FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 157-163
https://doi.org/10.22190/FUMI200420013K
Original Scientific Paper

PARALLELISM OF DISTRIBUTIONS AND GEODESICS ON $F(\pm a^2, \pm b^2)$-STRUCTURE LAGRANGIAN MANIFOLD

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Abstract. This paper deals with the Lagrange vertical structure on the vertical tangent space $T_V(N)$ endowed with a non-zero (1,1) tensor field F_v satisfying $(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2) = 0$. The similar structure on the horizontal subspace $T_H(N)$ and on $T(N)$ is investigated if the $F(\pm a^2, \pm b^2)$-structure on $T_V(N)$ is given. Furthermore, we have proved some theorems and obtained conditions under which the distribution P and Q are ∇-parallel, $\bar{\nabla}$ anti half parallel when $\nabla = \bar{\nabla}$. Finally, certain theorems on geodesics on the Lagrange manifold are established.
Keywords: Distribution, Parallelism, Geodesic, Almost product structure.

1. Introduction

Let M and N be two differentiable manifolds of dimension n and $2n$ respectively and (N, π, M) be vector bundle with $\pi(N) = M$. The local coordinate systems $(x^1, x^2, ..., x^n)$ about x in M and $(y^1, y^2, ..., y^n)$ about y in N. Let $(x^i, y^\alpha), 1 \leq i \leq n, 1 \leq \alpha \leq n$ be system of local coordinates in the open set $\pi^{-1}(U)$ and called induced coordinates in $\pi^{-1}(U)$, where U is a coordinate neighborhood in M. Let $T_p(N)$ be tangent space and $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha} \right\}$ canonical basis for $T_p(N)$ such that $p \in \pi^{-1}(U)$ and it is also denoted by $\{\partial_i, \partial_\alpha\}$ where $\partial_i = \frac{\partial}{\partial x^i}$. If (x^h, x^{α^1}) be coordinates of a point in the interesting region $\pi^{-1}(U) \cap \pi^{-1}(U)$, then [2, 6]

$$(1.1) \qquad x^{i^1} = x^{i^1}(x^i),$$

Received April 20, 2020; accepted December 5, 2020.
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2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C22

(1.2) $$y^{\alpha^1} = \frac{\partial x^{\alpha^1}}{\partial x^{\alpha}} y^{\alpha},$$

and another canonical basis in the intersecting region are given by

(1.3) $$\partial_{i^1} = \frac{\partial x^i}{\partial x^{i^1}} \partial_i$$

(1.4) $$\partial_{\alpha^1} = \frac{\partial y^{\alpha}}{\partial y^{\alpha^1}} \partial_{\alpha}.$$

The tangent space of N is denoted by $T(N)$ and spanned by $\{\partial_i, \partial_\alpha\}$ and its subspaces by $T_V(N)$ and $T_H(N)$ spanned by $\{\partial_\alpha\}$ and $\{\partial_i\}$ respectively [8]. Then we have,

(1.5) $$dimT_V(N) = dimT_H(N) = n.$$

The Riemannian material structure on $T(N)$ is given by

(1.6) $$G = g_{ij}(x^i, y^\alpha) dx^i \otimes dx^j + g_{ab}(x^i, y^\alpha) \delta y^\alpha \otimes \delta y^b,$$

where $g_{ij}(x^i, y^\alpha) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2}\partial_a \partial_b L(x^i, y^\alpha)$ and $L(x^i, y^\alpha)$ denotes the Lagrange function. The manifold referred as Lagrangian manifold [2].

Let X be an element of $T(N)$, then

(1.7) $$X = \bar{X}^i \partial_i + X^\alpha \partial_\alpha.$$

The automorphism $J : \chi(T(N)) \to \chi(T(N))$ given as

(1.8) $$JX = \bar{X}^i \partial_i + X^\alpha \partial_\alpha$$

is a natural almost product structure on $T(N)$ that is $J^2 = I, I$ denotes the identity operator. The projection morphisms of $T(N)$ onto $T_V(N)$ and $T_H(N)$ denoted by v and h respectively, then we have

(1.9) $$J_0 h = v_0 J.$$

2. The $F(\pm a^2, \pm b^2)$-structure

Let $T_V(N)$ be the vertical space and F_v a non-zero tensor field of type (1,1) satisfying [10]

(2.1) $$(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2) = 0,$$

where a, b are real or complex constants, then the vertical space $T_V(N)$ admits $F(\pm a^2, \pm b^2)$-structure. The rank $(F_v) = r$ and such structure is called Lagrange vertical structure on $T_V(N)$.

