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COINCIDENCE POINT RESULTS FOR WEAK $\psi - \varphi$ CONTRACTION ON PARTIALLY ORDERED METRIC SPACES WITH APPLICATION

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Abstract. We establish coincidence point theorem for g-non-decreasing mappings satisfying weak $\psi - \varphi$ contraction on partially ordered metric spaces. With the help of our result, we indicate the formulation of a coupled coincidence point theorem of a generalized compatible pair of mappings $F, G : X^2 \to X$. We also deduce certain coupled fixed point results without mixed monotone property of F. We also give an example and an application to integral equation to support our results presented here. Our results generalize, extend, modify, improve, sharpen, enrich and complement several well-known results of the existing literature.

Keywords: Coincidence point, coupled coincidence point, weak $\psi - \varphi$ contraction, partially ordered metric space, O-compatible, generalized compatibil.ity, g-non-decreasing mapping, mixed monotone mapping, commuting mapping.

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1. Introduction and Preliminaries

In the sequel, we denote by X a non-empty set. Given a natural number $n \in \mathbb{N}$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). We employ mappings $T, g : X \to X$ and $F : X^n \to X$. For simplicity, if $x \in X$, we denote $T(x)$ by Tx.

In [14], Guo and Lakshmikantham introduced the following notion of coupled fixed point for single-valued mappings:

Definition 1.1. Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of F if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

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Following this paper, in 2006, Bhaskar and Lakshmikantham [4] introduced the notion of mixed monotone mappings for single-valued mappings and established some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces.

In [4], Bhaskar and Lakshmikantham introduced the following:

Definition 1.2. Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ is a given mapping. We say that F has the mixed monotone property if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

After that, Lakshmikantham and Ciric [20] extended the notion of mixed monotone property to mixed $g-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Bhaskar and Lakshmikantham [4].

In [20], Lakshmikantham and Ciric introduced the following:

Definition 1.3. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy.$$

Definition 1.4. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a common coupled fixed point of the mappings F and g if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy.$$

Definition 1.5. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if

$$gF(x,\ y) = F(gx,\ gy), \text{ for all } (x,\ y) \in X^2.$$

Definition 1.6. Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed g-monotone property if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Later, Choudhury and Kundu [6] introduced the notion of compatibility in the context of coupled coincidence point and used this notion to improve the results of Lakshmikantham and Ciric [20].

Definition 1.7. [6] The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{aligned}\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) &= 0,\\ \lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) &= 0,\end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n, y_n) &= \lim_{n\to\infty} gx_n = x,\\ \lim_{n\to\infty} F(y_n, x_n) &= \lim_{n\to\infty} gy_n = y, \text{ for some } x, y \in X.\end{aligned}$$

A great deal of these studies investigate contractions on partially ordered metric spaces because of their applicability to initial value problems defined by differential or integral equations.

Hussain et al. [16] introduced a new concept of generalized compatibility of a pair of mappings $F, G : X^2 \to X$ defined on a product space and proved some coupled coincidence point results. Hussain et al. [16] also deduce some coupled fixed point results without the mixed monotone property.

In [16], Hussain et al. introduced the following:

Definition 1.8. Suppose that $F, G : X^2 \to X$ are two mappings. F is said to be G-increasing with respect to $\preceq$ if for all $x, y, u, v \in X$, with $G(x, y) \preceq G(u, v)$ we have $F(x, y) \preceq F(u, v)$.

Example 1.1. Let $X = (0, +\infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $F, G : X^2 \to X$ by $F(x, y) = \ln(x + y)$ and $G(x, y) = x + y$ for all $(x, y) \in X^2$. Note that F is G-increasing with respect to $\leq$.

Example 1.2. Let $X = \mathbb{N}$ endowed with the partial order defined by $x, y \in X^2$, $x \preceq y$ if and only if y divides x. Define the mappings $F, G : X^2 \to X$ by $F(x, y) = x^2y^2$ and $G(x, y) = xy$ for all $(x, y) \in X^2$. Then F is G-increasing with respect to $\preceq$.

Definition 1.9. Suppose that $F, G : X^2 \to X$ are two mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of mappings F and G if

$$F(x, y) = G(x, y) \text{ and } F(y, x) = G(y, x).$$

Example 1.3. Let $F, G : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = xy$ and $G(x, y) = \frac{2}{3}(x + y)$ for all $(x, y) \in X^2$. Note that $(0, 0)$, $(1, 2)$ and $(2, 1)$ are coupled coincidence points of F and G.

Definition 1.10. Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is g-increasing with respect to $\preceq$ if for any $x, y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2).$$

Definition 1.11. Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is increasing with respect to $\preceq$ if for any $x, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1 \preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2).$$

Definition 1.12. Let $F, G : X^2 \to X$ are two mappings. We say that the pair $\{F, G\}$ is generalized compatible if

$$\begin{aligned}
\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\
\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0,
\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}
\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x,\\
\lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y, \text{ for some } x,\ y \in X.
\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Erhan et al. [10], remarked that the results established in Hussain et al. [16] can be derived from the coincidence point results in the literature.

In [10], Erhan et al. recalled the following basic definitions:

Definition 1.13. ([1], [11]) A coincidence point of two mappings $T, g : X \to X$ is a point $x \in X$ such that $Tx = gx$.

Definition 1.14. [10] A partially ordered metric space $(X, d, \preceq)$ is a metric space (X, d) provided with a partial order $\preceq$.

Definition 1.15. ([4], [16]) A partially ordered metric space $(X, d, \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all n, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all n. $(X, d, \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

Definition 1.16. [11] Let$(X, \preceq)$ be a partially ordered set and let $T, g : X \to X$ be two mappings. We say that T is $(g, \preceq)$-non-decreasing if $Tx \preceq Ty$ for all $x, y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$-non-decreasing.

