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ON THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In the present paper, we study three-dimensional trans-Sasakian manifolds admitting the Schouten-van Kampen connection. Also, we have proved some results on ϕ-projectively flat, ξ-projectively flat and ξ-concircularly flat three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection. Locally ϕ-symmetric trans-Sasakian manifolds of dimension three have been studied with respect to Schouten-van Kampen connection. Finally, we construct an example of a three-dimensional trans-Sasakian manifold admitting Schouten-van Kampen connection which verifies Theorem 4.1. and Theorem 5.2.
Key words: General geometric structures on manifolds, Schouten-van Kampen connection, Special Riemannian manifolds

1. Introduction

The Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection. Solov'ev investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection ([18], [19], [20], [21]). In 2014, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [17]. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection. Recently, G. Ghosh [10], Yildiz [26], Nagaraja [15] and D. L. Kiran Kumar [12] have studied the

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Schouten-van Kampen connection in Sasakian manifolds, f-Kenmotsu manifolds and Kenmotsu manifolds respectively.

A transformation of an n-dimensional differentiable manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [27], [13]. A concircular transformation is always a conformal transformation [13]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor $\mathbb{W}$ with respect to Levi-Civita connection. It is defined by [27], [28]

$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where $X, Y, Z \in \chi(M), R$ and r are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection.
The concircular curvature tensor $\tilde{\mathbb{W}}$ with respect to the Schouten-van Kampen connection is defined by

$$\tilde{\mathbb{W}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

where $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and the scalar curvature with respect to the Schouten-van Kampen connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In 1985, a new class of n-dimensional almost contact manifold namely trans-Sasakian manifold was introduced by J. A. Oubina [16] and further study about the local structures of trans-Sasakian manifolds was carried by J. C. Marrero [14]. Trans-Sasakian manifolds of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are, called the cosymplectic, α-Sasakian and β-Kenmotsu respectively ([2], [11]). In particular, if $\alpha = 0, \beta = 1; \alpha = 1, \beta = 0$; then a trans-Sasakian manifold becomes Kenmotsu and Sasakian manifolds respectively. Hence, trans-Sasakian structures give a large class of generalized Quasi-Sasakian structures. It has been proven that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha-$Sasakian and $\beta-$Kenmotsu manifold. Three-dimnesional trans-Sasakian manifolds with different restrictions on curvature and smooth functions α, β are studied in ([7], [8], [5], [6]).

In the present paper, we have studied three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection.

The present paper is organized as follows: After the introduction in Section 1, we give some required preliminaries in Section 2. Section 3 is devoted to the study of the curvature tensor, the Ricci tensor, scalar curvature of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Section 4 is devoted to the study of ξ-projectively and ϕ-projectively flat trans-Sasakian manifolds of dimension three with respect to the Schouten-van Kampen connection. In this section, we have proved that a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is ξ-projectively flat if and only if the scalar curvature of the manifold vanishes. In Section 5, we study ξ-concircularly flat trans-Sasakian manifold of dimension three admitting Schouten-van Kampen connection. In the next section, we study locally ϕ-symmetric trans-Sasakian manifolds of dimensional three with respect to Schouten-van Kampen connection. In Section 7, we study Weyl ξ-conformally flat in three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. In the last section, we construct an example of a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection to support the results obtained in Section 4 and Section 5.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [16] if $(M \times R, J, G)$ belongs to the class W_4 [9], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X) d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [3]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. Here ∇ is Levi-Civita connection on M. We say M as the trans-Sasakian manifold of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

In a three-dimensional trans-Sasakian manifold following relations hold [7], [8]:

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$\begin{aligned} S(X,Y) &= \{\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(X,Y) \\ &\quad -\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y) - \{Y\beta + (\phi X)\alpha\}\eta(Y), \end{aligned} \tag{2.9}$$

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\ &\quad -g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &\quad -\eta(X)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\ &\quad +g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &\quad -\eta(Y)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ &\quad -[(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ &\quad +[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.10}$$

where S is the Ricci tensor of type $(0,2)$, and r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

From here after we consider α and β are constants, then the above relations become

$$\begin{aligned} R(X,Y)Z &= \{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &\quad +\{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= \{\frac{r}{2} - (\alpha^2 - \beta^2)\}g(X,Y) \\ &\quad -\{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y), \end{aligned} \tag{2.12}$$