Theorem 2.1. *Let $T_V(N)$ be a vertical space ad F_v Lagrange vertical structure on $T_V(N)$. Then the structure define on the subspace $T_H(N)$ with respect to almost product strcture of $T(N)$.*

Proof: Suppose that

$$F_h = JF_vJ, \tag{2.2}$$

then F_h is a tensor field of type (1,1) on $T_H(N)$, where J is an almost product structure on $T(N)$.

Apply F_h on both sides we get

$$F_h^2 = (JF_vJ)(JF_vJ) = JF_v^2J,$$

$$F_h^3 = JF_v^3J$$

and so on.

In the view of equation (2.1), we have

$$\begin{aligned}(2.3)\qquad &(F_h^2 - a^2)(F_h^2 + a^2)(F_h^2 - b^2)(F_h^2 + b^2)\\ &= J((F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2))J\\ &= 0,\end{aligned}$$

Hence, F_h gives $F(\pm a^2, \pm b^2)$-structure on $T_H(N)$.

Theorem 2.2. *Let $T_V(N)$ be a vertical space ad F_v Lagrange vertical structure on $T_V(N)$. Then the similar structure define on the enveloping space $T(N)$ by using projection morphism of $T(N)$.*

Proof: In the view of Theorem (2.1), the projection morphisms of $T_V(N)$ and $T_H(N)$ on $T(N)$ denoted by v and h respectively then we have

$$F = F_vh + F_vv \tag{2.4}$$

As $hv = vh = 0$ and $h^2 = h, v^2 = v$, we obtain

$$F^2 = F_h^2h + F_v^2v$$

Now,

$$\begin{aligned}&(F^2 - a^2)(F^2 + a^2)(F^2 - b^2)(F^2 + b^2)\\ &= (F_h^2 - a^2)(F_h^2 + a^2)(F_h^2 - b^2)(F_h^2 + b^2)h\\ (2.5)\qquad &+(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2)v\end{aligned}$$

By theorem 2.1, we have

$$(F^2 - a^2)(F^2 + a^2)(F^2 - b^2)(F^2 + b^2) = 0.$$

As $rank(F_v) = rank(F_h) = r$,

Hence, $rank(F) = 2r$.

Let us define tensor fields p and q of type (1,1) on $T(N)$ with $F(\pm a^2, \pm b^2)$-structure of rank $2r$ as follows

$$p = \frac{(F^2 + a^2)(F^2 - a^2)}{b^4 - a^4}$$

$$q = \frac{(F^2 + b^2)(F^2 - b^2)}{a^4 - b^4} \tag{2.6}$$

Then it is easy to show that

$$p^2 = p, \quad q^2 = q, \quad pq = qp = 0, \quad p + q = I. \tag{2.7}$$

This implies that p and q are complementary projection operators [4, 5, 7].

3. Parallelism of distributions

Suppose that N be Lagrangian manifold with $F(\pm a^2, \pm b^2)$-structure on $T(N)$ and let P and Q complementary distributions corresponding to complementary projection operators p and q respectively. The linear connection $\bar{\nabla}$ and $\tilde{\nabla}$ are given by [2]

$$\bar{\nabla}_X Y = p\nabla_X(pY) + q\nabla_X(qY) \tag{3.1}$$

and

$$\tilde{\nabla}_X Y = p\nabla_{pX}(pY) + q\nabla_{qX}(qY) + p[qX, pY] + q[pX, qY]. \tag{3.2}$$

We have the following definitions [3, 6]:

∇-parallel: The distribution P is said ∇-parallel if $\forall X \in P, Y \in T(N)$ implies that $\nabla_Y X \in P$.

∇-half parallel: The distribution P is said ∇-half parallel if $\forall X \in P, Y \in T(N)$, $(\Delta F)(X, Y) \in P$ where

$$(\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y(FX) \tag{3.3}$$

∇-anti half parallel: The distribution P is said ∇-anti half parallel if for all $X \in P, Y \in T(N), (\Delta F)(X, Y) \in Q$.

Theorem 3.1. *On the $F(\pm a^2, \pm b^2)$-structure manifold, the complementary distributions namely P and Q are $\bar{\nabla}$-parallel and $\tilde{\nabla}$-parallel.*

Proof: By using the equations (3.1), (3.2) and $pq = qp = 0, \quad q^2 = q$, we obtain

$$q\bar{\nabla}_X Y = q\nabla_X(qY)$$

If $Y \in P, qY = 0$ so $q\bar{\nabla}_X Y = 0 \to \bar{\nabla}_X Y = 0$, as $qY = 0$ because Y is an element of P.

This implies that $\bar{\nabla}_X Y \in P$.

Thus, $\forall Y \in P, \forall X \in T(N) \Rightarrow \bar{\nabla}_X Y \in P$.

Hence P is $\bar{\nabla}$-parallel.

