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SPACES OF FIBONACCI DIFFERENCE IDEAL CONVERGENT SEQUENCES IN RANDOM 2–NORMED SPACE

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Abstract. In this article, by using the same Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence of sequences in random 2–normed space in the same technique, we have introduced new spaces of Fibonacci difference ideal convergent sequences with respect to random 2 –norm and studied some inclusion relations, as well as topological and algebraic properties of these spaces.
Key words: ideals, statistical convergence, probabilistic metric spaces.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers respectively. By ω we denote the linear space of sequence of real numbers. c_0, c and ℓ_∞ represent sequence spaces of null convergent, convergent and bounded sequences respectively. The approach to statistical convergence was done by Fast [6] and Steinhaus [19] in 1951 independently. In 1999, Kostryko et al. [14] generalised the notion of statistical convergence to ideal convergence and some properties of this interesting generalization have been studied by Šalát et al. [17]. An ideal is a non–empty subset of the set of natural numbers $\mathbb{N}$ which satisfies hereditary and additivity property,

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i.e., $I \subseteq 2^{\mathbb{N}}$ such that $A \in I$ with $B \subset A$ implies $B \in I$ and $A \cup B \in I$ whenever $A, B \in I$. A non–empty family of sets $F \subseteq 2^{\mathbb{N}}$ is said to be a filter on $\mathbb{N}$ if only if $\phi \notin F, A \cap B \in F$ for $A, B \in F$ and any superset of an element of F is in F. An ideal I is non–trivial if $I \neq 2^{\mathbb{N}}$. A non–trivial ideal I is admissible if it contains all singletons. A sequence $x = (x_n) \in \omega$ is said to be $I-$ convergent to $L \in \mathbb{R}$ if the set $\{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in I$ for every $\epsilon > 0$. If $L = 0$, then we say the sequence is $I-$ null. The concept of ideal convergence was studied from the sequence point of view and linked with the summability theory by Hazarika and Savaş [11, 10]. The approach to construct sequence spaces by means of the domain of an infinite matrix and with the help of the notion of ideal convergence was firstly used by Śalát et. al [18] to introduce the sequence spaces $(c^I)_A$ and $(m^I)_A$. The theory of random 2–normed space was introduced by Gölet and studied some properties of convergence and Cauchy sequence with respect to random 2–norm as well. Recently, the notion of ideal convergence of sequences in the framework of random 2–normed spaces defined by Mursaleen and Alotaibi [15].

In 2013, Kara defined the double band matrix matrix $\hat{F} = (\hat{f}_{nk})$ by:

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & \text{if } k = n-1 \\ \frac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci sequence defined by the recurrence relation $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ satisfying some basic properties and addressed the approach to construct sequence spaces by means of an infinite matrix of particular limitation methods to introduced the Fibonacci difference sequence space

$$\ell_\infty(\hat{F}) = \Big\{x = (x_n) \in \omega : \sup_{n \in \mathbb{N}} \Big|\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}\Big| < \infty\Big\}.$$

The domains $c_0(\Delta^F), c(\Delta^F)$ and $l_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and l_∞ are introduced by Kizmaz [13]. Aftermore, the domain bv_p of the backward difference matrix Δ^B in the space l_p have recently been investigated for $0 < p < 1$ by Altay and Başar [1], and for $1 \leq p \leq \infty$ by Başar and Altay [2]. Quite recently, by combining the definitions of ideal convergence and the Fibonacci difference matrix $\hat{F}$, Khan et al. [12] have introduced some new Fibonacci difference sequence spaces

$$\lambda(\hat{F}) = \{x = (x_n) \in \omega : \hat{F}x = ((\hat{F}x)_n) \in \lambda\},$$

for $\lambda = c_0^I, c^I$ and ℓ_∞^I, the spaces of all I–null and I–convergent sequences,where the sequence $\hat{F}x = ((\hat{F}x)_n)$ is the $\hat{F}$–transform of the sequence $x = (x_n) \in \omega$ defined as follows:

$$\hat{F}(x) = ((\hat{F}x)_n) = \begin{cases} \frac{f_0}{f_1} x_0, & n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \geq 1. \end{cases}$$

For more work on difference sequence spaces and Fibonacci difference sequence space please see the references [16, 4, 5].

In this article, by using Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence in random 2–normed space, we introduce new sequence spaces and study their topological and algebraic properties.

We recall some definitions which will be used throughout this article.

