Proof is clear that by Lemma 3.1. □

Lemma 3.2. *For the operators (2.1),*

$$S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)\leq M^*\frac{x^2+x+1}{(n+\beta)^2}$$

where $M_i=(n,a,\beta,\alpha)$, $i=1,2,...;\quad M^*=\max(M_i)$.

Proof. From linearity of the operator (2.1) and Lemma3.1, since $\frac{x^s}{(1+x)^l}\leqq x^s$ for all $x\geqq 0,\quad l<s\quad(l,s=1,2,3,4)$, we can write

$$\begin{aligned}S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) &\leq \frac{1}{(n+\beta)^2}\left(\frac{n\left(n+\alpha^2+2an+n^2\right)-(n-1)(n+\beta)(n+2\alpha-\beta)}{n-1}\right)x^2\\ &+\frac{1}{(n+\beta)^2}\left(\frac{(n-2\alpha+2n\alpha)(a+n)-2\alpha(n-1)(n+\beta)}{n-1}\right)x+\frac{\alpha^2}{(n+\beta)^2}\\ &=\frac{x^2}{(n+\beta)^2}M_1+\frac{x}{(n+\beta)^2}M_2+\frac{1}{(n+\beta)^2}M_3\\ &\leq M^*\frac{x^2+x+1}{(n+\beta)^2}.\end{aligned}$$

□

4. Rates of Convergence

We can show the approximation of the operator with the help of the modulus of continuity.

Theorem 4.1. *Let $x \in (0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B$, then we have*

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq M^{**} w\left(f; \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}\right).$$

Proof. By the definition of the operators (2.1) and properties of modulus of continuity, we may write

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq \\
&\leq e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} \left| f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) - f(x)\right| dy
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} w\left(f; \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right|\right) dy \\
&= w(f,\delta) + e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \frac{1}{\delta} w(f;\delta) \left(\int_0^\infty y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right| dy\right)
\end{aligned}$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq w(f,\delta)$$

$$\begin{aligned}
&+\frac{1}{\delta} w(f;\delta) \left\{ \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^\infty \frac{x^{n+1}}{n!} y^n e^{-xy} \left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right)^2 dy \right)^{\frac{1}{2}} \right. \\
&\left. \times \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^\infty \frac{x^{n+1}}{n!} y^n e^{-xy} dy \right)^{\frac{1}{2}} \right\} \\
&\leq w(f,\delta) + \frac{1}{\delta} w(f;\delta) \sqrt{M^*} \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}
\end{aligned}$$

If we take $\delta = \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}$, then it follows

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq M^{**} w\left(f; \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}\right).$$

which ends the proof where

$$M^{**} = 1 + \sqrt{M^*}.$$

□

Let $C_B(0,\infty)$ denote the space of real valued continuous and bounded functions on the interval $(0,\infty)$, with the norm

$$\|f\| = \sup_{0\le x<\infty} |f(x)| .$$

For every $\delta > 0$, Peetre's K- functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2(0,\infty)} \{\|f-g\| + \delta \|g''\|\}$$

where

$$C_B^2(0,\infty) = \{g \in C_B(0,\infty) : g', g'' \in C_B(0,\infty)\} .$$

There exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C w_2\left(f;\sqrt{\delta}\right) \tag{4.1}$$

holds where w_2 is the second order modulus of smoothness of f, defined by

$$w(f;\delta) = \sup_{0<h\le\delta} \sup_{0<x<\infty} |f(x+2h) - 2f(x+h) + f(x)| .$$

Now, we consider the following $\hat{S}_{n,a}^{\alpha,\beta}(f;x)$ by means of operator $S_{n,a}^{\alpha,\beta}$

$$\hat{S}_{n,a}^{\alpha,\beta}(f;x) = S_{n,a}^{\alpha,\beta}(f;x) - f\left(\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x) . \tag{4.2}$$

Then, the following Lemma can be given.