Definition 1.17. [11] If T is $(g, \preceq)$-non-decreasing and $gx = gy$, then $Tx = Ty$. It follows that

$$gx = gy \Rightarrow \left\{ \begin{array}{c} gx \preceq gy, \\ gy \preceq gx \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} Tx \preceq Ty, \\ Ty \preceq Tx \end{array} \right\} \Rightarrow Tx = Ty.$$

Definition 1.18. [24] Let $(X, \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u,\ v),\ (x,\ y) \in X^2. \tag{1.1}$$

Definition 1.19. ([6], [15], [22], [24]) Let $(X, d, \preceq)$ be a partially ordered metric space. Two mappings $T, g : X \to X$ are said to be O-compatible if

$$\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Our basic references are ([2], [3], [5], [7], [8], [9], [12], [17], [18], [19], [21], [26], [27], [28], [29], [30], [31]).

Recently Samet et al. [30] claimed that most of the coupled fixed point theorems for single-valued mappings on partially ordered metric spaces are consequences of the well-known fixed point theorems.

In this paper, we establish coincidence point theorem for g-non-decreasing mappings satisfying weak $\psi - \varphi$ contraction on partially ordered metric spaces. With the help of our result, we indicate the formulation of a coupled coincidence theorem of generalized compatible pair of mappings $F, G : X^2 \to X$. We also deduce certain coupled fixed point results without mixed monotone property of F. We also give an example and an application to integral equation to support our results presented here. We generalize, extend, modify, improve, sharpen, enrich and complement the results of Bhaskar and Lakshmikantham [4], Gordji et al. [13], Lakshmikantham and Ciric [20] and several well-known results of the existing literature.

2. Main results

Lemma 2.1. *Let (X, d) be a metric space. Suppose $Y = X^2$ and define $\delta : Y \times Y \to [0, +\infty)$, for all $(x, y), (u, v) \in Y$, by*

$$\delta((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}.$$

Then δ is metric on Y and (X, d) is complete if and only if (Y, δ) is complete.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$,
(iii_ψ) $\limsup_{s\to 0+} \frac{s}{\psi(s)} < \infty$.

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is lower semi-continuous and non-increasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,
(iii_φ) for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$.

Let Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying

(i_θ) θ is continuous and non-decreasing,
(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $T, g : X \to X$ be two mappings such that the following properties are fulfilled:*

(i) *$T(X) \subseteq g(X)$,*

(ii) *T is $(g, \preceq)$-non-decreasing,*

(iii) *there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,*

(iv) *there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi\left(d(T(x), T(y))\right) \leq \psi\left(M(x, y)\right) - \varphi\left(\psi\left(M(x, y)\right)\right) + \theta\left(N(x, y)\right),$$

where

$$M(x, y) = \max\left\{\begin{array}{c} d(gx, gy),\ d(gx, Tx), \\ d(gy, Ty),\ \frac{d(gx, Ty)+d(gy, Tx)}{2} \end{array}\right\},$$

and

$$N(x, y) = \min\{d(gx, gy), d(gy, Tx)\}.$$

for all $x, y \in X$ such that $gx \preceq gy$. Also assume that, at least, one of the following conditions holds.

(a) *(X, d) is complete, T and g are continuous and the pair (T, g) is O-compatible,*

(b) (X, d) *is complete, T and g are continuous and commuting,*

(c) $(g(X), d)$ *is complete and* $(X, d, \preceq)$ *is non-decreasing-regular,*

(d) (X, d) *is complete,* $g(X)$ *is closed and* $(X, d, \preceq)$ *is non-decreasing-regular,*

(e) (X, d) *is complete, g is continuous and the pair* (T, g) *is O-compatible and* $(X, d, \preceq)$ *is non-decreasing-regular.*

Then T and g have, at least, a coincidence point.

Proof. We divide the proof into four steps.

Step 1. We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$-non-decreasing and $gx_{n+1} = Tx_n$, for all $n \geq 0$. Starting from $x_0 \in X$ given in (iii) and taking into account that $Tx_0 \in T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g, \preceq)$-non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g, \preceq)$-non-decreasing, $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$-non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n, \text{ for all } n \geq 0.$$

Step 2. We claim that $\{d(gx_n, gx_{n+1})\} \to 0$. Suppose first that $gx_{n_0} = gx_{n_0+1}$ for some n_0 implies that $gx_{n_0} = Tx_{n_0}$. This proves that x_{n_0} is a coincidence point of T and g. Also, the sequence $\{gx_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = k$, then $gx_k = gx_{k+1}$. Now, by contractive condition (iv) and (i_ψ), we obtain

$$\begin{aligned} & \psi\left(d(gx_{k+1}, gx_{k+2})\right) \\ = \; & \psi\left(d(Tx_k, Tx_{k+1})\right) \\ \leq \; & \psi\left(M(x_k, x_{k+1})\right) - \varphi\left(\psi\left(M(x_k, x_{k+1})\right)\right) + \theta\left(N(x_k, x_{k+1})\right), \end{aligned}$$

where

$$\begin{aligned} & M(x_k, x_{k+1}) \\ = \; & \max\left\{ \begin{array}{c} d(gx_k, gx_{k+1}), d(gx_k, Tx_k), \\ d(gx_{k+1}, Tx_{k+1}), \frac{d(gx_k, Tx_{k+1})+d(gx_{k+1}, Tx_k)}{2} \end{array} \right\} \\ = \; & \max\left\{ \begin{array}{c} d(gx_k, gx_{k+1}), d(gx_k, gx_{k+1}), d(gx_{k+1}, gx_{k+2}), \\ \frac{d(gx_k, gx_{k+2})+d(gx_{k+1}, gx_{k+1})}{2} \end{array} \right\} \\ = \; & \max\left\{ d(gx_{k+1}, gx_{k+2}), \frac{d(gx_k, gx_{k+2})}{2} \right\} \\ = \; & d(gx_{k+1}, gx_{k+2}), \end{aligned}$$

and

$$\begin{aligned} & N(x_k, x_{k+1}) \\ = \; & \min\left\{ d(gx_k, gx_{k+1}), d(gx_{k+1}, Tx_k) \right\} \\ = \; & 0. \end{aligned}$$

