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FIXED POINTS OF GENERALIZED (α, ψ, φ)-CONTRACTIVE MAPS AND PROPERTY(P) IN S-METRIC SPACES

Gutti Venkata Ravindranadh Babu[1] **and Leta Bekere Kumssa**[2]

[1] **Department of Mathematics, Andhra University,**
Visakhapatnam-530 003, India
[2] **Department of Mathematics, Madda Walabu University,**
Bale-Robe, P. O. Box 247, Ethiopia

Abstract. In this paper, we have introduced generalized (α, ψ, φ)-contractive maps and proved the existence and uniqueness of fixed points in complete S-metric spaces. We have also proved that these maps satisfy property (P). The results presented in this paper extend several well known comparable results in metric and G-metric spaces. We have provided an example in support of our result.

Keywords: S-metric space, property(P), generalized contractive maps, fixed points

1. Introduction and Preliminaries

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in various areas. Finding the existence of fixed points of a self map by considering more general ambient spaces is an interesting aspect. In this course of development, some authors have tried to give generalizations of metric spaces in various ways. In 2005, Mustafa and Sims [13] introduced a new structure of metric spaces which are called G-metric spaces as a generalization of metric spaces to develop and introduce new concepts on contraction maps and proved the existence of fixed points of various mappings in this new space. For more works on G-metric spaces, we refer [3, 14, 21]. In 2007, Sedghi [18] introduced D^*-metric spaces which is a probable modification of the definition

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Corresponding Author: Leta Bekere Kumssa, Department of Mathematics, Madda Walabu University, Bale-Robe, P. O. Box 247, Ethiopia | E-mail: letabekere@yahoo.com
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of D-metric spaces introduced by Dhage [7] and proved some basic properties of D^*-metric spaces [17, 18]. In 2012, Sedghi, Shobe and Aliouche [19] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Sedghi, Shobe and Aliouche [19] asserted that S-metric space is a generalization of G-metric space. But, very recently Dung, Hieu and Radojevic [8] have verified by example (Example 2.1 and Example 2.2) that S-metric space is not a generalization of G- metric space or vice versa. Therefore, the classes of G-metric spaces and S- metric spaces are different. Recent papers dealing with fixed point theorems for mappings satisfying certain contractive conditions on S-metric spaces can be referred in [1, 2, 8, 12, 15, 16, 20].

Now we provide some preliminaries and basic definitions which we use throughout this paper. We start with a G- metric spaces introduced by Mustafa and Sims [13].

Definition 1.1. [13] Let X be a non-empty set, $G : X^3 \to [0,\infty)$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leqslant G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

(G4) $G(x,y,z) = G(x,z,y) = G(z,x,y) = ...$ (symmetry in all three variables),

(G5) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric (G-metric) and the pair (X,G) is called a G-metric space.

Definition 1.2. [11] A mapping $\psi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if it satisfies: (i) ψ is continuous (ii) ψ non-decreasing and (iii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Ψ.

We denote $\Phi = \{\varphi : [0,\infty) \to [0,\infty)\ \varphi$ is continuous and non-decreasing$\}$.

Remark 1.1. [4] If $\psi \in \Psi$ and $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then $\varphi(0) = 0$. Therefore $\varphi \in \Psi$.

Definition 1.3. [9] Let X be a non-empty set and T be a self map of X. We denote the set of all fixed points of T by $F(T)$, where $F(T) \neq \emptyset$. Then, T is said to satisfy property (P) if $F(T) = F(T^n)$ for all $n \in \mathbb{N}$.

Here we note that even though, a map $f : X \to X$ has a unique fixed point, it may not have property (P).

In [4] Bousselsal et.al proved the existence and uniqueness of fixed points and property (P) in G-metric spaces.

Theorem 1.1. [4] *Let (X, G) be a complete G-metric space and $f : X \to X$ be a mapping. If there exists $\psi \in \Psi$ and $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, such that*

$$\psi(G(fx, fy, fz)) \leqslant \varphi\big(\max\big\{G(x, y, y), G(x, fx, fx), G(y, fy, fy), \tag{1.1}$$

$$G(z, fz, fz), \alpha G(fx, fx, y)+(1-\alpha)G(fy, fy, z), \beta G(x, fx, fx)+(1-\beta)G(y, fy, fy)\big\}\big)$$

for all $x, y, z \in X$, where $\alpha, \beta \in (0, 1)$.
Then f has a unique fixed point (say u) and f is G-continuous at u. Further, f has property (P).