Definition 1.1. [7] *A sequence $x = (x_n) \in \omega$ is said to be statistically convergent to $L \in \mathbb{R}$ if for every $\epsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ is zero. We write st–$\lim x_n = L$.*

Definition 1.2. [12] *An ideal is a subset of the set of natural numbers $\mathbb{N}$ which satisfies hereditary and additivity property, i.e., $I \subseteq 2^{\mathbb{N}}$ such that $A \in I$ with $B \subset A$ implies $B \in I$ and $A \cup B \in I$ whenever $A, B \in I$. A non–empty family of sets $F \subseteq 2^{\mathbb{N}}$ is said to be a filter on $\mathbb{N}$ if only if $\phi \notin F, A \cap B \in F$ for $A, B \in F$ and any superset of an element of F is in F. An ideal I is non–trivial if $I \neq 2^{\mathbb{N}}$. A non–trivial ideal I is admissible if it contains all singletons. A sequence $x = (x_n) \in \omega$ is said to be I–convergent to $L \in \mathbb{R}$ if $n \in \mathbb{N} : |x_n - L| \geq \epsilon \in I$ for every $\epsilon > 0$. If $L = 0$, then we say that the sequence is I–null.*

Definition 1.3. [15] *A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is said to be a distribution function if it is non–decreasing and left continuous such that $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions with $f(0) = 0$. For $a \in \mathbb{R}_0^+, H_a \in D^+$*

$$H_a(t) = \begin{cases} 1, & t > a \\ 0, & t \leq a \end{cases}$$

Definition 1.4. [15] *A triangular norm is a continuous map $* : [0,1] \times [0,1] \to [0,1]$, $([0,1], *)$ is an abelian monoid with unit one and $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$ for all $a, b, c, d \in [0,1]$. A triangle τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.*

Definition 1.5. [8] *Let X be a vector space with dimension more than 1. A function $\|.,.\| : X \times X \to \mathbb{R}$ with the following properties:*

(1) $\|x_1, x_2\| = 0$ *if and only if x_1, x_2 are linearly dependent,*

(2) $\|x_1, x_2\| = \|x_2, x_1\|$,

(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|, \alpha \in \mathbb{R}$,

(4) $\|x_1 + x_2, x_3\| \leq \|x_1, x_3\| + \|x_2, x_3\|$.

Then $(X, \|.,.\|)$ is called a 2–normed space.

Definition 1.6. [9] *Let X be a linear space of dimension greater than 1, $*$ denote a t norm. $\mathcal{F} : X \times X \to D^+$ is said to be random 2–norm if the following conditions are satisfied:*

(1) $\mathcal{F}(x_1, x_2; t) = H_0(t)$ *if x_1, x_2 are linearly dependent,*

(2) $\mathcal{F}(x_1, x_2; t) \neq H_0(t)$ *if x_1, x_2 are linearly independent,*

(3) $\mathcal{F}(x_1, x_2; t) = \mathcal{F}(x_2, x_1; t)$ *for all $x_1, x_2 \in X$,*

(4) $\mathcal{F}(\alpha x_1, x_2; t) = \mathcal{F}(x_1, x_2; \frac{t}{|\alpha|})$ *for $t > 0, \alpha \neq 0$,*

(5) $\mathcal{F}(x_1, x_2, x_3; t_1 + t_2) \geq \mathcal{F}(x_1, x_3; t_1) * \mathcal{F}(x_2, x_3; t_2)$ *for all $x_1, x_2, x_3 \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$.*

Then $(X, \mathcal{F}, *)$ is called a random 2–normed space (R2NS).

Definition 1.7. [15] *A sequence $x = (x_n) \in X$ is $\mathcal{F}-$ convergent to L in $(X, \mathcal{F}, *)$ if there exists $n_0 > 0$ such that $\mathcal{F}(x_n - L, z; \epsilon) > 1 - \theta$ whenever $n \geq n_0$ for every $\epsilon > 0, \theta \in (0, 1)$ and non–zero $z \in X$. We denote it as $\mathcal{F}-\lim_n x_n = L$.*

Definition 1.8. [15] *Let $(X, \mathcal{F}, *)$ be a R2NS. A sequence $x = (x_n) \in X$ is $I-$ convergent to L in $(X, \mathcal{F}, *)$ if for every $\epsilon > 0, \theta \in (0, 1)$ and non–zero $z \in X$ if the set $\{n \in \mathbb{N} : \mathcal{F}(x_n - L, z; \epsilon) \leq 1 - \theta\} \in I$. We write I^{R2N}–lim $x = L$.*

Definition 1.9. [17] *A sequence space E is said to be solid if $(\alpha_n x_n) \in E$ for $(x_n) \in E$ where (α_n) is a sequence of scalars such that $|\alpha_n| \leq 1$.*

Definition 1.10. [17] *Let $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and E be a sequence space. A K step space of E is a sequence space $\lambda_k^E = \{(x_{k_n} \in \omega : (x_n) \in E\}$. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_k^E$ is a sequence $(y_n) \in \omega$ defined as follows:*

$$y_n = \begin{cases} x_n, & if\ n \in K, \\ 0, & otherwise. \end{cases}$$

A canonical preimage of a step space λ_k^E is a set of canonical preimages of all elements in λ_k^E, i.e., y is in canonical preimage of λ_k^E if and only if y is canonical preimage of some $x \in \lambda_k^E$.