Thus, by (ii_θ), we get

$$\psi\left(d(gx_{k+1},\ gx_{k+2})\right) \leq \psi\left(d(gx_{k+1},\ gx_{k+2})\right) - \varphi\left(\psi\left(d(gx_{k+1},\ gx_{k+2})\right)\right).$$

It follows, by (ii_φ) and (ii_ψ), that $gx_{k+1} = gx_{k+2}$. Thus the sequence $\{gx_n\}$ is constant (starting from some n_0). Suppose that for each $n \in \mathbb{N}$, $d(gx_n, gx_{n+1}) > 0$. It is clear that $N(x_n, x_{n+1}) = 0$ for all $n \in \mathbb{N}$. Now, by contractive condition (iv), (i_ψ) and (ii_θ), we have

$$\begin{aligned} & \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\ = \ & \psi\left(d(Tx_n,\ Tx_{n+1})\right) \\ \leq \ & \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right) + \theta\left(N(x_n,\ x_{n+1})\right) \\ \leq \ & \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right). \end{aligned}$$

Thus

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) \leq \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right), \tag{2.1}$$

which by the fact that $\varphi \geq 0$ implies

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) < \psi\left(M(x_n,\ x_{n+1})\right).$$

Since ψ is non-decreasing, therefore we obtain

$$d(gx_{n+1},\ gx_{n+2}) \leq M(x_n,\ x_{n+1}). \tag{2.2}$$

Again

$$\begin{aligned} & M(x_n,\ x_{n+1}) \\ = \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ Tx_n), \\ d(gx_{n+1},\ Tx_{n+1}),\ \frac{d(gx_n,\ Tx_{n+1}) + d(gx_{n+1},\ Tx_n)}{2} \end{array} \right\} \\ = \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_n,\ gx_{n+1}), \\ d(gx_{n+1},\ gx_{n+2}),\ \frac{d(gx_n,\ gx_{n+2}) + d(gx_{n+1},\ gx_{n+1})}{2} \end{array} \right\} \\ \leq \ & \max\left\{ \begin{array}{c} d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2}), \\ \frac{d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2})}{2} \end{array} \right\} \\ \leq \ & \max\left\{d(gx_n,\ gx_{n+1}),\ d(gx_{n+1},\ gx_{n+2})\right\}. \end{aligned}$$

If $d(gx_{n+1}, gx_{n+2}) \geq d(gx_n, gx_{n+1})$ for some n. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_{n+1},\ gx_{n+2}). \tag{2.3}$$

From (2.2) and (2.3), we get

$$M(x_n,\ x_{n+1}) = d(gx_{n+1},\ gx_{n+2}).$$

Thus, by (2.1), we have

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) \leq \psi\left(d(gx_{n+1},\ gx_{n+2})\right) - \varphi\left(\psi\left(d(gx_{n+1},\ gx_{n+2})\right)\right),$$

which is only possible when $d(gx_{n+1}, gx_{n+2}) = 0$, it is a contradiction. Hence, $d(gx_n, gx_{n+1}) \geq d(gx_{n+1}, gx_{n+2})$ for all n. Then

$$M(x_n,\ x_{n+1}) \leq d(gx_n,\ gx_{n+1}). \tag{2.4}$$

Notice that

$$M(x_n,\ x_{n+1}) \geq d(gx_n,\ gx_{n+1}). \tag{2.5}$$

From (2.4) and (2.5), we get

$$M(x_n,\ x_{n+1}) = d(gx_n,\ gx_{n+1}).$$

This shows that the sequence $\{d(gx_n, gx_{n+1})\}_{n=0}^{\infty}$ is a non-increasing sequence. Thus there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = \delta. \tag{2.6}$$

Then

$$\lim_{n\to\infty} M(x_n,\ x_{n+1}) = \delta. \tag{2.7}$$

We shall prove that $\delta = 0$. Assume to the contrary that $\delta > 0$. Now, by contractive condition (iv) and (i_ψ), we have

$$\begin{aligned} & \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\ = \ & \psi\left(d(Tx_n,\ Tx_{n+1})\right) \\ \leq \ & \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right) + \theta\left(N(x_n,\ x_{n+1})\right), \end{aligned}$$

which implies, by (ii_θ), that

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) \leq \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right).$$

Letting $n \to \infty$ in the above inequality, by using (i_ψ), (i_φ), (2.6) and (2.7), we get

$$\psi\left(\delta\right) \leq \psi\left(\delta\right) - \varphi\left(\psi\left(\delta\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies

$$\delta = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = 0.$$

Step 3. We claim that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since

$$\lim_{n\to\infty} M(x_n,\ x_{n+1}) = 0,$$

and ψ is continuous. Then, by (iii_φ), there exist $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that

$$\varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right) \geq k\psi\left(M(x_n,\ x_{n+1})\right),$$

for all $n \geq n_0$. For any natural number $n \geq n_0$, we have

$$\begin{aligned}
& \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\
= \quad & \psi\left(d(Tx_n,\ Tx_{n+1})\right) \\
\leq \quad & \psi\left(M(x_n,\ x_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ x_{n+1})\right)\right) + \theta\left(N(x_n,\ x_{n+1})\right) \\
\leq \quad & (1-k)\psi\left(M(x_n,\ x_{n+1})\right) \\
\leq \quad & (1-k)\psi\left(d(gx_n,\ gx_{n+1})\right).
\end{aligned}$$

Thus, for all $n \geq n_0$, we have

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) \leq (1-k)\psi\left(d(gx_n,\ gx_{n+1})\right). \tag{2.8}$$

Denote

$$a_n = \psi\left(d(gx_n,\ gx_{n+1})\right), \text{ for all } n \geq 0.$$

From (2.8), we have

$$a_{n+1} \leq (1-k)a_n, \text{ for all } n \geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} (1-k)^{n-n_0} a_{n_0} < \infty. \tag{2.9}$$

On the other hand, by (iii_ψ), we have

$$\limsup_{n\to\infty} \frac{d(gx_n,\ gx_{n+1})}{\psi\left(d(gx_n,\ gx_{n+1})\right)} < \infty. \tag{2.10}$$

Thus, by (2.9) and (2.10), we have $dsumd(gx_n, gx_{n+1}) < \infty$. It means that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X.