Note: In view of Remark 1.1, we can choose $\varphi \in \Psi$ in Theorem 1.1.

Remark 1.2.

$$\text{Since } \max\big\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)$$
$$+(1-\alpha)G(fy, fy, z), \beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)\big\}$$
$$= \max\big\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)$$
$$+(1-\alpha)G(fy, fy, z)\big\}$$

so that we need not consider the β terms in the inequality (1.1).

In 2012, Sedghi, Shobe and Aliouche [19] introduced S-metric spaces as follows:

Definition 1.4. [19] Let X be a non-empty set. An $S-$metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x, y, z) \geqslant 0$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$ and

(S3) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 1.1. (Example 2.4 [19]). Let (X, d) be a metric space. Define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then S is an S-metric on X. This S-metric is called the S-metric induced by the metric d.

Example 1.2. (Example 1.9 [8]). Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 1.3. (Example 2.2 [8]). There exists an S-metric which is not a G-metric. Let (X, S) be the S-metric space in Example 1.2. We have
$S(1, 0, 2) = |0 + 2 - 2| + |0 - 2| = 2, S(2, 0, 1) = |0 + 1 - 4| + |0 - 1| = 4$.
Then $S(1, 0, 2) \neq S(2, 0, 1)$. So that $(G4)$ fails. Hence S is not a G-metric.

Example 1.4. (Example 2.1, [8]). There exists a G-metric which is not an S-metric. Let $X = \{a, b\}$. Define $G : X^3 \to [0, \infty)$ by $G(a, a, a) = G(b, b, b) = 0$, $G(a, b, b) = 2$, $G(a, a, b) = 1$ and extend G to all X^3 by using $(G4)$. Then G is a G-metric but not an S-metric. Since $2 = G(a, b, b) \nleq 1 = G(a, a, b) + G(b, b, b) + G(b, b, b)$. This shows that G is not an S-metric on X.

Remark 1.3. From Example 1.3 and Example 1.4, we can conclude that the class of S-metrics and the class of G-metrics are distinct.

The following lemmas are very useful in our subsequent discussions in proving our main results.

Lemma 1.1. [19] *In an S-metric space, we have* $S(x, x, y) = S(y, y, x)$.

Lemma 1.2. [8] *Let (X, S) be an S-metric space. Then*
(i) $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ *and*
(ii) $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$.

Definition 1.5. [19] Let (X, S) be an S-metric space. We define the following:

(i) A sequence $\{x_n\}$ in X converge to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geqslant n_0$.

(iii) The S-metric space (X, S) is said to be complete if each Cauchy sequence in x is convergent.

Definition 1.6. [12] Let (X, S) and (Y, S') be two S-metric spaces. Then the function $f : X \to Y$ is S-continuous at $x \in X$ if it is S-sequentially continuous at x, that is, whenever $\{x_n\}$ is S-convergent to x, we have $f(x_n)$ is S'-convergent to $f(x)$.

Lemma 1.3. [19] *Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 1.4. [19] *Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*
$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.5. [1] *Any S-metric space is a Hausdorff space.*

In 2012, Sedghi [19] proved an analogue of Banach's contraction principle in S-metric space.

Definition 1.7. [19] Let (X, S) be an S-metric space. A map $f : X \to X$ is said to be an S-contraction if there exists a constant $0 \leqslant \lambda < 1$ such that $S(f(x), f(x), f(y)) \leq \lambda S(x, x, y)$ for all $x, y \in X$.

Theorem 1.2. [19] *Let (X, S) be a complete $S-$metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} f^n(x) = u$ with $S(f^n(x), f^n(x), u) \leq \frac{2\lambda^n}{1-\lambda}(Sx, x, f(x))$.*

We now introduce the following definition.