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.13}$$

$$QX = \{\frac{r}{2} - (\alpha^2 - \beta^2)\}X - \{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}\eta(X)\xi, \tag{2.14}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.15}$$

$$R(\xi,X)Y = 2(\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(Y)X). \tag{2.16}$$

From (2.8) it follows that if α and β are constants, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

3. Curvature tensor of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection

For an almost contact metric manifold M, the Schouten-van Kampen connection $\tilde{\nabla}$ is given by [17]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{3.1}$$

Let M be a three-dimensional trans-Sasakian manifold. Then from above equation we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)\phi X) - g(\phi X, Y)\xi\} + \beta\{g(X,Y)\xi - \eta(Y)X\}. \tag{3.2}$$

We define the curvature tensor $\tilde{R}$ of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{3.3}$$

In view of (3.2) and (3.3) we obtain

$$\begin{aligned} \tilde{R}(X,Y)Z \;=\;& R(X,Y)Z + \alpha^2\{g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\} \\ & +\beta^2\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.4}$$

Taking inner product in both sides of (3.4) with W, we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) \;=\;& R(X,Y,Z,W) + \alpha^2\{g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) \\ & +g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z) \\ & -g(Y,Z)\eta(X)\eta(W) + g(X,Z)\eta(Y)\eta(W)\} \\ & +\beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{3.5}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$.

Taking a frame field from (3.5), we get

$$\tilde{S}(Y,Z) = S(Y,Z) + 2\beta^2 g(Y,Z) - 2\alpha^2\eta(Y)\eta(Z). \tag{3.6}$$

From above equation we have

$$\tilde{Q}Y = QY + +2\beta^2 Y - 2\alpha^2\eta(Y)\xi. \tag{3.7}$$

Again putting $Y = Z = e_i$ $(i = 1,2,3)$ and taking summation over i in (3.6), we obtain

$$\tilde{r} = r - 2\alpha^2 + 6\beta^2, \tag{3.8}$$

where $\tilde{r}$ and r are the scalar curvatures of the Schouten-van Kampen connection $(\tilde{\nabla})$ and Levi-Civita connection (∇) respectively.

Hence we have the following :

Proposition 3.1. A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection following statements are equivalent
(a) The curvature tensor $\tilde{R}$ is given by (3.4),
(b) The Ricci tensor $\tilde{S}$ is given by (3.6),
(c) $\tilde{r} = r - 2\alpha^2 + 6\beta^2$,
(d) The Ricci tensor $\tilde{S}$ is symmetric,
provided α and β are constants.

4. ξ-Projectively and ϕ-projectively flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

In this section, we study projectively flat three-dimensional trans-Sasakian manifold M with respect to the Schouten-van Kampen connection. In a three-dimensional trans-Sasakian manifold, the projective curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{4.1}$$

Definition 4.1. A three-dimensional trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ξ-projectively flat if

$$\tilde{P}(X,Y)\xi = 0,$$

for all vector fields X, Y on M. This notion was first defined by Tripathi and Dwivedi [22]. If $\tilde{P}(X,Y)\xi = 0$, just holds for X, Y orthogonal to ξ, we call such a manifold a horizontal ξ-projectively flat manifold.

Using (3.4) in (4.1) we have

$$\begin{aligned}\tilde{P}(X,Y)Z &= R(X,Y)Z+\alpha^2\{g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\ &+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ &-g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi\}\\ &+\beta^2\{g(Y,Z)X-g(X,Z)Y\}\\ &-\frac{1}{2}\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y\}. \end{aligned} \tag{4.2}$$

Putting $Z=\xi$ and using (2.1), (2.3), (2.15) and (3.6) in (4.2), we get

$$\tilde{P}(X,Y)\xi = 0. \tag{4.3}$$

Thus we can state the following:

Theorem 4.1. A three-dimensional trans-Sasakian manifold is ξ-projectively flat with respect to the Schouten-van Kampen connection provided α and β are constants.

Again putting (3.6) in (4.2) we get

$$\begin{aligned}\tilde{P}(X,Y)Z &= P(X,Y)Z+\alpha^2\{g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\ &-g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{4.4}$$

Putting $Z=\xi$ in (4.4) and using (2.1) and (2.3), it follows that

$$\tilde{P}(X,Y)\xi = P(X,Y)\xi. \tag{4.5}$$

In view of above discussion we state the following theorem:

Theorem 4.2. A three-dimensional trans-Sasakian manifold is ξ-projectively flat with respect to the Schouten-van Kampen connection if and only if the manifold is ξ-projectively flat with respect to the Levi-Civita connection provided α and β are constants.