Definition 1.11. [17] *A sequence space E is said to be monotone if it contains the canonical preimage of all its step spaces i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_n) \in E$ the sequence $(\alpha_n x_n)$, where*

$$\alpha_n = \begin{cases} 1, & if\, n \in K \\ 0, & otherwise. \end{cases}$$

Lemma 1.1. *Every solid sequence space is monotone.*

2. Main Results

2.1. Some New Fibonacci Difference Ideal Convergent Sequence Spaces

In the present section, we define Fibonacci difference spaces of I–convergent and I–null sequences in a random 2–normed space. Also, we discuss some inclusion relations topological and algebraic properties of these spaces. Throughout this paper, ideal I is admissible ideal. For $\epsilon > 0$, $0 < \theta < 1$ and non zero z in X, define

$$c_0^{I_{R2N}}(\hat{F}) := \{x = (x_n) \in X : \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n), z; \epsilon) \leq 1 - \theta\} \in I\},$$

$$c^{I_{R2N}}(\hat{F}) := \{x = (x_n) \in X : \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) \leq 1 - \theta\} \in I\}.$$

Remark 2.1. *We introduce an open ball with respect to R2N by means of the domain of the Fibonacci matrix, as follows:*

$$B(((\hat{F}x)_n), r, \epsilon) := \{y \in X : \mathcal{F}((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r \text{ for } \epsilon > 0, 0 < r < 1\}.$$

Theorem 2.1. *The spaces $c_0^{I_{R2N}}(\hat{F})$ and $c^{I_{R2N}}(\hat{F})$ are vector spaces over $\mathbb{R}$.*

Proof. We shall prove the result for $c^{I_{R2N}}(\hat{F})$. Let $x = (x_n)$ and $y = (y_n) \in c^{I_{R2N}}(\hat{F})$, then there exist $L_1, L_2 \in X$ such that for $\epsilon > 0, \theta \in (0,1)$ and non–zero $z \in X$, we have

$$A = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L_1, z; \frac{\epsilon}{2|\alpha|}) \leq 1 - \theta\} \in I,$$

$$B = \{n \in \mathbb{N}; \mathcal{F}(((\hat{F}y)_n) - L_2, z; \frac{\epsilon}{2|\beta|}) \leq 1 - \theta\} \in I,$$

where α and β are non–zero scalars in $\mathbb{R}$. Choose $\eta \in (0,1)$ such that $(1-\theta)*(1-\theta) > 1 - \eta$. Consider

$$C = \{n \in \mathbb{N} : \mathcal{F}((\alpha(\hat{F}x)_n) + (\beta(\hat{F}y)_n)) - (\alpha L_1 + \beta L_2)) \leq 1 - \eta\}.$$

We show $C \subseteq A \cup B$ or equivalently $A^c \cap B^c \subseteq C^c$. Since $A^c \cap B^c \in F(I)$ so is non–empty. Let $m \in A^c \cap B^c \in F(I)$, then

$$\begin{aligned}
& \mathcal{F}((\alpha(\hat{F}x)_n) + (\beta(\hat{F}y)_n) - (\alpha L_1 + \beta L_2), z; \epsilon) \\
\geq\ & \mathcal{F}((\alpha(\hat{F}x)_m) - L_1), z; \frac{\epsilon}{2}) * \mathcal{F}((\beta(\hat{F}y)_m) - L_2), z; \frac{\epsilon}{2}) \\
=\ & \mathcal{F}((\hat{F}x)_m) - L_1, z; \frac{\epsilon}{2|\alpha|}) * \mathcal{F}((\hat{F}y)_m) - L_2, z; \frac{\epsilon}{2|\beta|}) \\
>\ & (1-\theta) * (1-\theta) \\
>\ & 1 - \eta.
\end{aligned}$$

Thus $m \in C^c$ and therefore $A^c \cap B^c \subseteq C^c$. Hence $C \in I$. The proof for $c_0^{I_{R2N}}(\hat{F})$ can be given in the same manner. $\square$

