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SOME RESULTS ON $*-$RICCI FLOW

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Abstract. In this paper we have introduced the notion of $*$-Ricci flow and shown that $*$-Ricci soliton which was introduced by Kaimakamis and Panagiotidou in 2014 is a self similar soliton of the $*$-Ricci flow. We have also found the deformation of geometric curvature tensors under $*$-Ricci flow. In the last two section of the paper, we have found the $\mathfrak{F}$-functional and ω-functional for $*$-Ricci flow respectively.
Keywords: $*-$ Ricci flow, Conformal Ricci flow, $\mathfrak{F}$ functionals, ω functionals.

1. Introduction

A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\pounds$ denotes the Lie derivative operator, λ is a constant and S is the Ricci tensor of the metric g. Tachibana [3] first introduced $*$-Ricci tensor on almost Hermitian manifolds and Hamada [1] apply this to almost contact manifolds by defining

$$S^*(X, Y) = \frac{1}{2} trace(Z \to R(X, \phi Y)\phi Z),$$

for any $X, Y \in TM$. In 2014, Kaimakamis and Panagiotidou [2] introduced the concept of $*$-Ricci solitons within the background of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (1.1) with the $*$-Ricci tensor S^*. More precisely, a $*$-Ricci soliton on (M, g) is defined by

$$\pounds_V g + 2S^* + 2\lambda g = 0. \tag{1.2}$$

Inspired by the work of Kaimakamis and Panagiotidou [2], we introduced and studied $*$-Ricci flow on Riemannian manifold and further studied $*$-Ricci solitons. We

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have obtained deformation of geometric curvature tensor under $*$-Ricci flow. We have also provided the rate of change of F-functionals and ω-entropy functional with respect to time under this flow.
We have defined $*$-Ricci flow as follows

$$\frac{\partial g}{\partial t} = -2S^*(X,Y). \tag{1.3}$$

In this paper we have shown that just like Ricci soliton; $*$-Ricci soliton is a self-similar soliton of the $*$-Ricci flow. We have also found the deformation of geometric curvature tensors under $*$-Ricci flow.

Proposition 1.1. Defining $g\bar{(t)} = \sigma(t)\phi_t^*(g) + \sigma(t)\phi_t^*(\frac{\partial g}{\partial t}) + \sigma(t)\varphi_t^*(\pounds_X g)$, we have

$$\frac{\partial \hat{g}}{\partial t} = \acute{\sigma}(t)\psi_t^*(g) + \sigma(t) + \psi_t^*(\frac{\partial g}{\partial t}) + \sigma(t)\psi_t^*(\pounds_X g). \tag{1.4}$$

Proof: This follows from the definition of Lie derivative. If we have a metric g, a vector field Y and $\lambda \in R$ such that

$$-2Ric^*(g_0) = \pounds_Y g_0 - 2\lambda g_0$$

after setting $g(t) = g_0$ and $\sigma(t) = 1-2\lambda t$ and then integrating the t-dependent vector filed $X(t) = \frac{1}{\sigma(t)}Y$. To give a family of deffeomorphism ψ_t with ψ_0 the identity then $\bar{g}$ defined previously is a Ricci flow with

$$\bar{g} = g_0 \frac{\partial \bar{g}}{\partial t} = \sigma'(t)\phi_t^*(g_0) + \sigma(t)\phi_t^*(\pounds_X g_0)$$

$$= \phi_t^*(-2\lambda g_0 + \pounds_Y g_0) = \phi_t^*(-2Ric^*(g_0)) = -2Ric^*(\bar{g}).$$

Proposition 1.2. Under $*-$Ricci flow

$$g(\frac{\partial}{\partial g}\nabla_X Y, Z) = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y, \nabla_X Z) + 2S^*(\nabla_X Y, Z).$$