Step 4. We claim that T and g have a coincidence point distinguishing between cases $(a) - (e)$.

Suppose now that (a) holds, that is, (X, d) is complete, T and g are continuous and the pair (T, g) is O-compatible. Since (X, d) is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$. Now $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Taking into account that the pair (T, g) is O-compatible, we deduce that $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. In such a case, we conclude that

$$d(gz,\ Tz) = \lim_{n\to\infty} d(ggx_{n+1},\ Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0,$$

that is, z is a coincidence point of T and g.

Suppose now that (b) holds, that is, (X, d) is complete, T and g are continuous and commuting. It is obvious because (b) implies (a).

Suppose now that (c) holds, that is, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X), d)$, so there exists $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$. In this case $\{gx_n\} \to gz$. Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv) and (i_ψ),

$$\begin{aligned} & \psi\left(d(gx_{n+1},\ Tz)\right) \\ = \ & \psi\left(d(Tx_n,\ Tz)\right) \\ \leq \ & \psi\left(M(x_n,\ z)\right) - \varphi\left(\psi\left(M(x_n,\ z)\right)\right) + \theta\left(N(x_n,\ z)\right), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} M(x_n,\ z) &= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ Tx_n), \\ d(gz,\ Tz),\ \frac{d(gx_n,\ Tz)+d(gz,\ Tx_n)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(gx_n,\ gz),\ d(gx_n,\ gx_{n+1}), \\ d(gz,\ Tz),\ \frac{d(gx_n,\ Tz)+d(gz,\ gx_{n+1})}{2} \end{array} \right\}, \end{aligned}$$

and

$$\begin{aligned} N(x_n,\ z) &= \min\left\{d(gx_n,\ gz),\ d(gz,\ Tx_n)\right\} \\ &= \min\left\{d(gx_n,\ gz),\ d(gz,\ gx_{n+1})\right\}. \end{aligned}$$

Letting $n \to \infty$ in (2.11), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi\left(d(gz,\ Tz)\right) \leq \psi\left(d(gz,\ Tz)\right) - \varphi\left(\psi\left(d(gz,\ Tz)\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g.

Suppose now that (d) holds, that is, (X, d) is complete, $g(X)$ is closed and $(X, d, \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. Thus (c) is applicable.

Suppose now that (e) holds, that is, (X, d) is complete, g is continuous, the pair (T, g) is O-compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all n, we also have that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair (T, g) is O-compatible, we deduce that $\lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. As $\{ggx_n\} \to gz$ the previous property means that $\{Tgx_n\} \to gz$.

Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$ for all $n \geq 0$. Applying the contractive condition (iv) and (i_ψ),

$$\begin{aligned} & \psi\left(d(Tgx_n,\ Tz)\right) \\ \leq \ & \psi\left(M(gx_n,\ z)\right) - \varphi\left(\psi\left(M(gx_n,\ z)\right)\right) + \theta\left(N(gx_n,\ z)\right), \end{aligned} \tag{2.12}$$

where

$$M(gx_n,\ z) = \max\left\{ \begin{array}{c} d(ggx_n,\ gz),\ d(ggx_n,\ Tgx_n), \\ d(gz,\ Tz),\ \frac{d(ggx_n,\ Tz)+d(gz,\ Tgx_n)}{2} \end{array} \right\},$$

and

$$N(x_n, z) = \min\{d(ggx_n, gz), d(gz, Tgx_n)\}.$$

Letting $n \to \infty$ in (2.12), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi(d(gz, Tz)) \leq \psi(d(gz, Tz)) - \varphi(\psi(d(gz, Tz))),$$

which, by (ii_φ) and (ii_ψ), implies $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g. $\square$

Next, we deduce the two dimensional version of Theorem 2.1. Given the partially ordered metric space $(X, d, \preceq)$, let us consider the partially ordered metric space $(X^2, \delta, \sqsubseteq)$, where δ was defined in Lemma 2.1 and $\sqsubseteq$ was introduced in (1.1). We define the mappings $T_F, T_G : X^2 \to X^2$, for all $(x, y) \in X^2$, by,

$$T_F(x, y) = (F(x, y), F(y, x)) \text{ and } T_G(x, y) = (G(x, y), G(y, x)).$$

Under these conditions, the following properties hold:

Lemma 2.2. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F, G : X^2 \to X$ be two mappings. Then*

(1) (X, d) is complete if and only if (X^2, δ) is complete.

(2) If $(X, d, \preceq)$ is regular, then $(X^2, \delta, \sqsubseteq)$ is also regular.

(3) If F is d-continuous, then T_F is δ-continuous.

(4) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G, \sqsubseteq)$-non-decreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(6) for any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) Assume there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that

$$\begin{aligned} (2.13) \qquad & \psi(d(F(x, y), F(u, v))) \\ \leq \; & \psi(M(x, y, u, v)) - \varphi(\psi(M(x, y, u, v))) + \theta(N(x, y, u, v)), \end{aligned}$$

where

$$M(x, y, u, v) = \max\left\{\begin{array}{c} d(G(x, y), G(u, v)), d(G(x, y), F(x, y)), \\ d(G(u, v), F(u, v)), d(G(y, x), G(v, u)), \\ d(G(y, x), F(y, x)), d(G(v, u), F(v, u)), \\ \frac{d(G(x, y), F(u, v)) + d(G(u, v), F(x, y))}{2}, \\ \frac{d(G(y, x), F(v, u)) + d(G(v, u), F(y, x))}{2} \end{array}\right\},$$

and

$$N(x, y, u, v) = \min\left\{\begin{array}{c} d(G(x, y), G(u, v)), d(G(u, v), F(x, y)), \\ d(G(y, x), G(v, u)), d(G(v, u), F(y, x)) \end{array}\right\},$$