Definition 1.8. Let (X, S) be an S-metric space. Let $f : X \to X$ be a self map of X. If there exists $\alpha \in (0, 1)$ and $\psi, \varphi \in \Psi$ such that

$$\begin{aligned}(1.2)\qquad \psi(S(fx, fy, fz)) \leqslant \varphi\big(\max\{&S(x, y, z), S(x, x, fx), S(y, y, fy),\\ &S(z, z, fz), \alpha S(fx, fx, y) + (1 - \alpha)S(fy, fy, z)\}\big)\end{aligned}$$

for all $x, y, z \in X$. Then we say that f is a generalized (α, ψ, φ)-contractive map on X.

Remark 1.4. We note that S-contraction map is a generalized (α, ψ, φ)-contraction map with $\psi(t) = t$, for all $t \geqslant 0$ and $\varphi(t) = \lambda t$, for all $t \geqslant 0$ where λ is an S-contraction constant. But its converse is not true (Example 3.1). Thus the class of S-contraction map is a proper subset of the class of all generalized (α, ψ, φ)-contraction map.

Hence we study the existence of fixed points of generalized (α, ψ, φ)-contractions in S-metric spaces.

2. Main Results

We start this section with following lemma which is useful in proving our main results.

Lemma 2.1. *Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X such that*

$$(2.1)\qquad \lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$(2.2)\qquad S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon, S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon \text{ and}$$

(i) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon,$

(ii) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon,$

(*iii*) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon$.

Proof. Let $\{x_n\} \subset X$ be not Cauchy. Then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon. \tag{2.3}$$

We choose m_k, the least positive integer satisfying (2.3). Then $m_k > n_k > k$ with $S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon$ and $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$. Hence (2.2) holds.

From (2.2), we have

$$\epsilon \leqslant S(x_{m_k}, x_{m_k}, x_{n_k}). \tag{2.4}$$

On taking the lower limit in (2.4), we get

$$\epsilon \leq \liminf_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}). \tag{2.5}$$

By Lemma 1.2, we have

$$\begin{aligned} S(x_{m_k}, x_{m_k}, x_{n_k}) &\leqslant 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}) \\ &= 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \\ &< 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon. \end{aligned} \tag{2.6}$$

From (2.4), (2.6) and on taking the upper limit as $k \to \infty$, we have

$$\limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon. \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon. \tag{2.8}$$

Hence (*i*) is proved.

Again, from (2.2), by Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} \epsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k}) \\ &\leqslant 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \\ &= 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}). \end{aligned} \tag{2.9}$$

From (2.9) and on taking the upper limit as $k \to \infty$, we obtain

$$\epsilon \leq \limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.10}$$

Once again, by Lemma 1.2 and (2.3), we get

$$S(x_{m_k}, x_{m_k}, x_{n_k-1}) = S(x_{n_k-1}, x_{n_k-1}, x_{m_k})$$

$$
\begin{aligned}
&\leqslant 2S(x_{n_k-1},x_{n_k-1},x_{n_k})+S(x_{n_k},x_{n_k},x_{m_k})\\
&= 2S(x_{n_k-1},x_{n_k-1},x_{n_k})+S(x_{m_k},x_{m_k},x_{n_k}). \qquad (2.11)
\end{aligned}
$$

Now, on taking the upper limit as $k\to\infty$ in (2.11), we obtain

$$\limsup_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1})\leqslant \epsilon. \qquad (2.12)$$

By (2.10) and (2.12), we get

$$\limsup_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1})= \epsilon. \qquad (2.13)$$

From (2.9), we obtain

$$S(x_{m_k},x_{m_k},x_{n_k-1})\geq \epsilon-2S(x_{n_k},x_{n_k},x_{n_k-1}). \qquad (2.14)$$

Hence on taking the lower limit as $k\to\infty$ in (2.14), we get

$$\epsilon\leq \liminf_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1}). \qquad (2.15)$$

Therefore from (2.13) and (2.15), we obtain

$$\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1})=\lim_{k\to\infty} S(x_{n_k-1},x_{n_k-1},x_{m_k})=\epsilon. \qquad (2.16)$$

So, (ii) is proved.