Lemma 4.1. *Let $g \in C_B^2(0,\infty)$. Then we have*

$$\left|\hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x)\right| \le \delta_n(x) \|g''\|$$

where

$$\delta_n(x) = S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) + \left(\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2 .$$

Proof. For the operators $\hat{S}_{n,a}^{\alpha,\beta}(f;x)$, we get

$$\begin{aligned}\hat{S}_{n,a}^{\alpha,\beta}(t-x;x) &= S_{n,a}^{\alpha,\beta}(t-x;x) - \left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right) \\ &= S_{n,a}^{\alpha,\beta}(t;x) - xS_{n,a}^{\alpha,\beta}(1;x) - S_{n,a}^{\alpha,\beta}(t;x) + xS_{n,a}^{\alpha,\beta}(1;x) = 0.\end{aligned}$$

□

Let $g\in C_B^2(0,\infty)$ and $x\in(0,\infty)$. By Taylor's formula of g, we may write

$$g(t)-g(x)=(t-x)g\prime(x)+\int_x^t(t-u)g''(u)\,du \qquad ; \qquad t\in[0,\infty).$$

If we apply the operator $\hat{S}_{n,a}^{\alpha,\beta}$ to this equality, we obtain

$$\hat{S}_{n,a}^{\alpha,\beta}(g(t)-g(x);x)=g'(x)\hat{S}_{n,a}^{\alpha,\beta}((t-x);x)+\hat{S}_{n,a}^{\alpha,\beta}\left(\int_x^t(t-x)g''(u)\,du;x\right)$$

$$\begin{aligned}
&= \hat{S}_{n,a}^{\alpha,\beta}\left(\int_x^t(t-x)g''(u)\,du;x\right)\\
&= S_{n,a}^{\alpha,\beta}\left(\int_x^t(t-x)g''(u)\,du;x\right)-\\
&- \left(\int_x^{\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}}\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}-u\right)g''(u)\,du;x\right)\\
&\quad+\int_x^x(x-u)\,du.
\end{aligned}$$

By using the following inequality

$$\left|\int_x^t(t-u)g''(u)\,du\right|\le(t-x)^2\left\|g''(u)\right\|$$

we can write

$$\int_x^{\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}}\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}-u\right)g''(u)\,du\le\left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2\left\|g''(u)\right\|.$$

In wiev of this inequality, we can conclude that

$$\begin{aligned}
\left|\hat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)\right| &\le\left\{S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)+\left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2\right\}\left\|g''\right\|\\
&=\delta_n(x)\left\|g''\right\|.
\end{aligned}$$

Theorem 4.2. *Let $f\in C_B(0,\infty)$. For all $x\in(0,\infty)$, there exists a constant $B>0$ such that*

$$\left|S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right|\le Bw_2\left(f;\sqrt{\delta_n(x)}\right)+w\left(f;\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)$$

where

$$\delta_n(x) = S_{n,a}^{\alpha,\beta}\left((t-x)^2; x\right) + \left(\frac{ax + (1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right)^2.$$

Proof. For the operators $\hat{S}_{n,a}^{\alpha,\beta}$, we write

$$(4.3) \qquad \hat{S}_{n,a}^{\alpha,\beta}(f;x) - f(x) = \hat{S}_{n,a}^{\alpha,\beta}(f-g;x) + (f-g)(x) + \hat{S}_{n,a}^{\alpha,\beta}(g-g(x);x)$$

from the equality(4.1), it follows

(4.4)

$$\begin{aligned} S_{n,a}^{\alpha,\beta}(f;x) - f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x) - f(x) = \hat{S}_{n,a}^{\alpha,\beta}(f-g;x) + (f-g)(x) \\ + \hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x) \end{aligned}$$

and

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \le \left|\hat{S}_{n,a}^{\alpha,\beta}(f-g;x)\right| + |(f-g)(x)| \\ + \left|\hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x)\right| + \left|f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) - f(x)\right|. \end{aligned}$$

By taking the supremum of $\hat{S}_{n,a}^{\alpha,\beta}$operators, we get

$$\begin{aligned} \left|\hat{S}_{n,a}^{\alpha,\beta}(f;x)\right| &= \left|S_{n,a}^{\alpha,\beta}(f;x) - f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x)\right| \\ &\le \left|S_{n,a}^{\alpha,\beta}(f;x)\right| + 2\|f\| \\ &\le 3\|f\|. \end{aligned}$$

Now if equality (4.3) is replaced by inequality (4.4), we have

$$\begin{aligned} |S_n^a(f;x) - f(x)| &\le 4\|f-g\| + \left|\hat{S}_n^a(g;x) - g(x)\right| \\ &\quad + \left|f\left(\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) - f(x)\right| \end{aligned}$$

from Lemma4.1 we obtain

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| &\le 4\{\|f-g\| + \delta_n(x)\|g''\|\} \\ &\quad + w\left(f; \frac{ax+(1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right). \end{aligned}$$