for all $x, y, u, v \in X$*, where* $G(x, y) \leq G(u, v)$ *and* $G(y, x) \geq G(v, u)$*, then*

$$\begin{aligned} & \psi\left(\delta(T_F(x, y), T_F(u, v))\right) \\ \leq \; & \psi\left(M_\delta((x, y), (u, v))\right) - \varphi\left(\psi\left(M_\delta((x, y), (u, v))\right)\right) \\ & + \theta\left(N_\delta((x, y), (u, v))\right), \end{aligned}$$

where

$$M_\delta((x, y), (u, v)) = \max \left\{ \begin{array}{c} \delta(T_G(x, y), T_G(u, v)), \\ \delta(T_G(x, y), T_F(x, y)), \\ \delta(T_G(u, v), T_F(u, v)), \\ \frac{\delta(T_G(x, y), T_F(u, v)) + \delta(T_G(u, v), T_F(x, y))}{2} \end{array} \right\},$$

and

$$N_\delta((x, y), (u, v)) = \min\left\{\delta(T_G(x, y), T_G(u, v)), \delta(T_G(u, v), T_F(x, y))\right\}.$$

(8) If the pair $\{F, G\}$ *is generalized compatible, then the mappings* T_F *and* T_G *are* O*-compatible in* $(X^2, \delta, \sqsubseteq)$.

(9) A point $(x, y) \in X^2$ *is a coupled coincidence point of* F *and* G *if and only if it is a coincidence point of* T_F *and* T_G.

Proof. Item (1) follows from Lemma 2.1 and items (2), (3), (5), (6) and (9) are obvious.

(4) Assume that F is $G-$increasing with respect to $\preceq$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Then $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Since F is $G-$increasing with respect to $\preceq$, we deduce that $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Therefore $T_F(x, y) \sqsubseteq T_F(u, v)$ and this means that T_F is $(T_G, \sqsubseteq)$-non-decreasing.

(7) Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that

$$\begin{aligned} & \psi\left(d(F(x, y), F(u, v))\right) \\ \leq \; & \psi\left(M(x, y, u, v)\right) - \varphi\left(\psi\left(M(x, y, u, v)\right)\right) + \theta\left(N(x, y, u, v)\right), \end{aligned}$$

for all $x, y, u, v \in X$, where $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Therefore $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Using (2.13), we have

$$\begin{aligned} & \psi\left(d(F(x, y), F(u, v))\right) \\ \leq \; & \psi\left(M(x, y, u, v)\right) - \varphi\left(\psi\left(M(x, y, u, v)\right)\right) + \theta\left(N(x, y, u, v)\right). \end{aligned} \tag{2.14}$$

Furthermore taking into account that $G(y, x) \succeq G(v, u)$ and $G(x, y) \preceq G(u, v)$, the contractive condition (2.13) also guarantees that

$$\begin{aligned} & \psi\left(d(F(y, x), F(v, u))\right) \\ \leq \; & \psi\left(M(x, y, u, v)\right) - \varphi\left(\psi\left(M(x, y, u, v)\right)\right) + \theta\left(N(x, y, u, v)\right). \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$\begin{aligned} & \max\left\{\psi\left(d(F(x, y), F(u, v))\right), \psi\left(d(F(y, x), F(v, u))\right)\right\} \\ \leq \; & \psi\left(M(x, y, u, v)\right) - \varphi\left(\psi\left(M(x, y, u, v)\right)\right) + \theta\left(N(x, y, u, v)\right). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned}(2.16)\qquad & \psi\left(\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}\right)\\ \leq\ & \psi\left(M(x,y,u,v)\right)-\varphi\left(\psi\left(M(x,y,u,v)\right)\right)+\theta\left(N(x,y,u,v)\right).\end{aligned}$$

Thus, it follows from (2.16) that

$$\begin{aligned} & \psi\left(\delta(T_F(x,y),T_F(u,v))\right)\\ =\ & \psi\left(\delta\left((F(x,y),F(y,x)),(F(u,v),F(v,u))\right)\right)\\ =\ & \psi\left(\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}\right)\\ \leq\ & \psi\left(M(x,y,u,v)\right)-\varphi\left(\psi\left(M(x,y,u,v)\right)\right)+\theta\left(N(x,y,u,v)\right)\\ \leq\ & \psi\left(M_\delta((x,y),(u,v))\right)-\varphi\left(\psi\left(M_\delta((x,y),(u,v))\right)\right)\\ & +\theta\left(N_\delta((x,y),(u,v))\right).\end{aligned}$$

(8) Let $\{(x_n,y_n)\}\subseteq X^2$ be any sequence such that $T_F(x_n,y_n)\xrightarrow{\delta}(x,y)$ and $T_G(x_n,y_n)\xrightarrow{\delta}(x,y)$ (notice that we do not need to suppose that $\{T_G(x_n,y_n)\}$ is $\sqsubseteq$-monotone). Therefore,

$$\begin{aligned} & (F(x_n,y_n),F(y_n,x_n))\xrightarrow{\delta}(x,y)\\ \Rightarrow\ & \left[F(x_n,y_n)\xrightarrow{d}x \text{ and } F(y_n,x_n)\xrightarrow{d}y\right],\end{aligned}$$

and

$$\begin{aligned} & (G(x_n,y_n),G(y_n,x_n))\xrightarrow{\delta}(x,y)\\ \Rightarrow\ & \left[G(x_n,y_n)\xrightarrow{d}x \text{ and } G(y_n,x_n)\xrightarrow{d}y\right].\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}F(x_n,y_n) &= \lim_{n\to\infty}G(x_n,y_n)=x\in X,\\ \lim_{n\to\infty}F(y_n,x_n) &= \lim_{n\to\infty}G(y_n,x_n)=y\in X.\end{aligned}$$