Definition 4.2. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ϕ-projectively flat if

$$\phi^2\tilde{P}(\phi X,\phi Y)\phi Z = 0.$$

It can be easily seen that $\phi^2\tilde{P}(\phi X,\phi Y)\phi Z = 0$ holds if and only if

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{4.6}$$

for $X,Y,Z,W \in T(M)$.

Using (4.1) and (4.6), ϕ-projectively flat means

$$\begin{aligned} g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) \\ &\quad -\tilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{4.7}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (4.7) and summing up with respect to i, we have

$$\begin{aligned} \sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \frac{1}{2}\sum_{i=1}^{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &\quad -\tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\}. \end{aligned} \tag{4.8}$$

Using (2.1), (2.2), (2.3) and (3.5) it can be easily verified that

$$\begin{aligned} \sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \sum_{i=1}^{2} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) \\ &\quad +(\alpha^2+\beta^2)g(Y,Z)+(\beta^2-3\alpha^2)\eta(Y)\eta(Z) \\ &= S(\phi Y,\phi Z)+(\alpha^2+\beta^2)g(Y,Z) \qquad (4.9)\\ &\quad +(\beta^2-3\alpha^2)\eta(Y)\eta(Z). \qquad (4.10) \end{aligned}$$

$$\sum_{i=1}^{2} g(\phi e_i,\phi e_i) = 2. \tag{4.11}$$

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = \tilde{S}(\phi Y,\phi Z). \tag{4.12}$$

Using (4.9), (4.10) and (4.11), the equation (4.8) becomes

$$\tilde{S}(\phi Y,\phi Z) = 2\{S(\phi Y,\phi Z)+(\alpha^2+\beta^2)g(Y,Z)+(\beta^2-3\alpha^2)\eta(Y)\eta(Z)\}. \tag{4.13}$$

Using (3.6) in (4.12), we get

$$S(\phi Y,\phi Z) = -2\alpha^2 g(Y,Z)+2(3\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{4.14}$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (4.13) and using (2.1) (2.2) and (2.13), we obtain

$$S(Y,Z) = -2\alpha^2 g(Y,Z)+2(2\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{4.15}$$

Conversely, let S be of the form (4.14), then obviously

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0.$$

Thus we can state the following:

Theorem 4.3. A three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is ϕ-projectively flat if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection provided α, β are constants with $\beta \neq \pm\sqrt{2}\alpha, (\alpha \neq 0)$.

5. ξ-Concircularly flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

Definition 5.1. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ξ-concircularly flat if

$$\tilde{\mathbb{W}}(X,Y)\xi = 0, \tag{5.1}$$

for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Theorem 5.1. A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is horizontally ξ-concircularly flat if and only if the manifold with respect to the Levi-Civita connection is also ξ-concircular flat provided α, β are constants.

Proof. Combining (1.1),(1.2) and using (3.4), (3.6) (3.8), we get

$$\begin{aligned}\tilde{\mathbb{W}}(X,Y)Z &= \mathbb{W}(X,Y)Z + \alpha^2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\quad -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ &\quad -\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\}.\end{aligned} \tag{5.2}$$

Putting $Z = \xi$ in (5.2) we get

$$\tilde{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi + \frac{2\alpha^2}{3}\{\eta(X)Y - \eta(Y)X\}. \tag{5.3}$$

From (5.3), implies that

$$\tilde{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi; \quad \text{for all } X, Y \text{orthogonal to } \xi. \tag{5.4}$$

Hence the proof of theorem is complete.

Theorem 5.2. A three-dimensional trans-Sasakian manifold is ξ-concircularly flat with respect to the Schouten-van Kampen connection if and only if the scalar curvature $\tilde{r}$ is zero, provided α and β are constants.

Proof. Putting $Z = \xi$ in (1.2) and using (2.1), (2.3), (2.3), (2.15) and (3.4), we have

$$\tilde{\mathbb{W}}(X,Y)\xi = -\frac{\tilde{r}}{6}\{\eta(Y)X - \eta(X)Y\}. \tag{5.5}$$

Thus the theorem is proved.

6. Locally ϕ-symmetric trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

Definition 6.1. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is called to be locally ϕ-symmetric if

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = 0, \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ on M. This notion was introduced by Takahashi [24], for Sasakian manifolds.

