

NUMERICAL SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS VIA LAPLACE TRANSFORM

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Abstract. In this study, solutions of time-space fractional partial differential equations (FPDEs) are obtained by utilizing the Laplace transform iterative method. The utility of the technique is shown by getting numerical solutions of various FPDEs.

Key words: differential equations, Laplace transform, Fractional derivatives and integrals, Functional-differential equations

1. Introduction

Mathematical models by fractional differential equations for various physical phenomena play important roles in all applied sciences such as mathematics physics, biology, dynamical systems, control systems, engineering and so on [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Also, there are various studies on fractional diffusion equations. Exact analytical solutions of heat equations are obtained by using operational method [13]. The existence, uniqueness and regularity of solution of impulsive sub-diffusion equation are established by means of eigenfunction expansion [14]. The anomalous diffusion models with non-singular power-law kernel have been investigated and constructed [15]. Moreover, nonlinear fractional partial differential equations (FPDEs) are employed in modeling various nonlinear phenomena, mainly dealing with memory, and they present a crucial role in technology and science. Taking

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physical knowledge and physical properties of the nonlinear problem into account the exact solution of nonlinear FPDEs can be obtained. This knowledge gives us the idea about how numerical solutions of the nonlinear FPDEs can be constructed by the combination of Daftardar-Jafari method (DJM) and Laplace transform. In this study, Laplace Transform iterative method (LTIM) is extended to obtain solutions for time-space FPDEs. The LTIM method is employed to solve a variety of linear and nonlinear FPDEs. LTIM generally generates an accurate solution of FPDEs, which can be represented in terms of the fractional trigonometric functions or Mittag-Leffler functions. Moreover, it has been shown that semi-analytical methods with Laplace transform need fewer CPU time to compute the solutions of nonlinear fractional models, which are utilized in engineering and applied science. LTIM is a robust method to obtain solutions for distinct types of nonlinear and linear FPDEs. LTIM can decrease the time of calculation as well as error margin of the approximate solution.

2. Preliminaries

In this section, preliminaries, notations and features of the fractional calculus are given [1, 2]. Riemann-Liouville time-fractional integral of a real valued function $u(x, t)$ is defined as

$$(2.1) \quad I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x, s) ds$$

where $\alpha > 0$ denotes the order of the integral.

α^{th} order the Liouville-Caputo time-fractional derivative operator of $u(x, t)$ is defined as

$$(2.2) \quad \begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= I_t^{m-\alpha} \left[\frac{\partial^m u(x, t)}{\partial t^m} \right] \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1} \frac{\partial^m u(x, y)}{\partial y^m} dy, & m-1 < \alpha < m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m \end{cases} \end{aligned}$$

Mittag-Leffler function with two parameters is defined as

$$(2.3) \quad E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \quad z, \beta \in \mathbb{C}$$

where α and β are parameters.

The following set of functions has Laplace transformation

$$\left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

and it is defined as

$$(2.4) \quad L[f(t)] = F(p) = \int_0^\infty e^{-pt} f(t) dt$$

which has the following property

$$(2.5) \quad L[t^\alpha] = \int_0^\infty e^{-pt} t^\alpha dt = \Gamma(\alpha + 1) \left(\frac{1}{p}\right)^{\alpha+1}, \quad Re(\alpha) > 0$$

inverse Laplace inverse transform of $\left(\frac{1}{p}\right)^{n\alpha+1}$ is defined as

$$(2.6) \quad L^{-1}\left[\left(\frac{1}{p}\right)^{n\alpha+1}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad Re(\alpha) > 0$$

where $n > 0$ [2].

For α^{th} order the Liouville-Caputo time-fractional derivative of $f(x, t)$, the Laplace transformation has the following form:

$$(2.7) \quad L\left[\frac{\partial^\alpha f(x, t)}{\partial t^\alpha}\right] = p^\alpha L[f(x, t)] - \sum_{k=0}^{n-1} \left[p^{\alpha-k-1} \frac{\partial^k f(x, 0)}{\partial t^k}\right], \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}.$$

3. Methodology

In this section, we take the general time and space FPDE

$$(3.1) \quad \frac{\partial^\zeta f}{\partial t^\zeta} = F\left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right), \quad j-1 < \zeta \leq j, \quad i-1 < \eta \leq i, \quad l, j, i \in \mathbb{N}$$

along with the initial conditions

$$(3.2) \quad \frac{\partial^m f(x, 0)}{\partial t^m} = h_m(x), \quad k = 0, 1, 2, \dots, j-1,$$

into account where $F\left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right)$ could be linear or nonlinear and the function $f = f(x, t)$ is unknown.