Again, from (2.2), by Lemma 1.1 and Lemma 1.2, we have

$$
\begin{aligned}
\epsilon &\leq S(x_{m_k},x_{m_k},x_{n_k})=S(x_{n_k},x_{n_k},x_{m_k})\\
&\leqslant 2S(x_{n_k},x_{n_k},x_{n_k-1})+S(x_{n_k-1},x_{n_k-1},x_{m_k})\\
&= 2S(x_{n_k},x_{n_k},x_{n_k-1})+S(x_{m_k},x_{m_k},x_{n_k-1})\\
&\leq 2S(x_{n_k},x_{n_k},x_{n_k-1})+2S(x_{m_k},x_{m_k},x_{m_k-1})+S(x_{m_k-1},x_{m_k-1},x_{n_k-1}). \qquad (2.17)
\end{aligned}
$$

From (2.17) and on taking the upper limit as $k\to\infty$, we obtain

$$\epsilon\leq \limsup_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}). \qquad (2.18)$$

Again, by Lemma 1.1 and Lemma 1.2, we have

$$S(x_{m_k-1},x_{m_k-1},x_{n_k-1})\leq 2S(x_{m_k-1},x_{m_k-1},x_{m_k})+S(x_{m_k},x_{m_k},x_{n_k-1}). \qquad (2.19)$$

On taking the upper limit as $k\to\infty$ in (2.19) and by using (2.16), we have

$$\limsup_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1})\leq \epsilon. \qquad (2.20)$$

From (2.17), we obtain

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \geq \epsilon - 2S(x_{m_k}, x_{m_k}, x_{m_k-1}). \tag{2.21}$$

Hence on taking the lower limit as $k \to \infty$ in (2.21), we get

$$\epsilon \leq \liminf_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.22}$$

Therefore by combining (2.18), (2.20) and (2.22), we obtain

$$\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{2.23}$$

So, *(iii)* is proved. Hence the lemma follows. □

In the following we prove the main result of this paper.

Theorem 2.1. *Let (X, S) be a complete S-metric space and let f be a generalized (α, ψ, φ)-contractive map. If there exists $\psi, \varphi \in \Psi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then f has a unique fixed point (say u) and f is S-continuous at u.*

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and we are through.

Now, we assume that $x_n \neq x_{n+1}$ for all n. By (1.2) and substituting $x = y = x_{n-1}, z = x_n$, we have

$$\begin{aligned}
\psi(S(x_n, x_n, x_{n+1})) = \psi(S(fx_{n-1}, fx_{n-1}, fx_n)) &\leqslant \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n),\\
&S(x_{n-1}, x_{n-1}, fx_{n-1}), S(x_{n-1}, x_{n-1}, fx_n), S(x_n, x_n, fx_n),\\
&\alpha S(fx_{n-1}, fx_{n-1}, x_{n-1}) + (1-\alpha)S(fx_{n-1}, fx_{n-1}, x_n)\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n),\\
&S(x_n, x_n, x_{n+1}), \alpha S(x_n, x_n, x_{n-1}) + (1-\alpha)S(x_n, x_n, x_n)\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \alpha S(x_n, x_n, x_{n-1})\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\big\}\big).
\end{aligned} \tag{2.24}$$

If $S(x_n, x_n, x_{n+1}) > S(x_{n-1}, x_{n-1}, x_n)$, then (2.24) becomes

$$\psi(S(x_n, x_n, x_{n+1})) \leqslant \varphi(S(x_n, x_n, x_{n+1})) < \psi(S(x_n, x_n, x_{n+1})), \tag{2.25}$$

a contradiction. Hence $S(x_{n-1}, x_{n-1}, x_n)$ is the maximum. Therefore

$$\psi(S(x_n, x_n, x_{n+1})) \leqslant \varphi\big(S(x_{n-1}, x_{n-1}, x_n) \big) < \psi(S(x_{n-1}, x_{n-1}, x_n)). \tag{2.26}$$

By using the property of ψ, φ and from (2.26), we obtain

$$S(x_n, x_n, x_{n+1}) \leqslant S(x_{n-1}, x_{n-1}, x_n) \text{ for all } n.$$

Hence $\{S(x_n, x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Then there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = r. \tag{2.27}$$

On letting $n \to \infty$ in (2.26) and using (2.27), we have $\psi(r) \leqslant \varphi(r) < \psi(r)$, a contradiction. Hence $r = 0$.