By taking the infimum for all $g \in C_B^2(0,\infty)$ on the right-hand side of the last inequality and considering (4.1), we get that

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| &\leq 4K_2(f;\delta_n) + w\left(f; \frac{ax + (1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right) \\ &\leq 4Cw_2\left(f;\sqrt{\delta_n}\right) + w\left(f; \frac{ax + (1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right) \\ &= Bw_2\left(f;\sqrt{\delta_n}\right) + w\left(f; \frac{ax + (1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right), \end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $0 < \gamma \leq 1$ and $f \in C_B(0,\infty)$. Then if $f \in Lip_M(\gamma)$, that is, the inequality*

$$|f(t) - f(x)| \leq M|t - x|^\gamma, \qquad x, t \in (0,\infty)$$

holds, then for each $x \in (0,\infty)$ we have

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq M\delta_n^{\frac{\gamma}{2}}(x)$$

where
$\delta_n = S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)$ and $M > 0$ is a constant.

Proof. Let $f \in C_B(0,\infty) \cap Lip_M(\gamma)$. By the linearity and monotonicity of the $S_{n,a}^{\alpha,\beta}$ operators, we get

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| &\leq S_{n,a}^{\alpha,\beta}(|f(t) - f(x)|;x) \\ &\leq MS_{n,a}^{\alpha,\beta}(|t-x|^\gamma;x) \\ &= M\sum_{k=0}^{\infty} e^{-\frac{ax}{1+x}} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right|^\gamma dy. \end{aligned}$$

By applying the Hölder inequality two times succesively to the right side with $p = \frac{2}{\gamma}, q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} &\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \\ \leq\ & M\left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right|^2 dy\right)^{\frac{\gamma}{2}} \\ \leq\ & MS_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{\gamma}{2}} \\ =\ & M\delta_n^{\frac{\gamma}{2}}(x), \end{aligned}$$

which is the desired result.

□

5. Weighted Approximation Properties

Firstly, we give some definitions and theorem

Let $\rho(x) = 1 + x^2$ and $B_\rho[0,\infty)$ denote the space of all functions having the property

$$|f(x)| \leq M_f \rho(x)$$

where $x \in [0,\infty)$ and M_f is a positive constant on f functions. The norm on $B_\rho[0,\infty)$ is defined as follows

$$\|f\|_\rho = \sup_{0 \leq x < \infty} \frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$ and $C_\rho^0[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = 0.$$

The basic theorem for approximation of weighted spaces is given by Gadjiev in[4].

Theorem 5.1. *Let $\{A_n\}$ be a sequence of positive linear operators defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$,and satisfying the conditions*

$$\lim_{n\to\infty} \|A_n(t^v;x) - x^v\|_\rho = 0, \qquad v = 0,1,2.$$

Then for any $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} \|A_n(f;x) - f(x)\|_\rho = 0.$$

It is shown in [4] that, a sequence of linear positive operators A_n is defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$\|A_n(\rho;x)\|_\rho \leq M_\rho$$

where M_ρ is a positive constant.

Theorem 5.2. *Let $\left\{S_{n,a}^{\alpha,\beta}\right\}$ be the sequence of positive linear operators. For each $f \in C_\rho^0(0,\infty)$, we have*

$$\lim_{n\to\infty} \left\|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right\|_\rho = 0.$$

Proof. Using Lemma3.2, we get

$$\begin{aligned}\sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}(\rho;x)\right|}{1+x^2} &= \sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}\left(1+t^2;x\right)\right|}{1+x^2}\\ &\leq 1+\frac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha(n-1)(2a+2n+\alpha)}{(n+\beta)^2(n-1)}.\end{aligned}$$

There exists a positive contant D such that for each n and $\alpha, a, \beta < \infty$

$$\frac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha\left(n-1\right)\left(2a+2n+\alpha\right)}{\left(n+\beta\right)^2\left(n-1\right)} < D.$$

Hence we may write

$$\sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}\left(\rho;x\right)\right|}{1+x^2} = \left\|S_{n,a}^{\alpha,\beta}\left(\rho;x\right)\right\|_{\rho} \leq 1+D.$$

which shows that $\{S_n^a\}$ is a sequence of positive linear operators defined from $C_\rho^0(0,\infty)$ to $B_\rho(0,\infty)$.