Applying the Laplace transform to both sides of Eq. (3.1) and rearranging leads to

$$(3.3) \quad L[f(x, t)] = \sum_{m=0}^{j-1} \left[\left(\frac{1}{p} \right)^{m+1} \frac{\partial^m f(x, 0)}{\partial t^m} \right] + \left(\frac{1}{p} \right)^{\zeta+1} L \left[F \left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}} \right) \right].$$

Employing the inverse Laplace transform of Eq. (3.3), we obtain

$$(3.4) \quad f(x, t) = L^{-1} \left[\sum_{m=0}^{j-1} \left[\left(\frac{1}{p} \right)^{m+1} \frac{\partial^m f(x, 0)}{\partial t^m} \right] \right] + L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} L \left[F \left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}} \right) \right] \right].$$

Equation (3.4) can be rearranged as

$$(3.5) \quad f(x, t) = g(x, t) + G \left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}} \right),$$

where

$$(3.6) \quad g(x, t) = L^{-1} \left[\sum_{m=0}^{j-1} \left[\left(\frac{1}{p} \right)^{m+1} \frac{\partial^m f(x, 0)}{\partial t^m} \right] \right] \\ G \left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}} \right) = L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} L \left[F \left(x, f, \frac{\partial^n f}{\partial x^n}, \dots, \frac{\partial^{l\eta} f}{\partial x^{l\eta}} \right) \right] \right]$$

Here G is a nonlinear / linear operator and g is known function. The solution of Eq. (3.5) can be obtained by the DJM introduced by Daftardar-Gejji and Jafari [16]. The solution is represented as an infinite series:

$$(3.7) \quad f = \sum_{n=0}^{\infty} f_n,$$

where the terms f_n are recursively computed. Decomposing the operator G leads to

$$\begin{aligned}
 & G\left(x, \sum_{n=0}^{\infty} f_n, \frac{\partial^\eta (\sum_{n=0}^{\infty} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{\infty} f_n)}{\partial x^{l\eta}}\right) \\
 &= G\left(x, f_0, \frac{\partial^\eta f_0}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} f_0}{\partial x^{l\eta}}\right) \\
 (3.8) \quad &+ \sum_{c=1}^{\infty} \left(G\left(x, \sum_{n=0}^c f_n, \frac{\partial^\eta (\sum_{n=0}^c f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^c f_n)}{\partial x^{l\eta}}\right) \right) \\
 &- \sum_{c=1}^{\infty} \left(G\left(x, \sum_{n=0}^{c-1} f_n, \frac{\partial^\eta (\sum_{n=0}^{c-1} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{c-1} f_n)}{\partial x^{l\eta}}\right) \right)
 \end{aligned}$$

$$\begin{aligned}
 (3.9) \quad & L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, \sum_{n=0}^{\infty} f_n, \frac{\partial^\eta (\sum_{n=0}^{\infty} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{\infty} f_n)}{\partial x^{l\eta}}\right) \right] \right] \\
 &= L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, f_0, \frac{\partial^\eta f_0}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} f_0}{\partial x^{l\eta}}\right) \right] \right] \\
 &+ \sum_{c=1}^{\infty} L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, \sum_{n=0}^c f_n, \frac{\partial^\eta (\sum_{n=0}^c f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^c f_n)}{\partial x^{l\eta}}\right) \right] \right] \\
 &- \sum_{c=1}^{\infty} L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, \sum_{n=0}^{c-1} f_n, \frac{\partial^\eta (\sum_{n=0}^{c-1} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{c-1} f_n)}{\partial x^{l\eta}}\right) \right] \right].
 \end{aligned}$$

Using Eqs. (3.7), (3.9) in Eq. (3.5), we get

$$\begin{aligned}
 \sum_{n=0}^{\infty} f_n &= L^{-1} \left[\sum_{m=0}^{j-1} \left[\left(\frac{1}{p}\right)^{m+1} \frac{\partial^m f(x, 0)}{\partial t^m} \right] \right] \\
 &+ L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, f_0, \frac{\partial^\eta f_0}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} f_0}{\partial x^{l\eta}}\right) \right] \right] \\
 &+ \sum_{c=1}^{\infty} L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, \sum_{n=0}^c f_n, \frac{\partial^\eta (\sum_{n=0}^c f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^c f_n)}{\partial x^{l\eta}}\right) \right] \right] \\
 &- \sum_{c=1}^{\infty} L^{-1} \left[\left(\frac{1}{p}\right)^{\zeta+1} L \left[F\left(x, \sum_{n=0}^{c-1} f_n, \frac{\partial^\eta (\sum_{n=0}^{c-1} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{c-1} f_n)}{\partial x^{l\eta}}\right) \right] \right].
 \end{aligned}$$

The recurrence relation is defined by as follows:

(3.10)

$$\begin{aligned}
f_0 &= L^{-1} \left[\sum_{m=0}^{j-1} \left[\left(\frac{1}{p} \right)^{m+1} \frac{\partial^m f(x, 0)}{\partial t^m} \right] \right], \\
f_1 &= L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} L \left[F \left(x, f_0, \frac{\partial^\eta f_0}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} f_0}{\partial x^{l\eta}} \right) \right] \right], \\
f_{r+1} &= L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} L \left[F \left(x, \sum_{n=0}^c f_n, \frac{\partial^\eta (\sum_{n=0}^c f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^c f_n)}{\partial x^{l\eta}} \right) \right] \right] \\
&\quad - L^{-1} \left[\left(\frac{q}{p} \right)^{\zeta+1} L \left[F \left(x, \sum_{n=0}^{c-1} f_n, \frac{\partial^\eta (\sum_{n=0}^{c-1} f_n)}{\partial x^\eta}, \dots, \frac{\partial^{l\eta} (\sum_{n=0}^{c-1} f_n)}{\partial x^{l\eta}} \right) \right] \right].
\end{aligned}$$

The r - term truncated solution of Eqs. (3.1),(3.2) is constructed as $f \approx f_0 + f_1 + \dots + f_{r-1}$. We lead the reader to [17] for the convergence of DJM.

4. Illustrative Example

Let us consider time and space fractional equation below

$$(4.1) \quad \frac{\partial^\zeta f}{\partial t^\zeta} = \left(\frac{\partial^\eta f}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f}{\partial x^\eta} \right), \quad t > 0, \quad \zeta, \eta \in (0, 1],$$

along with the initial condition

$$(4.2) \quad u(x, 0) = 3 + \frac{5}{2} E_\eta(x^\eta).$$

Let us apply the Laplace transform on both sides of (4.1).

$$L \left[\frac{\partial^\zeta f}{\partial t^\zeta} \right] = L \left[\left(\frac{\partial^\eta f}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f}{\partial x^\eta} \right) \right].$$

By means of the property (2.7), we obtain

$$(4.3) \quad L[f(x, t)] = \left(\frac{1}{p} \right) f(x, 0) + \left(\frac{1}{p} \right)^{\zeta+1} \left(L \left[\left(\frac{\partial^\eta f}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f}{\partial x^\eta} \right) \right] \right).$$

Applying the inverse Laplace transform to both sides of Eq. (4.3)

$$f(x, t) = L^{-1} \left[\left(\frac{1}{p} \right) f(x, 0) \right] + L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} \left(L \left[\left(\frac{\partial^\eta f}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f}{\partial x^\eta} \right) \right] \right) \right]$$

is obtained. Using the recurrence relation (3.10)

$$\begin{aligned} f_0 &= L^{-1} \left[\left(\frac{1}{p} \right) f(x, 0) \right] = 3 + \frac{5}{2} E_\eta(x^\eta), \\ f_1 &= L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} \left(L \left[\left(\frac{\partial^\eta f_0}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f_0}{\partial x^\eta} \right) \right] \right) \right] = -\frac{15t^\zeta E_\eta(x^\eta)}{2\Gamma(\zeta + 1)}, \\ f_2 &= L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} \left(L \left[\left(\frac{\partial^\eta (f_0 + f_1)}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta (f_0 + f_1)}{\partial x^\eta} \right) \right] \right) \right] \\ &\quad - L^{-1} \left[\left(\frac{1}{p} \right)^{\zeta+1} \left(L \left[\left(\frac{\partial^\eta f_0}{\partial x^\eta} \right)^2 - f \left(\frac{\partial^\eta f_0}{\partial x^\eta} \right) \right] \right) \right] = \frac{45t^{2\zeta} E_\eta(x^\eta)}{2\Gamma(2\zeta + 1)}, \\ f_3 &= -\frac{135t^{3\zeta} E_\eta(x^\eta)}{2\Gamma(3\zeta + 1)}, \\ f_4 &= -\frac{405t^{4\zeta} E_\eta(x^\eta)}{2\Gamma(4\zeta + 1)}. \end{aligned}$$

As a result, the series solution of the problem (4.1)-(4.2) is obtained by

$$f(x, t) = f_0 + f_1 + f_2 + f_3 + \dots = 3 + \left[\frac{5}{2} E_\zeta(-3t^\zeta) \right] E_\eta(x^\eta),$$

which gives the same solution as in [18].

5. Conclusion

LTIM is developed by taking the combination of DJM [16] and Laplace transform. This new approach is convenient for acquiring numerical solutions of time and space FPDEs. Its applicability is illustrated by an example in this study. As a result, the combination of DJM with Laplace transform provides a better and more effective approach than combination of Laplace transformation and homotopy, Sumudu or Adomian polynomials.

The results of this paper can be rewritten easily and trivially for any of the many rather inconsequential parameter and/or variable changes in the integral in Eq. (2.4) which defines the classical Laplace transform.

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