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W_2-CURVATURE TENSOR ON K-CONTACT MANIFOLDS

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Abstract. The object of the present paper is to obtain sufficient conditions for a K-contact manifold to be a Sasakian manifold.
Keywords: Sasakian manifold; K-contact manifold; W_2 curvature tensor.

1. Introduction

The inclination of existent mathematics is abstractions, generalizations and applications. In the offering exposition, we are entering an era of new concepts and some generalizations which play a functional role in contemporary mathematics. Contact geometry has been matured from the mathematical formalism of classical mechanics. A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Kähler structure (J, G) of the base manifold M^{2n}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n}, J, G) is Kählerian. If (M^{2n}, J, G) is only almost Kähler, then (ϕ, ξ, η, g) is only K-contact [3]. It is to be noted that a K-contact manifold is intermediate between a contact metric manifold and a Sasakian manifold. K-contact and Sasakian manifolds have been studied by several authors such as ([2], [7], [8], [10],[18], [20],) and many others. It is well known that every Sasakian manifold is K-contact, but the converse is not true, in general. However, a three-dimensional K-contact manifold is Sasakian [9].

On the other hand, Pokhariyal and Mishra [14] have introduced new tensor fields, called W_2 and E-tensor fields, in a Riemannian manifold, and studied their relativistic properties. Then, Pokhariyal [13] and De [6] have studied some properties of this tensor fields in a Sasakian manifold and Trans-Sasakian manifold respectively.

The curvature tensor W_2 is defined by

$$W_2(X,Y,U,V) = R(X,Y,U,V)$$

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$$+\frac{1}{n-1}[g(X,U)S(Y,V)-g(Y,U)S(X,V)], \tag{1.1}$$

where S is a Ricci tensor of type (0,2).

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([16],[11]) if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y).W_2 = 0$, then the manifold is said to be W_2 semi-symmetric manifold.

The object of the present paper is to enquire under what conditions a K contact manifold will be a Sasakian manifold.

The present paper is organized as follows:

After a brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. Section 3 is devoted to the study of K-contact manifolds satisfying $\tilde{Z}(X,Y).W_2 = 0$ and prove that the manifold is Sasakian. In section 4, we consider K-contact manifolds satisfying $R(\xi,X).W_2 = 0$ and $W_2(\xi,X).R = 0$.

2. Priliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ,ξ,η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([3], [4])

$$\phi^2 = -I + \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta\circ\phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ,ξ,η), structure, that is,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X,\phi Y) = -g(\phi X, Y) \tag{2.4}$$

and

$$g(X,\xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g).

An almost contact metric structure becomes a contact metric structure if

$$g(X,\phi Y) = d\eta(X,Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.

If ξ is a Killing vector field, then M^{2n+1} is said to be a K-contact manifold ([3], [15]). A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.6}$$

Besides the above relations in K-contact manifold the following relations hold ([1], [3], [15]):

$$\nabla_X \xi = -\phi X. \tag{2.7}$$

$$\tilde{R}(\xi, X, Y, \xi) = \eta(R(\xi, X)Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi. \tag{2.9}$$

$$S(X, \xi) = 2n\eta(X). \tag{2.10}$$

$$(\nabla_X \phi)Y = R(\xi, X)Y, \tag{2.11}$$

for any vector fields X, Y.
Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

A transformation of a n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([12], [21]). A concircular transformation is always a conformal transformation [12]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{Z}$. It is defined by ([19], [22])

$$\tilde{Z}(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}(g(Y,U)X - g(X,U)Y). \tag{2.12}$$

where $X, Y, W \in T(M)$. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In a K-contact manifold, using (2.6) equation (2.12) reduce to

$$\tilde{Z}(\xi, X)Y = (1 - \frac{r}{n(n-1)})\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.13}$$

A K-contact manifold is said to be W_2 flat if W_2 curvature vanishes at each point of the manifold. From the definition of the W_2 curvature tensor, it can be easily proved that a W_2 flat manifold implies the manifold is an Einstein manifold. It is known that [5] a compact K-contact Einstein manifold is Sasakian. Thus we have the following:

Proposition 2.1. *A W_2 flat compact K-contact manifold is Sasakian.*

3. K-contact manifolds satisfying $\tilde{Z}(X,Y).W_2 = 0$

In this section we consider a K-contact manifolds satisfying the curvature condition

$$\tilde{Z}(X,Y).W_2 = 0. \tag{3.1}$$

This equation implies

$$\begin{aligned} &\tilde{Z}(X,Y)W_2(Z,U)V - W_2(\tilde{Z}(X,Y)Z,U)V \\ &-W_2(Z,\tilde{Z}(X,Y)U)V - W_2(Z,U)\tilde{Z}(X,Y)V = 0. \end{aligned} \tag{3.2}$$

Putting $X = \xi$ in (3.2) we obtain

$$\begin{aligned} &\tilde{Z}(\xi,Y)W_2(Z,U)V - W_2(\tilde{Z}(\xi,Y)Z,U)V \\ &-W_2(Z,\tilde{Z}(\xi,Y)U)V - W_2(Z,U)\tilde{Z}(\xi,Y)V = 0. \end{aligned} \tag{3.3}$$