We now prove that $\{x_n\}$ is an S-Cauchy sequence. If possible $\{x_n\}$ is not S-Cauchy. By Lemma 2.1, there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that $S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon$, $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(ii) of Lemma 2.1.
Putting $x = y = x_{m_k-1}, z = x_{n_k-1}$ and applying (1.2), we get

$$\begin{aligned}
\psi(S(x_{m_k}, x_{m_k}, x_{n_k})) &= \psi(S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k-1}))\\
&\leqslant \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}),\\
&\quad S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}), S(x_{n_k-1}, x_{n_k-1}, fx_{n_k-1}),\\
&\quad \alpha S(fx_{m_k-1}, fx_{m_k-1}, x_{m_k-1}) + (1-\alpha)S(fx_{m_k-1}, fx_{m_k-1}, x_{n_k-1})\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k}, x_{m_k}, x_{m_k-1}),\\
&\quad S(x_{m_k}, x_{m_k}, x_{m_k-1}), S(x_{n_k}, x_{n_k}, x_{n_k-1}),\\
&\quad \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\big\}\big)\\
&= \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k}, x_{m_k}, x_{m_k-1}), S(x_{n_k}, x_{n_k}, x_{n_k-1}),\\
&\quad \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\big\}\big).
\end{aligned} \tag{2.28}$$

On letting $k \to \infty$ in (2.28), using (2.27) and Lemma 2.1, we obtain

$$\psi(\epsilon) \leqslant \varphi\big(\max\big\{\epsilon,\ 0,\ 0,\ (1-\alpha)\epsilon\big\}\big) = \varphi(\epsilon) < \psi(\epsilon),$$

a contradiction.
Hence $\{x_n\}$ is an S-Cauchy sequence. Since (X, S) is complete, there exists $u \in X$ such that $x_n \to u$.

We now show that u is a fixed point of f. Here by Lemma 1.4, we note that

$$\lim_{n\to\infty} S(x_n, x_n, fu) = S(u, u, fu).$$

Suppose that $f(u) \neq u$ and we consider

$$\begin{aligned}
\psi(S(fu, fu, x_n)) &= \psi(S(fu, fu, fx_{n-1})) \leqslant \varphi\big(\max\big\{S(u, u, x_{n-1}), S(u, u, fu), S(u, u, fu),\\
&\quad S(x_{n-1}, x_{n-1}, fx_{n-1}), \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\big\}\big)\\
&= \varphi\big(\max\big\{S(u, u, x_{n-1}), S(u, u, fu), S(x_{n-1}, x_{n-1}, x_n),\\
&\quad \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1}), S(fu, fu, u)\big\}\big).
\end{aligned} \tag{2.29}$$

On letting $n \to \infty$ in (2.29), we have

$$\psi(S(fu, fu, u)) \leqslant \varphi\big(\max\big\{S(u, u, u), S(u, u, fu), S(u, u, u),$$

$$\alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,u)\}\big)=\varphi(S(fu,fu,u))<\psi(S(fu,fu,u)),$$

a contradiction. Hence $fu=u$.

Next we prove uniqueness of fixed point. Suppose u and v are two distinct fixed points of f. Now, we consider

$$\begin{aligned}\psi(S(u,u,v)) &= \psi(S(fu,fu,fv))\\ &\leqslant \varphi\big(\max\big\{(S(u,u,v),S(u,u,fu),S(u,u,fu),S(v,v,fv),\\ &\quad \alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,v)\big\}\big)\\ &= \varphi\big(\max\big\{S(u,u,v),(1-\alpha)S(u,u,v)\big\}\big)\\ &= \varphi(S(u,u,v))<\psi(S(u,u,v)),\end{aligned}$$

a contradiction. Therefore $u=v$.