We know that

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= \tilde{\nabla}_W(\tilde{R}(X,Y)Z) - \tilde{R}(\tilde{\nabla}_W X,Y)Z \\ &\quad -R(X,\tilde{\nabla}_W Y)Z - \tilde{R}(X,Y)\tilde{\nabla}_W Z.\end{aligned} \tag{6.2}$$

By virtue of (3.1), above equation is reduced to

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= (\nabla_W\tilde{R})(X,Y)Z + \eta(X)\tilde{R}(\nabla_W\xi,Y)Z + (\nabla_W\eta)(X)\tilde{R}(\xi,Y)Z \\ &\quad +\eta(Y)\tilde{R}(X,\nabla_W\xi)Z + (\nabla_W\eta)(Y)\tilde{R}(X,\xi)Z \\ &\quad +\eta(Z)\tilde{R}(X,Y)\nabla_W Z + (\nabla_W\eta)(Z)\tilde{R}(X,Y)\xi.\end{aligned} \tag{6.3}$$

Now differentiating (3.4) with respect to W, using (2.1), (2.2), (2.3), (2.5) and (2.7) we obtain

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z \\ &\quad +\alpha^3[\{g(X,Y)g(\phi Y,Z) - g(W,Y)g(\phi X,Z)\}\xi \\ &\quad +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}W] \\ &\quad +\alpha^2\beta[\{g(\phi W,X)g(\phi Y,Z) - g(\phi W,Y)g(\phi X,Z)\}\xi \\ &\quad +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}\phi W] \\ &\quad +(\alpha^2-\beta^2)[\{\alpha(g(\phi W,Y)X - g(\phi W,X)Y) \\ &\quad -\beta^2(g(\phi W,\phi Y)X + g(\phi W,\phi X)Y)\}\eta(Z) \\ &\quad +(\beta g(\phi W,\phi Z) - \alpha g(\phi W,Z))(\eta(X)Y - \eta(Y)X)] \\ &\quad +\alpha^2(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))(-\alpha\phi W + \beta(W - \eta(W)\xi)) \\ &\quad -\alpha^2[-\alpha(g(Y,Z)g(\phi W,X) + g(X,Z)g(\phi W,Y)) \\ &\quad +\beta(g(Y,Z)g(\phi W,\phi X) + g(X,Z)g(\phi W,\phi Y))]\xi \\ &\quad +\beta^2[\{-\alpha(g(W,\phi Z)\eta(Y) + g(W,\phi Y)\eta(Z)) \\ &\quad -\beta(g(\phi W,\phi Z)\eta(Y) + g(\phi W,\phi Y)\eta(Z))\}X \\ &\quad +\{\alpha(g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)) \\ &\quad -\beta(g(\phi W,\phi Z)\eta(X) + g(\phi W,\phi X)\eta(Z))\}Y].\end{aligned} \tag{6.4}$$

Using (6.4) in (6.3) we have

$$\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z \;=\; & (\nabla_W R)(X,Y)Z \\
& +\alpha^3[\{g(X,Y)g(\phi Y,Z) - g(W,Y)g(\phi X,Z)\}\xi \\
& +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}W] \\
& +\alpha^2\beta[\{g(\phi W,X)g(\phi Y,Z) - g(\phi W,Y)g(\phi X,Z)\}\xi \\
& +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}\phi W] \\
& +(\alpha^2-\beta^2)[\{\alpha(g(\phi W,Y)X - g(\phi W,X)Y) \\
& -\beta^2(g(\phi W,\phi Y)X + g(\phi W,\phi X)Y)\}\eta(Z) \\
& +(\beta g(\phi W,\phi Z) - \alpha g(\phi W,Z))(\eta(X)Y - \eta(Y)X)] \\
& +\alpha^2(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))(-\alpha\phi W + \beta(W - \eta(W)\xi)) \\
& -\alpha^2[-\alpha(g(Y,Z)g(\phi W,X) + g(X,Z)g(\phi W,Y)) \\
& +\beta(g(Y,Z)g(\phi W,\phi X) + g(X,Z)g(\phi W,\phi Y))]\xi \\
& +\beta^2[\{-\alpha(g(W,\phi Z)\eta(Y) + g(W,\phi Y)\eta(Z)) \\
& -\beta(g(\phi W,\phi Z)\eta(Y) + g(\phi W,\phi Y)\eta(Z))\}X \\
& +\{\alpha(g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)) \\
& -\beta(g(\phi W,\phi Z)\eta(X) + g(\phi W,\phi X)\eta(Z))\}Y] \\
& +\eta(X)\tilde{R}(\nabla_W\xi,Y)Z + (\nabla_W\eta)(X)\tilde{R}(\xi,Y)Z \\
& +\eta(Y)\tilde{R}(X,\nabla_W\xi)Z + (\nabla_W\eta)(Y)\tilde{R}(X,\xi)Z \\
& +\eta(Z)\tilde{R}(X,Y)\nabla_W Z + (\nabla_W\eta)(Z)\tilde{R}(X,Y)\xi. \qquad (6.5)
\end{aligned}$$