Using (2.13) in (3.3), we obtain

$$\begin{aligned} &(1 - \frac{r}{n(n-1)})\{g(Y,W_2(Z,U)V)\xi - g(W_2(Z,U)V,\xi)Y \\ &-g(Y,Z)W_2(\xi,U)V + \eta(Z)W_2(Y,U)V - g(Y,U)W_2(Z,\xi)V \\ &\eta(U)W_2(Z,U)V - g(Y,V)W_2(Z,U)\xi + \eta(V)W_2(Z,U)Y\} = 0. \end{aligned} \tag{3.4}$$

Taking the inner product with ξ and using (2.13) in (3.4), we have

$$(1 - \frac{r}{n(n-1)})g(Y,W_2(Z,U)V) = 0. \tag{3.5}$$

Again from (2.13) we have $(1 - \frac{r}{n(n-1)}) \neq 0$. Hence we have

$$W_2(Z,U,V,Y) = 0. \tag{3.6}$$

From the Proposition 2.1 we have

Theorem 3.1. *A K-contact manifold satisfying the curvature condition*

$$\tilde{Z}(X,Y).W_2 = 0,$$

is Sasakian.

4. K-contact manifolds satisfying $R(\xi,X).W_2 = 0$ and $W_2(\xi,X).R = 0$

In this section we first proof a proposition

Proposition 4.1. *In an n-dimensional K-contact manifold,* $\eta(W_2(X,Y)Z) = 0$.

Proof. From equation (1.1), we have

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(Y,Z)QX]. \quad (4.1)$$

Taking the inner product of above equation with ξ and using equations (2.8) and (2.10), we get

$$\eta(W_2(X,Y)Z) = 0. \quad (4.2)$$

□

Theorem 4.1. *In an n-dimensional K-contact manifold , $R(\xi, X)W_2 = 0$ if and only if $W_2 = 0$.*

Proof. Let in an n-dimensional K-contact manifold the curvature condition

$$R(\xi, X).W_2 = 0 \quad (4.3)$$

holds. This equation implies

$$\begin{aligned} &R(\xi,X)W_2(Y,Z)U - W_2(R(\xi,X)Y,Z)U \\ &-W_2(Y,R(\xi,X)Z)U - W_2(Y,Z)R(\xi,X)U = 0. \end{aligned} \quad (4.4)$$

Using equation (2.8) and taking the inner product of above equation with ξ, we get

$$\begin{aligned} &W_2(Y,Z,U,X) - \eta(W_2(Y,Z)U)\eta(X) - g(X,Y)\eta(W_2(\xi,Z)U) \\ &+\eta(Y)\eta(W_2(X,Z)U) + \eta(Z)\eta(W_2(Y,X)U) - g(X,Z)\eta(W_2(Y,\xi)U) \\ &+\eta(U)\eta(W_2(Y,Z)X) - g(X,U)\eta(W_2(Y,Z)\xi) = 0, \end{aligned} \quad (4.5)$$

which on using equation (4.2) gives
$W_2(Y,Z,U,X) = 0$,
that is $W_2 = 0$.
Conversely, suppose $W_2 = 0$, then from equation (4.4), we have $R(\xi, X)W_2 = 0$.
This completes the proof. □

Theorem 4.2. *An n-dimensional K-contact manifold satisfying $W_2(\xi, X).R = 0$, is an Einstein manifold.*

Proof. Let the curvature condition $W_2(\xi, X).R = 0$ holds, then we have

$$\begin{aligned} &W_2(\xi,X)R(Y,Z)U - R(W_2(\xi,X)Y,Z)U - R(Y,W_2(\xi,X)Z)U \\ &-R(Y,Z)W_2(\xi,X)U = 0. \end{aligned} \quad (4.6)$$

Now putting $X = \xi$ in equation (4.1) and using equations (2.8) and (2.10), we obtain

$$W_2(\xi,Y)Z = \eta(Z)[\frac{QY}{n-1} - Y]. \quad (4.7)$$

Now from equations (4.6) and (4.7), we have

$$\eta(R(Y,Z)U)[\frac{QX}{n-1}-X]-\eta(Y)[\frac{1}{n-1}R(QX,Z)U-R(X,Z)U]$$
$$-\eta(Z)[\frac{1}{n-1}R(Y,QX)U-R(Y,X,)U]-$$
$$\eta(U)[\frac{1}{n-1}R(Y,Z)QX-R(Y,Z)X]=0, \tag{4.8}$$

which on taking the inner product with ξ and using equations (2.10) gives

$$\eta(Y)\eta(X)g(Z,U)-\eta(Z)\eta(X)g(Y,U)+\eta(Y)\eta(U)g(X,Z)$$
$$-\eta(U)\eta(Z)g(X,Y)-\frac{1}{n-1}[S(X,Y)g(Z,U)-S(X,Z)g(Y,U)+$$
$$\eta(U)\eta(Y)S(Z,X)-\eta(U)\eta(Z)S(X,Y)]=0. \tag{4.9}$$

Putting $U=Z=\xi$ in above equation , we get

$$S(X,Y)=(n-1)g(X,Y), \tag{4.10}$$

which shows that the manifold is an Einstein Manifold.

□

It is known that [5] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

Corollary 4.1. *A compact K-contact manifold satisfying the curvature condition* $W_2(\xi,X).R=0$*, is Sasakian.*

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