Finally we prove that f is S-continuous at u. Let $\{x_n\}$ be a sequence in X such that $x_n\to u$ as $n\to\infty$. We show that $fx_n\to fu$ as $n\to\infty$. For this purpose, we consider

$$\begin{aligned}\psi(S(u,u,fx_n)) &= \psi(S(fu,fu,fx_n))\\ &\leqslant \varphi\big(\max\big\{S(u,u,x_n),S(u,u,fu),S(u,u,fu),S(x_n,x_n,fx_n),\\ &\quad \alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,x_n)\big\}\big),\\ &= \varphi\big(\max\big\{S(u,u,x_n),S(u,u,u),S(x_n,x_n,x_{n+1}),\\ &= \alpha S(u,u,u)+(1-\alpha)S(u,u,x_n)\big\}\big).\end{aligned}\tag{2.30}$$

By taking the limit on both sides of (2.30), and using the continuity of φ, we have $\lim_{n\to\infty}\psi(S(fu,fu,fx_n))=0$.
By the continuity of ψ, we have $\psi(\lim_{n\to\infty}S(fu,fu,fx_n))=0$.
i.e., $\psi(\lim_{n\to\infty}S(fx_n,fx_n,fu)=0$ (by Lemma 1.1).
Again, by property of ψ we have $\lim_{n\to\infty}S(fx_n,fx_n,fu))=0$.
Hence by the definition of continuity of f, it follows that $fx_n\to fu$ as $n\to\infty$. Therefore, f is S-continuous at u. □

Theorem 2.2. *Under the hypotheses of Theorem 2.1 f has Property (P).*

Proof. In view of the proof of Theorem 2.1, f has a fixed point. Therefore $F(f^n)\neq\emptyset$. Now, we fix $n>1$ and assume that $u\in F(f^n)$. That is $f^nu=u$. We show that $u\in F(f)$. Assume $f(u)\neq u$, we consider

$$\begin{aligned}\psi(S(u,u,fu)) &= \psi(S(f^nu,f^nu,f^{n+1}u))=\psi(S(ff^{n-1}u,ff^{n-1}u,ff^nu))\\ &\leqslant \varphi\big(\max\big\{S(f^{n-1}u,f^{n-1}u,f^nu),S(f^{n-1}u,f^{n-1}u,ff^{n-1}u),\\ &\quad S(f^{n-1}u,f^{n-1}u,ff^{n-1}u),S(f^nu,f^nu,ff^nu),\\ &\quad \alpha S(ff^{n-1}u,ff^{n-1}u,f^{n-1}u)\end{aligned}$$

$$
\begin{aligned}
& +(1-\alpha) S(ff^{n-1}u, ff^{n-1}u, f^n u)\big\}\big), \\
= \;& \varphi\big(\max\big\{S(f^{n-1}u, f^{n-1}u, u), S(u,u,f^{n-1}u), S(u,u,f^{n-1}u), \\
& S(u,u,fu), \alpha S(u,u,f^{n-1}u) + (1-\alpha) S(u,u,u)\big\}\big), \\
= \;& \varphi\big(\max\big\{S(u,u,f^{n-1}u), S(u,u,fu)\big\}\big). \qquad (2.31)
\end{aligned}
$$

If $S(u,u,fu)$ is the maximum, then from (2.31), we have
$\psi(S(u,u,fu)) \leqslant \varphi(S(fu,fu,u)) = \varphi(S(u,u,fu)) < \psi(S(u,u,fu))$,
a contradiction. Consequently, $S(u,u,f^{n-1}u)$ is the maximum. Therefore, from (2.31) and Lemma 1.1, we obtain

$$
\begin{aligned}
\psi(S(u,u,fu)) &= \psi(S(f^n u, f^n u, f^{n+1}u)) \leqslant \varphi(S(u,u,f^{n-1}u)) \\
&= \varphi(S(f^n u, f^n u, f^{n-1}u)) < \psi(S(f^n u, f^n u, f^{n-1}u)) \\
&= \psi(S(f^{n-1}u, f^{n-1}u, f^n u)). \qquad (2.32)
\end{aligned}
$$

Since ψ is non decreasing, from (2.32), it follows that

$$S(f^n u, f^n u, f^{n+1}u) \leqslant S(f^{n-1}u, f^{n-1}u, f^n u).$$

Hence $\{S(f^n u, f^n u, f^{n+1}u)\}$ is a decreasing sequence of positive real numbers. Then, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} S(f^n u, f^n u, f^{n+1}u) = r. \qquad (2.33)$$

On letting $n \to \infty$ in (2.32) and using (2.33), we get $\psi(r) \leqslant \varphi(r) < \psi(r)$,
a contradiction. Therefore $r = 0$.
Hence $\psi(S(u,u,fu)) = \lim_{n\to\infty} (\psi(S(f^n u, f^n u, f^{n+1}u)) = 0$. That is, $fu = u$. Therefore, $u \in F(f)$. Hence f has property (P). □

In Section 3 we draw some corollaries from our results and provide a supportive example.