For $v=0$, it is clear that

$$\left\|S_{n,a}^{\alpha,\beta}\left(1;x\right)-1\right\|_{\rho}=0.$$

For $v=1$, we have

$$\begin{aligned}\left\|S_{n,a}^{\alpha,\beta}\left(t;x\right)-x\right\|_{\rho} &= \sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}(t;x)-x\right|}{1+x^2}\\ &= \sup_{0\leq x<\infty}\left|\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\frac{1}{1+x^2}-\frac{x}{1+x^2}\right|\\ &\leq \left|\frac{A}{n+\beta}\right|\end{aligned}$$

holds. Similarly, for $v=2$, we get

$$\begin{aligned}\left\|S_{n,a}^{\alpha,\beta}\left(t^2;x\right)-x^2\right\|_{\rho} &\leq \sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}\left(t^2;x\right)-x^2\right|}{1+x^2}\\ &\leq \left|\frac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha(n-1)(2a+2n+\alpha)}{(n+\beta)^2(n-1)}-1\right|\\ &= \left|\frac{B}{(n+\beta)^2(n-1)}\right|.\end{aligned}$$

As a result, we obtain

$$\lim_{n\to\infty}\left\|S_{n,a}^{\alpha,\beta}\left(t^v;x\right)-x^v\right\|_{\rho}=0, \quad v=0,1,2.$$

Thus, the proof is completed.

□

Theorem 5.3. *Let $x \in (0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B$. For the operators*

$$S_{n,a}^{\alpha,\beta}(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) dy$$

and

$$L_{n,a}^{\alpha,\beta}(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k+\alpha}{n+\beta}\right),$$

the inequality

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \le w(f;\delta)\,\varphi(x)$$

is holds true, where

$$\varphi(x) = \left(1 + \frac{1}{\delta}\sqrt{\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2 + \frac{n+a}{(n-1)(n+\beta)^2}x}\right)$$

and

$$\delta = \sqrt{\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2 + \frac{n+a}{(n-1)(n+\beta)^2}x}.$$

Proof. From the definition and properties of modulus of continuity, we have

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \\
&\le e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left| f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) - f\left(\frac{k+\alpha}{n+\beta}\right)\right| dy \\
&\le w(f,\delta) + \frac{1}{\delta} w(f,\delta)\, e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \left[\int_0^{\infty} y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right|\right] dy.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \\
&\le w(f,\delta) + \frac{1}{\delta} w(f,\delta) \left\{ \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right)^2 dy \right)^{\frac{1}{2}} \right. \\
&\left. \times \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} dy \right)^{\frac{1}{2}} \right\} \\
&= w(f,\delta) + \frac{1}{\delta} w(f,\delta) \sqrt{S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right)^2; x\right)}.
\end{aligned}$$

If we calculate the $S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^2;x\right)$, we show that

$$\begin{aligned}
&S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^2;x\right)=S_{n,a}^{\alpha,\beta}\left(\frac{k^2\left(\frac{n}{xy}-1\right)^2}{(n+\beta)^2};x\right)\\
&=\frac{1}{(n-1)(n+\beta)^2}\left[\left(n(n+1)x^2+2an\frac{x^2}{(1+x)}+\frac{a^2x^2}{(1+x)}+nx+\frac{ax}{(1+x)}\right)\right]\\
&\leq\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2+\frac{n+a}{(n-1)(n+\beta)^2}x,
\end{aligned}$$

from which, it follows

$$\lim_{n\to\infty}S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^2;x\right)=0.$$

Thus, we have

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x)-L_{n,a}^{\alpha,\beta}(f;x)\right|\\
\leq\ &w(f,\delta)+\frac{1}{\delta}w(f,\delta)\sqrt{\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2+\frac{n+a}{(n-1)(n+\beta)^2}x}\\
\leq\ &w(f,\delta)\varphi(x).
\end{aligned}$$

□

6. Voronovskaya Type Theorem

Lemma 6.1. *For the operators $S_{n,a}^{\alpha,\beta}(f;x)$ defined (2.1), we have*

$$S_{n,a}^{\alpha,\beta}(t-x;x)=\frac{\alpha-\beta x}{n+\beta}+\frac{ax}{(n+\beta)(1+x)}.$$

$$\begin{aligned}
S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)=&\frac{2n^2+n\beta^2-\beta^2}{(n+\beta)^2(n-1)}x^2+\frac{2a(n+\beta-n\beta)}{(n+\beta)^2(n-1)}\frac{x^2}{1+x}+\frac{a^2n}{(n+\beta)^2(n-1)}\frac{x^2}{(1+x)^2}\\
&+\frac{\left(n^2-2\alpha\beta n+2\alpha\beta\right)}{(n+\beta)^2(n-1)}x+\frac{an+2a\alpha(n-1)}{(n+\beta)^2(n-1)}\frac{x}{1+x}+\frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Proof. By using the definition of $S_{n,a}^{\alpha,\beta}$, it can be proved easily. □