Since the pair $\{F,G\}$ is generalized compatible, we deduce that

$$\begin{aligned}\lim_{n\to\infty}d(F(G(x_n,y_n),G(y_n,x_n)),G(F(x_n,y_n),F(y_n,x_n))) &= 0,\\ \lim_{n\to\infty}d(F(G(y_n,x_n),G(x_n,y_n)),G(F(y_n,x_n),F(x_n,y_n))) &= 0.\end{aligned}$$

In particular,

$$\begin{aligned}
& \lim_{n\to\infty} \delta(T_G T_F(x_n, y_n), T_F T_G(x_n, y_n)) \\
= & \lim_{n\to\infty} \delta(T_G(F(x_n, y_n), F(y_n, x_n)), T_F(G(x_n, y_n), G(y_n, x_n))) \\
= & \lim_{n\to\infty} \delta\left(\begin{array}{c} (G(F(x_n, y_n), F(y_n, x_n)), G(F(y_n, x_n), F(x_n, y_n))), \\ (F(G(x_n, y_n), G(y_n, x_n)), F(G(y_n, x_n), G(x_n, y_n))) \end{array} \right) \\
= & \lim_{n\to\infty} \max \left\{ \begin{array}{c} d(G(F(x_n, y_n), F(y_n, x_n)), F(G(x_n, y_n), G(y_n, x_n))), \\ d(G(F(y_n, x_n), F(x_n, y_n)), F(G(y_n, x_n), G(x_n, y_n))) \end{array} \right\} \\
= & \; 0.
\end{aligned}$$

Hence, the mappings T_F and T_G are O-compatible in $(X^2, \delta, \sqsubseteq)$. □

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two generalized compatible mappings such that F is $G-$increasing with respect to $\preceq$, G is continuous and has the mixed monotone property, and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).$$

Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying (2.13) and for any $x, y \in X$, there exist $u, v \in X$ such that

$$F(x, y) = G(u, v) \text{ and } F(y, x) = G(v, u). \tag{2.17}$$

Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Proof. It is only necessary to apply Theorem 2.1 to the mappings $T = T_F$ and $g = T_G$ in the partially ordered metric space $(X^2, \delta, \sqsubseteq)$ taking into account all items of Lemma 2.2. □

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two commuting mappings such that F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).$$

Suppose that (2.13) and (2.17) hold and either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Now we deduce the results without mixed g-monotone property of F.

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$ and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} &\psi\left(d(F(x,\ y),\ F(u,\ v))\right) \\ \leq\ & \psi\left(M_g(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M_g(x,\ y,\ u,\ v)\right)\right) + \theta\left(N_g(x,\ y,\ u,\ v)\right), \end{aligned} \tag{2.18}$$

where

$$M_g(x,\ y,\ u,\ v) = \max \left\{ \begin{array}{c} d(gx,\ gu),\ d(gx,\ F(x,\ y)),\ d(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ d(gy,\ F(y,\ x)),\ d(gv,\ F(v,\ u)), \\ \frac{d(gx,\ F(u,\ v))+d(gu,\ F(x,\ y))}{2}, \\ \frac{d(gy,\ F(v,\ u))+d(gv,\ F(y,\ x))}{2} \end{array} \right\},$$

and

$$N_g(x,\ y,\ u,\ v) = \min \left\{ \begin{array}{c} d(gx,\ gu),\ d(gu, F(x,\ y)) \\ d(gy,\ gv),\ d(gv,\ F(y,\ x)) \end{array} \right\},$$

for all $x,\ y,\ u,\ v \in X$, where $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$ and the pair $\{F, g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$ and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying (2.18). Suppose that $F(X^2) \subseteq g(X)$, g is continuous, monotone increasing with respect to $\preceq$ and the pair $\{F, g\}$ is commuting. Also suppose that either*

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Now, we deduce the result without mixed monotone property of F.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ be an increasing mapping with respect to $\preceq$ and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} (2.19) \qquad & \psi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \psi\left(m(x, y, u, v)\right) - \varphi\left(\psi\left(m(x, y, u, v)\right)\right) + \theta\left(n(x, y, u, v)\right), \end{aligned}$$

where

$$m(x, y, u, v) = \max\left\{\begin{array}{c} d(x, u),\ d(x, F(x, y)),\ d(u, F(u, v)), \\ d(y, v),\ d(y, F(y, x)),\ d(v, F(v, u)), \\ \frac{d(x, F(u, v))+d(u, F(x, y))}{2}, \\ \frac{d(y, F(v, u))+d(v, F(y, x))}{2} \end{array}\right\},$$

and

$$n(x, y, u, v) = \min\left\{\begin{array}{c} d(x, u),\ d(u, F(x, y)), \\ d(y, v),\ d(v, F(y, x)) \end{array}\right\},$$

for all $x, y, u, v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Then F has a coupled fixed point.

Example 2.1. Suppose that $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ and with the natural ordering of real numbers $\leq$. Let $F, G : X^2 \to X$ be defined as

$$\begin{aligned} F(x, y) &= \begin{cases} \frac{x^2-y^2}{3}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases} \\ G(x, y) &= \begin{cases} x^2 - y^2, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases} \end{aligned}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{t}{2}, \text{ for all } t \geq 0,$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{t}{3}, \text{ for all } t \geq 0,$$

and $\theta : [0, +\infty) \to [0, +\infty)$ by

$$\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.$$

First, we shall show that the mappings F and G satisfy the condition (2.13). Let $x, y, u, v \in X$ such that $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$, we have

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
= & \left|\frac{x^2 - y^2}{3} - \frac{u^2 - v^2}{3}\right| \\
= & \frac{1}{3}\left|G(x,\ y) - G(u,\ v)\right| \\
= & \frac{1}{3} d(G(x,\ y),\ G(u,\ v)) \\
\leq & \frac{1}{3} M(x,\ y,\ u,\ v) \\
\leq & \psi\left(M(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right) + \theta\left(N(x,\ y,\ u,\ v)\right).
\end{aligned}$$

Thus the contractive condition (2.13) is satisfied for all $x, y, u, v \in X$. In addition, like in [16], all the other conditions of Theorem 2.2 are satisfied and $z = (0, 0)$ is a coupled coincidence point of F and G.