Now applying ϕ^2 on both sides of (6.5) and taking X, Y, Z, W are orthogonal to ξ and using (2.1), (2.3) we get from above equation

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \qquad (6.6)$$

Hence we can state the following:

Theorem 6.1. A three-dimensional trans-Sasakian manifold is locally ϕ-symmetry with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ if and only if the manifold is also locally ϕ-symmetry with respect to the Levi-Civita connection ∇ provided α, β are constants.

U. C. De and Avijit Sarkar [7] have proved that a trans-Sasakian manifold is locally ϕ-symmetry if and only if the scalar curvature is constant provided α, β are constants.

In view of above result we can state the following:

Theorem 6.2. A three-dimensional trans-Sasakian manifold is locally ϕ-symmetric with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ if and only if the scalar curvature is constant, provided α, β are constants.

7. Weyl conformally flat trans-Sasakian manifold with respect to Schouten-van Kampen connection

The Weyl conformal curvature tensor $\tilde{C}$ of type (1,3) of M, an $n-$dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is given by [23]

$$\begin{aligned}\tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z - \frac{1}{n-2}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X \\ &\quad -g(X,Z)\tilde{Q}Y] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \qquad (7.1)\end{aligned}$$

where $\tilde{Q}$ is the Ricci operator with respect to the Schouten-van Kampen connection.

Let us consider that a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is Weyl conformally flat, that is $\tilde{C} = 0$. Then from (7.1), we get

$$\begin{aligned}\tilde{R}(X,Y)Z &= [\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X \\ &\quad -g(X,Z)\tilde{Q}Y] - \frac{\tilde{r}}{2}[g(Y,Z)X - g(X,Z)Y]. \qquad (7.2)\end{aligned}$$

Let us take inner product of the equation (7.2) with W. Then we get

$$\begin{aligned}g(\tilde{R}(X,Y)Z,W) &= [\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W) + g(Y,Z)g(\tilde{Q}X,W) \\ &\quad -g(X,Z)g(\tilde{Q}Y,W)] - \frac{\tilde{r}}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \qquad (7.3)\end{aligned}$$

Using (2.1), (2.3), (3.5)-(3.8), we get

$$\begin{aligned}g(\tilde{R}(X,Y)Z,W) &= [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) \\ &\quad -g(X,Z)g(QY,W)] - \frac{r-2\alpha^2}{2}[g(Y,Z)g(X,W) \qquad (7.4) \\ &\quad -g(X,Z)g(Y,W)] \\ &\quad -\alpha^2[g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) \\ &\quad -g(Y,W)\eta(X)\eta(Z) + g(X,W)\eta(Y)\eta(Z) \\ &\quad +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)]. \qquad (7.5)\end{aligned}$$

Putting $X = W = \xi$ in (7.4) and using (2.1) and (2.3), we get

$$\begin{aligned}g(\tilde{R}(\xi,Y)Z,\xi) &= [S(Y,Z) - S(\xi,Z)\eta(Y) + g(Y,Z)S(\xi,\xi) \\ &\quad -\eta(Z)S(Y,\xi)] - \frac{r}{2}[g(Y,Z) - \eta(Z)\eta(Y)], \qquad (7.6)\end{aligned}$$

where $g(QY,Z)=S(Y,Z)$.

Now, using (2.13) and (2.16), we get

$$S(Y,Z)=\frac{r}{2}g(Y,Z)+[6(\alpha^2-\beta^2)-\frac{r}{2}]\eta(Y)\eta(Z). \tag{7.7}$$

Therefore

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a=\frac{r}{2}$ and $b=[6(\alpha^2-\beta^2)-\frac{r}{2}]$.

This shows that the manifold M is an η-Einstein manifold.

Thus we can state the following:

Theorem 7.1. A three-dimensional Weyl conformally flat trans-Sasakian manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is an η-Einstein manifold provided α,β are constants with $\alpha\neq\beta$.