Theorem 2.2. *Let $(X, \mathcal{F}, *)$ be a random 2–space. Every open ball $B((\hat{F}x)_n), r, \epsilon)$ is an open set.*

Proof.

$$B((\hat{F}x)_n, r, \epsilon) := \{y \in X : \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r, \epsilon > 0, 0 < r < 1\}$$

Let $y \in B((\hat{F}x)_n, r, \epsilon)$ then by definition $\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r$, there exists $\epsilon_0 \in (0, \epsilon)$ such that $\mathcal{F}(((\hat{F}x)_n - ((\hat{F}y)_n), z; \epsilon_0) > 1 - r$. Put $\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon_0) = r_0$, then for $r_0 > 1-r$ there exists $s \in (0, 1)$ such that $r_0 > 1-s > 1 - r$. For $r_0 > 1 - s$, there exists $r_1 \in (0, 1)$ with $r_0 * r_1 > 1 - s$. We show $B((\hat{F}y)_n), 1 - r_1, \epsilon - \epsilon_0) \subset B((\hat{F}x)_n), r, \epsilon)$.
Let $w \in B(((\hat{F}y)_n), 1-r_1, \epsilon-\epsilon_0)$. Then $\mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z; \epsilon - \epsilon_0) > r_1$. Now,

$$\begin{aligned} & \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \epsilon) \\ \geq\ & \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon_0) * \mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z, \epsilon - \epsilon_0) \\ \geq\ & r_0 * r_1 \\ >\ & 1 - s \\ >\ & 1 - r \end{aligned}$$

Thus we have, $w \in B(((\hat{F}x)_n), r, \epsilon)$ so that
$B(((\hat{F}y)_n), 1 - r_1, \epsilon - \epsilon_0) \subset B(((\hat{F}x)_n), r, \epsilon)$. □

Remark 2.2. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. Define $\tau^I_{\mathcal{F}}(\hat{F}) := \{A \subset c^{I_{R2N}}(\hat{F})$: for given $x \in A$, we can find $\epsilon > 0$ and $0 < r < 1$ such that $B((\hat{F}x)_n), r, \epsilon) \subset A\}$. Then $\tau^I_{\mathcal{F}}(\hat{F})$ is a topology on $c^{I_{R2N}}(\hat{F})$.*

Remark 2.3. *Since $\{B_x(\frac{1}{n}, \frac{1}{n})(\hat{F}) : n \in \mathbb{N}\}$ is a local base at x, the topology $\tau^I_{\mathcal{F}}(\hat{F})$ is first countable.*

Theorem 2.3. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. $c_0^{I_{R2N}}(\hat{F})$ and $c^{I_{R2N}}(\hat{F})$ are Hausdorff spaces.*

Proof. Let $x, y \in c^{I_{R2N}}(\hat{F})$ with $x \neq y$. For $\epsilon > 0$ and $z \neq 0 \in X, r = \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z, \epsilon) \in (0, 1)$. Given $r_0 \in (r, 1)$ there exists r_1 such that $r_1 * r_1 \geq r_0$. We show the open balls $B(((\hat{F}x)_n), 1 - r_1, \frac{\epsilon}{2})$ and $B(((\hat{F}y)_n), 1 - r_1, \frac{\epsilon}{2})$ are disjoint. Suppose on contrary $w \in B(((\hat{F}x)_n), 1 - r_1, \frac{\epsilon}{2}) \cap B(((\hat{F}y)_n), 1 - r_1, \frac{\epsilon}{2})$, then

$$\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) > r_1, \text{ and } \mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) > r_1$$

$$\begin{aligned} r &= \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) \\ &\geq \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) * \mathcal{F}(((\hat{F}w)_n) - ((\hat{F}y)_n), z\frac{\epsilon}{2}) \\ &> r_1 * r_1 \\ &> r_0 \\ &> r \end{aligned}$$

which is a contradiction. Hence $c^{I_{R2N}}(\hat{F})$ is Hausdorff. Similarly we can prove for $c_0^{I_{R2N}}(\hat{F})$. $\square$

Theorem 2.4. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. Then $c^{R2N}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$, where by $c^{R2N}(\hat{F})$ we denote the space of all Fibonacci convergent difference sequences defined as*

$$\{x = (x_n) \in X : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) > 1 - \theta\}$$

where $\epsilon > 0, \theta \in (0, 1)$ and z is non–zero element in X.

Proof. Let $\mathcal{F}\text{–}\lim((\hat{F}x)_n) = L$. Then for every $\theta \in (0, 1)$, $\epsilon > 0$ and non–zero $z \in X$, there exists $N > 0$ such that for all $n \geq N$ $\mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) > 1 - \theta$. The set $K(\epsilon) = \{k \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_k) - L, z; \epsilon) \leq 1 - \theta\} \subseteq \{1, 2, 3 \cdots\}$ and since I is admissible, we have $K(\epsilon) \in I$. Hence $I^{R2N}\text{–}\lim \hat{F}_n(x) = L$.