3. Corollaries and an Example

If ψ is the identity mapping on $[0,\infty)$ in Theorem 2.1, we have the following
.

Corollary 3.1. *Let (X,S) be a complete S-metric space and $f: X \to X$ be a mapping. Assume that there exists $\alpha \in (0,1)$, $\varphi \in \Psi$ satisfying $\varphi(t) < t$ for $t > 0$ such that*

$$
\begin{aligned}
S(fx,fy,fz) \leqslant \varphi\big(\max\big\{& S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \\
& \alpha S(fx,fx,y) + (1-\alpha) S(fy,fy,z)\big\}\big),
\end{aligned}
$$

for all $x,y,z \in X$. Then f has a unique fixed point (say u) and f is S-continuous at u.

Here we observe that the φ that is used in the inequality (3.1) is a Boyd-Wong [5] type contraction.

Corollary 3.2. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\lambda, \alpha \in (0,1)$, such that*

$$\begin{aligned} S(fx,fy,fz) \leqslant \lambda \max\big\{&S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),\\ &\alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\big\}, \end{aligned} \tag{3.1}$$

for all $x,y,z \in X$. Then f has a unique fixed point $(say\ u)$ and f is S-continuous at u.

Proof: By choosing $\varphi(t) = \lambda t$, for all $t \geqslant 0$ in Corollary 3.1, then the conclusion follows.

Corollary 3.3. *Let (X,S) be a complete $S-$metric space and $f : X \to X$ be a mapping. Assume there exist a constant $0 \leqslant \lambda < 1$, $\alpha \in (0,1), \psi$, such that $S(fx,fy,fz) \leq \lambda S(x,y,z)$ for all $x,y,z \in X$. Then f has a unique fixed point $u \in X$.*

If $\alpha = \frac{1}{2}$ in the inequality (1.2), we have the following corollary.

Corollary 3.4. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\psi, \varphi \in \Psi$ satisfying $\varphi(t) < \psi(t)$ for all $t > 0$ such that*

$$\begin{aligned} \psi(S(fx,fy,fz)) \leqslant \varphi\big(\max\big\{&S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),\\ &\tfrac{1}{2}[S(fx,fx,y) + S(fy,fy,z)]\big\}\big), \end{aligned} \tag{3.2}$$

for all $x,y,z \in X$. Then f has a unique fixed point $(say\ u)$ and f is S-continuous at u.

In the following, we provide an example in support of our result.

$$\text{Let } M_\alpha(x,y,z) = \max\big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),$$

$$\alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\big\}.$$

Example 3.1. Let $X = [0, \frac{7}{4}]$. We define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = |x-z| + |y-z|$ [19] and $f : X \to X$ by

$$f(x) = \begin{cases} \frac{7}{4} - x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x+1}{2} & \text{if } x \in (\frac{1}{2}, \frac{7}{4}]. \end{cases}$$

We define $\psi,\ \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{t}{2} \text{ for all } t \geqslant 0 \text{and} \qquad \varphi(t) = \begin{cases} \frac{t}{2} - \frac{t^2}{4} & \text{if } t \in [0, \frac{1}{2}] \\ \frac{t}{2} - \frac{1}{16} & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

We now show that f satisfies the inequality (1.2).
Case (i): Let $x,y,z\in[0,\frac{1}{2}]$. Here we consider

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi|\frac{7}{4}-x-(\frac{7}{4}-z)|+|\frac{7}{4}-y-(\frac{7}{4}-z)| = \psi\big(|z-x|+|z-y|\big)\\ &= \frac{1}{2}(|z-x|+|z-y|)\le\frac{1}{2}\le|2x-\frac{7}{4}|-\frac{1}{16}\\ &= \varphi(S(x,x,fx))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Case (ii): Let $x,y,z\in(\frac{1}{2},\frac{7}{4}]$.