Proof. Let us consider

$$\frac{\partial}{\partial t}\nabla_X Y = \pi(X,Y).$$

Now we can write

$$g(\frac{\partial}{\partial t}\nabla_X Y, Z) = g(\pi(X,Y), Z). \tag{1.5}$$

Again

$$g(\frac{\partial}{\partial t}\nabla_X Y, Z) = \frac{\partial}{\partial t} g(\nabla_X Y, Z) - \frac{\partial g}{\partial t}(\nabla_X Y, Z).$$

$$g(\pi(X,Y), Z) = \frac{\partial}{\partial t} g(\nabla_X, Z) + 2S^*(\nabla_X Y, Z). \tag{1.6}$$

We have

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{1.7}$$

From (1.5) we have

$$g(\pi(X,Y),Z) = \frac{\partial}{\partial t}[Xg(Y,Z) - g(Y,\nabla_X Z)] + 2S^*(\nabla_X Y, Z)$$

$$g(\pi(X,Y),Z) = X\frac{\partial g}{\partial t}(Y,Z) - (\frac{\partial g}{\partial t})(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z)$$

or

$$g(\pi(X,Y), Z = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z)$$

i.e.

$$g(\frac{\partial}{\partial g}\nabla_X Y, Z) = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z). \tag{1.8}$$

2. The $\mathfrak{F}$-functional for the $*$-Ricci flow

Let M be a fixed closed manifold, g a Riemannian metric and f a function defined on M to the set of real numbers $\mathbb{R}$.

Then the $\mathfrak{F}$-functional on pair (g,f) is defined as

$$\mathfrak{F}(g,f) = \int(-1 + |\nabla f|^2)e^{-f}dV. \tag{2.1}$$

Now, we will establish how the $\mathfrak{F}$-functional changes according to time under $*$-Ricci flow.

Theorem 2.1. *In $*$-Ricci flow the rate of change of $\mathfrak{F}$-functional with respect of time is given by*

$$\frac{d}{dt}\mathfrak{F}(g,f) = \int[-2Ric^*(\nabla f,\nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)$$

$$+ (-1 + |\nabla f|^2)(-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV$$

where

$$\mathfrak{F}(g,f) = \int(-1 + |\nabla f|^2)e^{-f}dV.$$

Proof. We may calculate

$$\frac{\partial}{\partial t}|\nabla f|^2 = \frac{\partial}{\partial t}g(\nabla f,\nabla f) = \frac{\partial g}{\partial t}(\nabla f,\nabla f) + 2g(\nabla\frac{\partial f}{\partial t},\nabla f). \tag{2.2}$$

So using proposition 2.3.12 of [13] we can write

$$\frac{d}{dt}\mathfrak{F}(g,f)=\int[\frac{\partial g}{\partial t}(\nabla f,\nabla f)+2g(\nabla\frac{\partial f}{\partial t},\nabla f)]e^{-f}dV$$
$$+\int(-1+|\nabla f|^2)[-\frac{\partial f}{\partial t}+\frac{1}{2}tr\frac{\partial g}{\partial t}]e^{-f}dV. \tag{2.3}$$

Using integration by parts of equation(2.2), we get

$$\int 2g(\nabla\frac{\partial f}{\partial t},\nabla f)e^{-f}dV=-2\int\frac{\partial f}{\partial t}(\Delta f-|\nabla f|^2)e^{-f}dV. \tag{2.4}$$

Now putting (2.4) in (2.3), we get

$$\frac{d}{dt}\mathfrak{F}(g,f)=\int[\frac{\partial g}{\partial t}(\nabla f,\nabla f)-2\frac{\partial f}{\partial t}(\Delta f-|\nabla f|^2)$$
$$+(-1+|\nabla f|^2)(-\frac{\partial f}{\partial t}+\frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV. \tag{2.5}$$

Using (1.3) in (2.5), we get the following result for conformal Ricci flow, as

$$\frac{d}{dt}\mathfrak{F}(g,f)=\int[-2Ric^*(\nabla f,\nabla f)-2\frac{\partial f}{\partial t}(\Delta f-|\nabla f|^2)$$
$$+(-1+|\nabla f|^2)(-\frac{\partial f}{\partial t}+\frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV. \tag{2.6}$$

Hence the proof.