To show the strictness of the inclusion let us consider $X = \mathbb{R}^2$ with 2–norm $\|x, y\| = |x_1y_2 - x_2y_1|, x = (x_1, x_2), y = (y_1, y_2)$ and $a * b = ab$ for all $a, b \in [0, 1]$. Define $\mathcal{F}(x, z; \epsilon) = \frac{\epsilon}{\epsilon + \|x,z\|}$, for all $x, z \in X$. Define a sequence $x = (x_n) \in X$ such that

$$((\hat{F}x)_n) = \begin{cases} (\sqrt{n}, 0) & \text{if } n \text{ is square,} \\ (0, 0) & \text{otherwise.} \end{cases}$$

For every $0 < \theta < 1$ and $\epsilon > 0$, write

$$A(\theta, \epsilon) = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) \leq 1 - \theta\}, L = (0, 0)$$

$$\mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) = \begin{cases} \frac{\epsilon}{\epsilon + \sqrt{n}z_2}, & \text{if } n \text{ is square,} \\ 1, & \text{otherwise.} \end{cases}$$

Hence

$$\lim_n \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) = \begin{cases} 0, & \text{if } n \text{ is square,} \\ 1, & \text{otherwise.} \end{cases}$$

Therefore $x = (x_n)$ is not convergent in $(X, \mathcal{F}, *)$. If we take $I = I_\delta = \{M \subseteq \mathbb{N} : \delta(M) = 0\}$, then since $A(\theta, \epsilon) \subseteq \{1, 4, 9, 16, \cdots\}$, $\delta(A(\theta, \epsilon)) = 0$. Thus $I^{R2N}\text{–}\lim((\hat{F}x)_n) = L$. $\square$

Theorem 2.5. *The inclusion $c_0^{I_{R2N}}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$ is strict.*

Proof. The inclusion $c_0^{I_{R2N}}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$ is obvious. To show the strictness of the inclusion, consider $X = \mathbb{R}^2$ with 2– norm $\|x, z\| = |x_1 z_2 - x_2 z_1|$ and $a * b = ab$. Define $\mathcal{F}(x, z) = \frac{\epsilon}{\epsilon + \|x, z\|}$ for $\epsilon > 0$. Define $x = (x_n) \in X$ such that $((\hat{F}x)_n) = (1, 1)$. Then I^{R2N}–$\lim((\hat{F}x)_n) = 1$, so $x = (x_n) \in c^{I_{R2N}}(\hat{F}) \setminus c_0^{I_{R2N}}(\hat{F})$. $\square$

Theorem 2.6. *The space $c_0^{I_{R2N}}(\hat{F})$ is solid and monotone.*

Proof. Let $x \in c_0^{I_{R2N}}(\hat{F})$. For $\theta \in (0, 1), \epsilon > 0$ and non–zero $z \in X$, we have

$$A = \{n \in \mathbb{N} : \mathcal{F}(\hat{F}x)_n), z; \frac{\epsilon}{|\alpha|}) \leq 1 - \theta\} \in I,$$

where $\alpha = (\alpha_n)$ is a sequence of scalars with $|\alpha| \leq 1$, then $A^c \in F(I)$. Consider

$$B = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}\alpha x)_n), z; \epsilon) \leq 1 - \theta\}.$$

If we show $A^c \subset B^c$, then we are done.

Let $m \in A^c$, then $\mathcal{F}(((\hat{F}x)_m), z; \epsilon) > 1 - \theta$. Now

$$\begin{aligned} \mathcal{F}(((\hat{F}\alpha\ x)_m), z; \epsilon) &= \mathcal{F}((\alpha(\hat{F}x)_m), z; \epsilon) = \mathcal{F}(((\hat{F}x)_m), z; \frac{\epsilon}{|\alpha|}) \\ &\geq \mathcal{F}(((\hat{F}x)_m), z; \epsilon) * \mathcal{F}(0, z; \frac{\epsilon}{|\alpha|} - \epsilon) \\ &> 1 - \theta * 1 = 1 - \theta \end{aligned}$$

Thus $B \in I$ so that $(\alpha\ x) \in c_0^{I_{R2N}}(\hat{F})$. Therefore $c_0^{I_{R2N}}(\hat{F})$ is solid. By Lemma 1.1, $c_0^{I_{R2N}}(\hat{F})$ is monotone. $\square$

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