Sub-case (i): $|x-z|+|y-z|\in[0,\frac{1}{2}]$. Therefore

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi|\frac{x+1}{2}-\frac{z+1}{2}|+|\frac{y+1}{2}-\frac{z+1}{2}| = \psi\big(\frac{1}{2}(|x-z|+|y-z|)\big)\\ &= \frac{1}{4}(|x-z|+|y-z|)\le\frac{1}{2}(|x-z|+|y-z|)-\frac{1}{4}(|x-z|+|y-z|)^2\\ &= \varphi(S(x,y,z))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Sub-case (ii): $|x-z|+|y-z|\geqslant\frac{1}{2}$. In this case

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi|\frac{x+1}{2}-\frac{z+1}{2}|+|\frac{y+1}{2}-\frac{z+1}{2}| = \psi\big(\frac{1}{2}(|x-z|+|y-z|)\big)\\ &= \frac{1}{4}(|x-z|+|y-z|)\le\frac{1}{2}(|x-z|+|y-z|)-\frac{1}{16}\\ &= \varphi(S(x,y,z))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Case (iii): Let $z\in[0,\frac{1}{2}]$ and $x,y\in(\frac{1}{2},\frac{7}{4}]$.

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi\big(S(\tfrac{x+1}{2},\tfrac{y+1}{2},\tfrac{7}{4}-z)\big)\\ &= \psi\big(|\tfrac{x+1}{2}-(\tfrac{7}{4}-z)|+|\tfrac{y+1}{2}-(\tfrac{7}{4}-z)|\big)\\ \tfrac{1}{2}\big(|\tfrac{x}{2}+z-\tfrac{5}{4}|+|\tfrac{y}{2}+z-\tfrac{5}{4}|\big) &\le |2z-\tfrac{7}{4}|-\tfrac{1}{16}=\tfrac{27}{16}-2z\\ &= \varphi(S(z,z,fz))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Case (iv): Let $x,y\in[0,\frac{1}{2}]$ and $z\in(\frac{1}{2},\frac{7}{4}]$.

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi\big(S(\tfrac{7}{4}-x,\tfrac{7}{4}-y,\tfrac{z+1}{2})\big)=\psi\big(|\tfrac{7}{4}-x-\tfrac{z+1}{2}|+|\tfrac{7}{4}-y-\tfrac{z+1}{2}|\big)\\ &= \psi\big(|\tfrac{5}{4}-x-\tfrac{z}{2}|+|\tfrac{5}{4}-y-\tfrac{z}{2}|\big)=\tfrac{1}{2}\big(|\tfrac{5}{4}-x-\tfrac{z}{2}|+|\tfrac{5}{4}-y-\tfrac{z}{2}|\big)\le 1\\ &\le |2x-\tfrac{7}{4}|-\tfrac{1}{16}=\tfrac{27}{16}-2x=\varphi(S(x,x,fx))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Case (v): Let $z,y\in[0,\frac{1}{2}]$ and $x\in(\frac{1}{2},\frac{7}{4}]$.

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \psi\big(S(\tfrac{x+1}{2},\tfrac{7}{4}-y,\tfrac{7}{4}-z)\big)\\ &= \psi\big(|\tfrac{x+1}{2}-(\tfrac{7}{4}-z)|+|\tfrac{7}{4}-y-(\tfrac{7}{4}-z)|\big)\\ &= \tfrac{1}{2}\big(|\tfrac{x}{2}+z-\tfrac{5}{4}|+|z-y|\big)\le\tfrac{5}{16}\le|2z-\tfrac{7}{4}|-\tfrac{1}{16}=\tfrac{27}{16}-2z\\ &= \varphi(S(z,z,fz))\le\varphi\big(M_\alpha(x,y,z)\big).\end{aligned}$$

Hence f, ψ, φ satisfy all the hypotheses of Theorem 2.1 and f has a unique fixed point $u=1$.

4. Summary

In our result, the concept generalized (α, ψ, φ)-contractive map was introduced with the proof of the existence and uniqueness of fixed points in complete S-metric spaces. The new idea, property (P), was also introduced and we proved that these maps satisfy property (P). The results presented in this paper extend several well known comparable results in metric and G-metric spaces. We derived corollaries and provided an example to show the validity of our result.

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