3. ω-entropy functional for the $*$- Ricci flow

Let M be a closed manifold, g a Riemannian metric on M and f a smooth function defined from M to the set of real numbers $\mathbb{R}$. We define ω-entropy functional as

$$\omega(g,f,\tau)=\int[\tau(R^*+|\nabla f|^2)+f-n]udV \tag{3.1}$$

where $\tau>0$ is a scale parameter and u is defined as $u(t)=e^{-f(t)}$;$\int_M udV=1$.

We would also like to define heat operator acting on the function $f: M\times[0,\tau]\longrightarrow\mathbb{R}$ by $\diamondsuit:=\frac{\partial}{\partial t}-\Delta$ and also, $\diamondsuit^*:=-\frac{\partial}{\partial t}-\Delta+R^*$, conjugate to $\diamondsuit$.

We choose u, such that $\diamondsuit^*u=0$.

Now we prove the following theorem.

Theorem 3.1: *If g, f, τ evolve according to*

$$\frac{\partial g}{\partial t}=-2Ric^* \tag{3.2}$$

$$\frac{\partial \tau}{\partial t} = -1 \tag{3.3}$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R^* + \frac{n}{2\tau} \tag{3.4}$$

and the function v *is defined as* $v = [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]u$, *the rate of change of* ω*-entropy functional for conformal Ricci flow is* $\frac{d\omega}{dt} = -\int_M \diamondsuit^* v$, *where*

$$\diamondsuit^* v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - u\tau[4 < Ric^*, Hessf >$$

$$-2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|Hessf|^2].$$

Proof: We find that

$$\diamondsuit^* v = \diamondsuit^*(\frac{v}{u}u) = \frac{v}{u}\diamondsuit^* u + u\diamondsuit^*(\frac{v}{u}).$$

We have defined previously that $\diamondsuit^* u = 0$,

so

$$\diamondsuit^* v = u\diamondsuit^*(\frac{v}{u})$$

$$\diamondsuit^* v = u\diamondsuit^*[\tau(2\nabla f - |\nabla f|^2 + R^*) + f - n].$$

We shall use the conjugate of heat operator, as defined earlier as $\diamondsuit^* = -(\frac{\partial}{\partial t} + \Delta - R^*)$.

Therefore

$$\diamondsuit^* v = -u(\frac{\partial}{\partial t} + \Delta - R^*)[\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]$$

$$\Rightarrow u^{-1}\diamondsuit^* v = -(\frac{\partial}{\partial t} + \Delta)[\tau(2\Delta f - |\nabla f|^2 + R^*)]$$

$$-(\frac{\partial}{\partial t} + \Delta)f - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n].$$

Using equation (3.3), we have

$$u^{-1}\diamondsuit^* v = (2\Delta f - |\nabla f|^2 + R^*) - \tau(\frac{\partial}{\partial t} + \Delta)(2\Delta f - |\nabla f|^2 + R^*)$$

$$-\frac{\partial f}{\partial t} - \Delta f - \frac{v}{u}. \tag{3.5}$$

Now
$$\frac{\partial}{\partial t}(2\Delta f-|\nabla f|^2+R^*)=2\frac{\partial}{\partial t}(\Delta f)-\frac{\partial}{\partial t}|\nabla f|^2.$$

Using proposition (2.5.6) of [13], we have
$$\frac{\partial}{\partial t}(2\Delta f-|\nabla f|^2+R^*)=2\Delta\frac{\partial f}{\partial t}+4<Ric^*,Hessf>$$
$$-\frac{\partial g}{\partial t}(\nabla f,\nabla f)-2g(\frac{\partial}{\partial t}\nabla f,\nabla f).$$