3. Application to integral equations

As an application of the results established in section 2 of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))\left[f(q,\ x(q)) + g(q,\ x(q))\right] dq + h(p), \tag{3.1}$$

for all $p \in I = [a, b]$.

Let Υ denote the set of all functions $\gamma : [0, +\infty) \to [0, +\infty)$ satisfying

(i_γ) γ is non-decreasing,
(ii_γ) $\gamma(p) \leq p$.

Condition 3.1. *We assume that the functions K_1, K_2, f, g fulfill the following conditions:*

(i) $K_1(p, q) \geq 0$ *and* $K_2(p, q) \leq 0$ *for all* $p, q \in I$,

(ii) *There exists positive numbers λ, μ and $\gamma \in \Upsilon$ such that for all $x, y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:*

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\gamma(x - y), \tag{3.2}$$

$$-\mu\gamma(x - y) \leq g(q,\ x) - g(q,\ y) \leq 0, \tag{3.3}$$

(*iii*)

$$\max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b [K_1(p,\ q) - K_2(p,\ q)]dq \le \frac{1}{6}. \tag{3.4}$$

Example 3.1. Let $P = \{a = x_0, x_1, x_2, ..., x_{r-1}, x_r, ..., x_n = b\}$ be any partition of the interval $I = [a, b]$ whose r^{th} sub-interval is $I_r = [x_{r-1}, x_r]$, its length is $\delta_r = x_r - x_{r-1}$, where $r = 1, 2, ...,$ n and $\|P\| = \max_{1\le r\le n} \delta_r$.

The functions K_1, K_2, f, g, h are defined as follows:

$$\begin{aligned} K_1(p,\ q) &= \begin{cases} \frac{e^{-\sigma(p-q)}}{24(1-e^{-\sigma T})}, & 0 \le q < p \le T, \\ \frac{e^{-\sigma(p-q+T)}}{24(1-e^{-\sigma T})}, & 0 \le p < q \le T, \end{cases} \\ K_2(p,\ q) &= \begin{cases} \frac{e^{\sigma(p-q)}}{24(1-e^{\sigma T})}, & 0 \le q < p \le T, \\ \frac{e^{\sigma(p-q+T)}}{24(1-e^{\sigma T})}, & 0 \le p < q \le T, \end{cases} \end{aligned}$$

where $\sigma = \max\{\lambda, \mu\}$, λ, μ are positive numbers and $T = |I| = b - a$. Define $f(q, x) = \delta x$ and $g(q, x) = -\delta x$, where δ denote the length of the sub-interval of I in which q lies and also $h(p) = p$ and $\gamma(p) = p$ forall $p \in I$. Then the functions K_1, K_2, f, g, h and γ satisfying condition 3.1.

Definition 3.1. [21]. A pair $(\alpha, \beta) \in X^2$ with $X = C(I, \mathbb{R})$, where $C(I, \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a coupled lower-upper solution of (3.1) if, for all $p \in I$,

$$\begin{aligned} \alpha(p) \le & \int_a^b K_1(p,\ q)\left[f(q,\ \alpha(q)) + g(q,\ \beta(q))\right] dq \\ & + \int_a^b K_2(p,\ q)\left[f(q,\ \beta(q)) + g(q,\ \alpha(q))\right] dq + h(p), \end{aligned}$$

and

$$\begin{aligned} \beta(p) \ge & \int_a^b K_1(p,\ q)\left[f(q,\ \beta(q)) + g(q,\ \alpha(q))\right] dq \\ & + \int_a^b K_2(p,\ q)\left[f(q,\ \alpha(q)) + g(q,\ \beta(q))\right] dq + h(p). \end{aligned}$$

Theorem 3.1. *Consider the integral equation (3.1) with $K_1, K_2 \in C(I \times I, \mathbb{R})$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and $h \in C(I, \mathbb{R})$. Suppose that there exists a coupled lower-upper solution (α, β) of (3.1) and Condition 3.1 is satisfied. Then the integral equation (3.1) has a solution in $C(I, \mathbb{R})$.*

Proof. Consider $X = C(I, \mathbb{R})$, the natural partial order relation, that is, for $x, y \in C(I, \mathbb{R})$,

$$x \preceq y \Longleftrightarrow x(p) \leq y(p), \text{ for all } p \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} \left| x(p) - y(p) \right|.$$

Now define on X^2 the following partial order: for $(x, y), (u, v) \in X^2$,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p), \text{ for all } p \in I.$$

Obviously, for any $(x, y) \in X^2$, the functions $\max\{x, y\}$ and $\min\{x, y\}$ are the upper and lower bounds of x and y respectively. Therefore for every $(x, y), (u, v) \in X^2$, there exists the element $(\max\{x,\ u\},\ \min\{y,\ v\})$ which is comparable to (x, y) and (u, v). Define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{t}{2}, \text{ for all } t \geq 0,$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{t}{3}, \text{ for all } t \geq 0,$$

and $\theta : [0, +\infty) \to [0, +\infty)$ by

$$\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.$$

Define now the mapping $F : X^2 \to X$ by

$$\begin{aligned} F(x,\ y)(p) \quad &= \int_a^b K_1(p,\ q)\left[f(q,\ x(q)) + g(q,\ y(q))\right] dq \\ &\quad + \int_a^b K_2(p,\ q)\left[f(q,\ y(q)) + g(q,\ x(q))\right] dq + h(p), \end{aligned}$$

for all $p \in I$. We can prove, like in [16], that F is increasing. Now for $x, y, u, v \in X$