8. Example of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen Connection

In this section, we wanted to construct an example of a three-dimensional trans-Sasakian manifold with respect to Schouten-van Kampen connection.

We have considered the three-dimensional manifold $M=\{(x,y,z)\in R^3, z\neq 0\}$, where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1=e^{-z}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\qquad e_2=e^{-z}(-\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\qquad e_3=\frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,\qquad g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1)=e_2$, $\phi(e_2)=-e_1$, $\phi(e_3)=0$. Then using the linearity of ϕ and g we have

$$\eta(e_3)=1,\qquad \phi^2Z=-Z+\eta(Z)e_3,\qquad g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any $Z,W\in\chi(M)$. Thus for $e_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1,e_2]=0,\qquad [e_2,e_3]=e_2,\qquad [e_1,e_3]=e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{8.1}$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From above we see that the manifold satisfies (2.6) for $\alpha = 0$, $\beta = 1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0,1)$. With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_2,e_3)e_3 = -e_2, & R(e_1,e_3)e_3 = -e_1, \\ R(e_1,e_2)e_2 = -e_1, & R(e_2,e_3)e_2 = e_3, & R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 = e_2, & R(e_2,e_3)e_1 = 0, & R(e_1,e_3)e_1 = e_3. \end{array}$$

Now we consider the Schouten-Van Kampen connection to this example.

Using (3.2) and above result we have

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_3 = (1-\beta)e_1 + \alpha e_2, & \tilde{\nabla}_{e_1} e_2 = -\alpha e_3, & \tilde{\nabla}_{e_1} e_1 = (\beta - 1)e_3, \\ \tilde{\nabla}_{e_2} e_3 = -\alpha e_1 + (1-\beta)e_2 & \tilde{\nabla}_{e_2} e_2 = (\beta-1)e_3, & \tilde{\nabla}_{e_2} e_1 = 0, \\ \tilde{\nabla}_{e_3} e_3 = 0 & \tilde{\nabla}_{e_3} e_2 = -\beta e_2, & \tilde{\nabla}_{e_3} e_1 = -\beta e_1. \end{array}$$

Using (3.4) we get

$$\begin{array}{ll} \tilde{R}(e_1,e_2)e_3 = 0, & \tilde{R}(e_2,e_3)e_3 = (\beta^2 - \alpha^2 - 1)e_2, \\ \tilde{R}(e_1,e_3)e_3 = (\beta^2 - \alpha^2 - 1)e_1, & \tilde{R}(e_1,e_2)e_2 = \alpha^2 e_2 + (\beta^2 + \alpha^2 - 1)e_1, \\ \tilde{R}(e_2,e_3)e_2 = (-\beta^2 + \alpha^2 + 1)e_3, & \tilde{R}(e_1,e_3)e_2 = 0, \\ \tilde{R}(e_1,e_2)e_1 = (1 - \beta^2 - \alpha^2)e_2, & \tilde{R}(e_2,e_3)e_1 = 0, \\ \tilde{R}(e_1,e_3)e_1 = (1 + \alpha^2 - \beta^2)e_3. & \end{array}$$

From the above expressions of the curvature tensor we obtain

$$S(e_1,e_1) = \sum_{i=1}^{3} g(R(e_i,e_1)e_1,e_i) = -2.$$

Similarly, we have

$$S(e_2,e_2) = -2 \quad \text{and} \quad S(e_3,e_3) = -2.$$

$$\tilde{S}(e_1, e_2) = \tilde{S}(e_2, e_2) = 2(\beta^2 - 1) \quad \tilde{S}(e_3, e_3) = 2(\beta^2 - \alpha^2 - 1).$$

$$r = -6 \quad \tilde{r} = 6\beta^2 - 2\alpha^2 - 6.$$

From above we see that $\tilde{r} = 0$ for $\alpha = 0, \beta = 1$. Therefore, the manifold under consideration satisfies the Theorem 5.2.
Using (4.1) and above relations, we get

$$P(e_1, e_2)e_3 = P(e_1, e_3)e_3 = P(e_2, e_3)e_3 = 0,$$

$$\tilde{P}(e_1, e_2)e_3 = \tilde{P}(e_1, e_3)e_3 = \tilde{P}(e_2, e_3)e_3 = 0.$$

Therefore, the manifold will be ξ-projectively flat on a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection which varifies the Theorem 4.1.

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