In a similar way $\forall X \in T(N), \forall Y \in P$

$\tilde{\nabla}_X Y = q\nabla_{qX}(qY) + q[pX, qY] = 0$ as $qY = 0$.

So $\tilde{\nabla}_X Y \in P$.

Thus P is $\tilde{\nabla}$-parallel.

In a similar way, it can be shown that distribution Q is $\bar{\nabla}$ as well as $\tilde{\nabla}$ parallel.

Theorem 3.2. *On the $F(\pm a^2, \pm b^2)$-structure manifold, the complementary distributions namely P and Q are ∇-parallel iff $\bar{\nabla} = \tilde{\nabla}$.*

Proof: Let distributions P and Q are ∇-parallel. By definition of ∇-parallel, we have

$$q\nabla_X(pY) = 0, \quad p\nabla_X(qY) = 0.$$

where X and Y are elements of $T(N)$.

Using equation (2.7), we get

$$\nabla_X(pY) = p\nabla_X(pY) \tag{3.4}$$

and

$$\nabla_X(qY) = q\nabla_X(qY) \tag{3.5}$$

Thus

$$\nabla_X Y = p\nabla_X(pY) + q\nabla_X(qY) = \bar{\nabla}_X Y.$$

This shows that $\nabla = \bar{\nabla}$.

The converse of the theorem showed easily.

Theorem 3.3. *On the $F(\pm a^2, \pm b^2)$-structure manifold N, the complementary distribution M is $\bar{\nabla}$-anti half parallel if*

$$q\bar{\nabla}_Y(FX) = q\nabla_{FX}qY.$$

where X is an element of Q and Y element of $T(N)$.

Proof: Let $\bar{\nabla}$ be linear connection on N. Then by using equations (3.3) and (2.7), we obtain

$$q(\Delta F)(X, Y) = q\bar{\nabla}_Y FX - q\bar{\nabla}_{FX} Y, \quad \textit{as} \quad qF = Fq = 0. \tag{3.6}$$

Making use of the equation (3.1), the obtained equation is

$$\bar{\nabla}_{FX} Y = p\nabla_{FX}(pY) + q\nabla_{FX}(qY)$$

operating q on both sides of above equation and using $pq = 0, q^2 = q$, we get

$$q\bar{\nabla}_{FX} Y = q\nabla_{FX}(qY)$$

and

$$q(\Delta F)(X,Y) = q\bar{\nabla}_Y FX - q\bar{\nabla}_{FX} Y,$$

as $(\Delta F)(X,Y) \in P$ so $q(\Delta F)(X,Y) = 0$.

Hence,

$$q\bar{\nabla}_Y (FX) = q\nabla_{FX}(qY),$$

This completes the proof.

3.1. Geodesics on the Lagrangian manifold

Let T be tangent to the curve γ in N. The curve γ is said the geodesic concernig to the connection ∇ if $\nabla_T T$ [6].

Theorem 3.4. *A curve γ is said to be geodesic concerning to connection $\bar{\nabla}$ if the vector fields $\nabla_T T - \nabla_T(qT) \in Q$ and $\nabla_T(qT) \in P$.*

Proof: The curve γ is said to be geodesic concerning to the connection $\bar{\nabla}$, we have $\bar{\nabla}_T T = 0$.

In the view of the equation (3.1), $\bar{\nabla}_T T = 0$ becomes

$$p\nabla_T(pT) + q\nabla_T(qT) = 0, \tag{3.7}$$

Using the equation (2.7), the equation (3.7) becomes

$$p\nabla_T(I - q)T + q\nabla_T(qT) = 0$$

or

$$p\nabla_T T - p\nabla_T(qT) + q\nabla_T(qT) = 0.$$

or

$$p(\nabla_T T - \nabla_T(qT)) \;\; and \;\; q\nabla_T(qT) = 0.$$

Hence, $\nabla_T T - \nabla_T(qT) \in Q$ and $\nabla_T(qT) \in P$.

This completes the proof.

Theorem 3.5. *The tensor fields p and q of type (1,1) are always covariantly constants concerning to connection $\bar{\nabla}$.*

Proof: Let X and Y be elements of $T(N)$, then

$$(\bar{\nabla}_X p)(Y) = \bar{\nabla}_X(pY) - p\bar{\nabla}_X Y. \tag{3.8}$$

From equation (3.1), we have

$$(\bar{\nabla}_X p)(Y) = p\nabla_X(p^2 Y) + q\nabla_X(qpY) - p\left\{p\nabla_X pY + q\nabla_X qY)\right\}$$

Using the properties $p^2 = p, q^2 = q, pq = qp = 0$, we have

$$(\bar{\nabla}_X p)(Y) = p\nabla_X(pY) - p\nabla_X pY = 0.$$

This shows that p is covariantly constant. In similar way, q is covariantly constant can be proved easily.

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