Now using the $*-$Ricci flow equation (1.3), we have
$$\frac{\partial}{\partial t}(2\Delta f-|\nabla f|^2+R^*)=2\Delta\frac{\partial f}{\partial t}+\ 4<Ric^*,Hessf>$$
$$+2Ric^*(\nabla f,\nabla f)-2g(\frac{\partial}{\partial t}\nabla f,\nabla f). \tag{3.6}$$

Using (3.4) in (3.6), we get
$$\frac{\partial}{\partial t}(2\Delta f-|\nabla f|^2+R^*)=2\Delta(-\Delta f+|\nabla f|^2-R^*+\frac{n}{2\tau})+4<Ric^*,Hessf>$$
$$+2Ric^*(\nabla f,\nabla f)-2g(\frac{\partial}{\partial t}\nabla f,\nabla f). \tag{3.7}$$

Now let us compute
$$\Delta(2\Delta f-|\nabla f|^2+R^*)=2\Delta^2 f-\Delta|\nabla f|^2. \tag{3.8}$$

Using (3.7) and (3.8) in (3.5) we obtain after a brief calculation
$$u^{-1}\Diamond^* v=(2\Delta f-|\nabla f|^2+R^*)-\tau[-2\Delta^2 f+2\Delta|\nabla f|^2+4<Ric^*,Hessf>$$
$$+2Ric^*(\nabla f,\nabla f)-2g(\tfrac{\partial}{\partial t}\nabla f,\nabla f)+2\Delta^2 f-\Delta|\nabla f|^2]-\tfrac{\partial f}{\partial t}-\Delta f-\tfrac{v}{u}$$
$$=\Delta f-|\nabla f|^2+R^*-\tau[\Delta|\nabla f|^2+4<Ric^*,Hessf>+2Ric^*(\nabla f,\nabla f)$$
$$-2g(\tfrac{\partial}{\partial t}\nabla f,\nabla f)]-\tfrac{\partial f}{\partial t}-\tfrac{v}{u}$$
$$=\Delta f-|\nabla f|^2+R^*-\tau[\Delta|\nabla f|^2+4<Ric^*,Hessf>+2Ric^*(\nabla f,\nabla f)$$
$$-2g(\tfrac{\partial}{\partial t}\nabla f,\nabla f)]+\Delta f-|\nabla f|^2+R^*-\tfrac{n}{2\tau}-\tfrac{v}{u}$$
$$=2(\Delta f-|\nabla f|^2+R^*)-\tfrac{n}{2\tau}-\tfrac{v}{u}-\tau[\Delta|\nabla f|^2+4<Ric^*,Hessf>$$
$$+2Ric^*(\nabla f,\nabla f)-2g(\tfrac{\partial}{\partial t}\nabla f,\nabla f)]$$

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \tfrac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n] - \tau[\Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f)].$$

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + \Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\nabla\frac{\partial f}{\partial t}, \nabla f)]. \tag{3.9}$$

Using (3.4), we get

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + \Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\nabla(-\Delta f + |\nabla f|^2 + \frac{n}{2\tau} - R^*), \nabla f)]. \tag{3.10}$$

We can rewrite (3.10) in the following way

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + 4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] + \tau[-\Delta|\nabla f|^2 - 2Ric^*(\nabla f, \nabla f) + 2g(\nabla(\Delta f), \nabla f)] \tag{3.11}$$

and using Bochner formula in (3.11) and simplifying it, we get

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + 4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2.$$

$$\Rightarrow u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n] - \tau[4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2.$$

i.e.

$$u^{-1}\diamondsuit^{*}v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - \frac{v}{u} - \tau[4 < Ric^*, Hessf > - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2. \tag{3.12}$$

So finally we have

$$\diamondsuit^* v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - u\tau[4 < Ric^*, Hessf > \\ - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|Hessf|^2]. \tag{3.13}$$

Now using remark (8.2.7) of [13], we get

$$\frac{d\omega}{dt} = -\int_M \diamondsuit^* v.$$

So the evolution of ω with respect to time can be found by this integration.

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