Theorem 6.1. *Let $a,x>0$, $0\leq\alpha\leq\beta$ and $n\in N$. For $f\in C^2(0,\infty)$ and bounded, we have*

$$\lim_{n\to\infty}(n+\beta)\left[S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right]=\left(\alpha-\beta x+\frac{ax}{1+x}\right)f'(x)+\frac{2x^2+x}{2}f''(x).$$

Proof. Let $x, t \in (0,\infty)$, $f \in C^2(0,\infty)$. By Taylor's formula for f, we have

$$f(t) = f(x) + (t-x) f'(x) + \frac{(t-x)^2}{2!} f''(x) + (t-x)^2 \phi(t;x) \tag{6.1}$$

where the function $\phi(t;x) \in C[0,\infty)$ and $\lim_{t\to x} \phi(t;x) = 0$. By applying the operator $S_{n,a}^{\alpha,\beta}$ to the both sides of (6.1), we have

(6.2)

$$\begin{aligned} S_{n,a}^{\alpha,\beta} f(t) = \;& f(x) S_{n,a}^{\alpha,\beta}(1;x) + f'(x) S_{n,a}^{\alpha,\beta}(t-x;x) + \tfrac{f''(x)}{2!} S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) \\ & + S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right). \end{aligned}$$

According to Lemma6.1, the equality (6.2) can be written as follows

$$\begin{aligned} &(n+\beta)\left[S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right] = (n+\beta)\left[\tfrac{\alpha-\beta x}{n+\beta} + \tfrac{ax}{(n+\beta)(1+x)}\right] f'(x) \\ &+ (n+\beta)\left[\tfrac{2n^2+n\beta^2-\beta^2}{(n+\beta)^2(n-1)} x^2 + \tfrac{2a(n+\beta-n\beta)}{(n+\beta)^2(n-1)} \tfrac{x^2}{1+x} + \tfrac{a^2 n}{(n+\beta)^2(n-1)} \tfrac{x^2}{(1+x)^2} \right. \\ &\left. + \tfrac{(n^2-2\alpha\beta n+2\alpha\beta)}{(n+\beta)^2(n-1)} x + \tfrac{an+2a\alpha(n-1)}{(n+\beta)^2(n-1)} \tfrac{x}{1+x} + \tfrac{\alpha^2}{(n+\beta)^2}\right] \tfrac{f''(x)}{2!} + S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right), \end{aligned}$$

where

$$\begin{aligned} & S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right) = \\ & = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right)^2 \phi(t;x)\, dy. \end{aligned}$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

(6.3)

$$(n+\beta) S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right) \le \sqrt{(n+\beta)^2 S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right)} \sqrt{S_{n,a}^{\alpha,\beta}\left(\phi^2(t;x);x\right)}.$$

From Lemma 6.1, we have $S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right) = O(n^{-2})$. Thus, we get

$$\lim_{n\to\infty} (n+\beta)^2 S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right) = 12x^4 + 12x^3 + 3x^2. \tag{6.4}$$

On the other hand, since $\phi(t;x) \in C[0,\infty)$ and $\lim_{t\to x} \phi(t;x) = 0$, then we conclude

$$\lim_{n\to\infty} S_{n,a}^{\alpha,\beta}\left(\phi^2(t;x);x\right) = \phi^2(x;x) = 0. \tag{6.5}$$

Hence, we get from (6.3), (6.4) and (6.5) that

$$\lim_{n\to\infty}(n+\beta)\,S_{n,a}^{\alpha,\beta}\left((t-x)^2\,\phi\,(t;x)\,;x\right)=0$$

and then, we find

$$\lim_{n\to\infty}(n+\beta)\left[S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right]=\left(\alpha-\beta x+\frac{ax}{1+x}\right)f'(x)+\frac{2x^2+x}{2}f''(x)$$

which completed the proof. [1] □

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