with $x \geq u$ and $y \leq v$, by using (3.2) and (3.3), we have

$$
\begin{aligned}
& F(x,\ y)(p) - F(u,\ v)(p) \\
= & \int_a^b K_1(p,\ q)\left[f(q,\ x(q)) + g(q,\ y(q))\right] dq \\
& + \int_a^b K_2(p,\ q)\left[f(q,\ y(q)) + g(q,\ x(q))\right] dq \\
& - \int_a^b K_1(p,\ q)\left[f(q,\ u(q)) + g(q,\ v(q))\right] dq \\
& - \int_a^b K_2(p,\ q)\left[f(q,\ v(q)) + g(q,\ u(q))\right] dq \\
= & \int_a^b K_1(p,\ q)\left[f(q,\ x(q)) - f(q,\ u(q)) + g(q,\ y(q)) - g(q,\ v(q))\right] dq \\
& + \int_a^b K_2(p,\ q)\left[f(q,\ y(q)) - f(q,\ v(q)) + g(q,\ x(q)) - g(q,\ u(q))\right] dq \\
= & \int_a^b K_1(p,\ q)\left[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))\right] dq \\
& - \int_a^b K_2(p,\ q)\left[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))\right] dq \\
\leq & \int_a^b K_1(p,\ q)\left[\lambda\gamma\,(x(q) - u(q)) + \mu\gamma\,(v(q) - y(q))\right] dq \\
& - \int_a^b K_2(p,\ q)\left[\lambda\gamma\,(v(q) - y(q)) + \mu\gamma\,(x(q) - u(q))\right] dq.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&F(x,\ y)(p) - F(u,\ v)(p) \\
\leq\ & \int_a^b K_1(p,\ q)\left[\lambda\gamma\left(x(q) - u(q)\right) + \mu\gamma\left(v(q) - y(q)\right)\right] dq \\
&- \int_a^b K_2(p,\ q)\left[\lambda\gamma\left(v(q) - y(q)\right) + \mu\gamma\left(x(q) - u(q)\right)\right] dq.
\end{aligned} \tag{3.5}
$$

Since the function γ is non-decreasing, $x \geq u$ and $y \leq v$, we have

$$
\begin{aligned}
\gamma\left(x(q) - u(q)\right) &\leq \gamma\left(\sup_{q\in I}\left|x(q) - u(q)\right|\right) = \gamma(d(x,\ u)), \\
\gamma\left(v(q) - y(q)\right) &\leq \gamma\left(\sup_{q\in I}\left|v(q) - y(q)\right|\right) = \gamma(d(y,\ v)).
\end{aligned}
$$

Hence by (3.5), in view of the fact that $K_2(p, q) \leq 0$, we obtain

$$
\begin{aligned}
&\left|F(x,\ y)(p) - F(u,\ v)(p)\right| \\
\leq\ & \int_a^b K_1(p,\ q)\left[\lambda\gamma(d(x,\ u)) + \mu\gamma(d(y,\ v))\right] dq \\
&- \int_a^b K_2(p,\ q)\left[\lambda\gamma(d(y,\ v)) + \mu\gamma(d(x,\ u))\right] dq \\
\leq\ & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\gamma(d(x,\ u)) + \max\{\lambda,\ \mu\}\gamma(d(y,\ v))\right] dq \\
&- \int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\gamma(d(y,\ v)) + \max\{\lambda,\ \mu\}\gamma(d(x,\ u))\right] dq,
\end{aligned}
$$

as all the quantities on the right hand side of (3.5) are non-negative. Now, taking the supremum with respect to p, by using (3.4), we get

$$
\begin{aligned}
&d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b \left(K_1(p,\ q) - K_2(p,\ q)\right) dq. \left[\gamma(d(x,\ u)) + \gamma(d(y,\ v))\right] \\
\leq\ & \frac{\gamma(d(x,\ u)) + \gamma(d(y,\ v))}{6}.
\end{aligned}
$$

Thus

$$\frac{1}{2}d(F(x, y), F(u, v)) \leq \frac{\gamma(d(x, u)) + \gamma(d(y, v))}{12}. \tag{3.6}$$

Now, since γ is non-decreasing, we have

$$\begin{aligned} \gamma(d(x, u)) &\leq \gamma\left(d(x, u) + d(y, v)\right), \\ \gamma(d(y, v)) &\leq \gamma\left(d(x, u) + d(y, v)\right), \end{aligned}$$

which implies, by (ii_γ), that

$$\begin{aligned} \frac{\gamma(d(x, u)) + \gamma(d(y, v))}{2} &\leq \gamma\left(d(x, u) + d(y, v)\right) \\ &\leq d(x, u) + d(y, v). \end{aligned}$$

Hence

$$\frac{\gamma(d(x, u)) + \gamma(d(y, v))}{12} \leq \frac{1}{6}d(x, u) + \frac{1}{6}d(y, v). \tag{3.7}$$

Thus by (3.6) and (3.7), we have

$$\begin{aligned} & \psi\left(d(F(x, y), F(u, v))\right) \\ = \;& \frac{1}{2}d(F(x, y), F(u, v)) \\ \leq \;& \frac{1}{6}d(x, u) + \frac{1}{6}d(y, v) \\ \leq \;& \frac{1}{3}m(x, y, u, v) \\ \leq \;& \psi\left(m(x, y, u, v)\right) - \varphi\left(\psi\left(m(x, y, u, v)\right)\right) \\ \leq \;& \psi\left(m(x, y, u, v)\right) - \varphi\left(\psi\left(m(x, y, u, v)\right)\right) + \theta\left(n(x, y, u, v)\right), \end{aligned}$$

which is the contractive condition (2.19) in Corollary 2.4. Now, let $(\alpha, \beta) \in X^2$ be a coupled upper-lower solution of (3.1), then we have $\alpha(p) \leq F(\alpha, \beta)(p)$ and $\beta(p) \geq F(\beta, \alpha)(p)$, for all $p \in I$, which shows that all hypothesis of Corollary 2.4 are satisfied. This proves that F has a coupled fixed point $(x, y) \in X^2$ which is the solution in $X = C(I, \mathbb{R})$ of the integral equation (3.1). $\square$

Remark 3.1. Using the same techniques that can be found in ([17]-[19], [24], [29], [30]), it is possible to deduce, from Theorem 2.1, tripled, quadruple and in general, multidimensional coincidence point theorems.

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