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ON $\mathcal{T}$-HYPERSURFACES OF A PARASASAKIAN MANIFOLD

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Abstract. The main purpose of this paper is to study transversal hypersurface (briefly, $\mathcal{T}$-hypersurface) P of a paraSasakian manifold M. We derive results allied with totally geodesic and totally umbilical $\mathcal{T}$-hypersurface of M. The necessary and sufficient condition for normality of $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure is established. Examples of $\mathcal{T}$-hypersurface are also illustrated.

Keywords: ParaSasakian manifold;Pseudo-metric; Hypersurface; $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure; Geodesic.

1. Introduction

The study of hypersurface in pseudo-Riemannian manifold is one of the potent aspects of the theory of pseudo-Riemannian geometry. It has ample significance in general theory of relativity, black holes and quantum mechanics ([1–3]). Therefore, several researchers showed interest in studying the geometry of hypersurface in different ambient spaces (c.f., [4–7]).

On the other hand, transversal hypersurface (briefly, $\mathcal{T}$-hypersurface) of contact Riemannian manifold is a hypersurface such that ξ, the *characteristic vector field* (or *Reeb vector field*) of manifold never tangent to the hyperplane. The concept of $\mathcal{T}$-hypersurface is introduced by K.Yano in 1972 [8]. After that transversal hypersurfaces were investigated by several authors in different ambient manifolds (c.f., [9–11]).

A systematic study of transversal hypersurfaces of paraSasakian manifold has not been undertaken yet, however paraSasakian manifolds have many analogies and differences with the Sasakian manifolds due to the fact that the geometry of hypersurfaces of pseudo-Riemannian manifold behave differently (for more details see, [12]). In the present paper, we consider an almost paracontact pseudo-metric manifold

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M. We obtain that every $\mathcal{T}$-hypersurface of M admits an almost paraHermitian structure as well as a $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure, and derive results allied with totally geodesic and totally umbilical transversal hypersurface. Finally, the condition of normality of $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure is obtained in a paraSasakian manifold. Examples of $\mathcal{T}$-hypersurface with $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure are also illustrated.

2. Preliminaries

Let a manifold M of dimension $(2n+1)$ be C^∞ and paracompact, and $\Gamma(TM)$ denotes the section of tangent bundle TM of manifold. Then M is said to be an *almost paracontact manifold* if it admits a tensor field φ of $(1,1)$-type, a 1-form η and a characteristic vector field ξ such that

$$\varphi^2 + \eta \otimes \xi = \mathcal{I} \text{ and } \eta(\xi) = 1, \tag{2.1}$$

where φ induces an *almost paracomplex structure* on the distribution $\mathcal{D} = ker(\eta)$, that is, the eigenspaces corresponding to eigenvalues ± 1 have equal dimension and $\mathcal{I}$ being the identity operator on tangent bundle of M. Equation (2.2) yields

$$\varphi\xi = 0, \ \text{rank}(\varphi) = 2n \text{ and } \eta \circ \varphi = 0. \tag{2.2}$$

A *pseudo-metric* $\tilde{\mathfrak{g}}$ is known as *compatible* with structure (φ, ξ, η) if for any vector fields Y and Z, we have

$$\tilde{\mathfrak{g}}(Y, Z) = \eta(Y)\eta(Z) - \tilde{\mathfrak{g}}(\varphi Y, \varphi Z) \tag{2.3}$$

where signature of $\tilde{\mathfrak{g}}$ is necessarily $(n+1, n)$ and $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ is known as an *almost paracontact pseudo-metric* $(2n+1)$*-manifold*. Here, $\tilde{\mathfrak{g}}(Y, \xi) = \eta(Y)$. In view of equations (2.1) and (2.2), we have

$$\tilde{\mathfrak{g}}(Y, \varphi Z) = -\tilde{\mathfrak{g}}(\varphi Y, Z). \tag{2.4}$$

Let us consider $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ be an almost paracontact pseudo-metric $(2n+1)$-manifold. Let $\left(Z, \nu\frac{d}{dx}\right)$ be any tangent vector on $M \times \mathbb{R}$, where $Z \in \Gamma(TM)$, x denotes standard coordinate on $\mathbb{R}$ and ν is a smooth function. Then the *almost paracomplex structure* J on product manifold $M \times \mathbb{R}$ is given by $J\left(Z, \nu\frac{d}{dx}\right) = \left(\varphi Z + \nu\xi, \eta(Z)\frac{d}{dx}\right)$ and M is called *normal* if and only if J is *integrable* i.e., M is *normal* if and only if

$$d\eta(Y, Z)\xi = \frac{1}{2}N_\varphi(Y, Z), \tag{2.5}$$

where N_φ being the *Nijenhuis torsion* of endomorphism φ which is given as follows:

$$N_\varphi(Y, Z) = (\nabla_{\varphi Y}\varphi)Z - (\nabla_{\varphi Z}\varphi)Y + \varphi((\nabla_Z\varphi)Y - (\nabla_Y\varphi)Z), \tag{2.6}$$

for any tangent vectors Y, Z on M. Let Φ denotes the fundamental 2-form on M then it is defined by $\Phi(Y, Z) = \tilde{\mathfrak{g}}(Y, \varphi Z)$. If $\Phi(Y, Z) = d\eta(Y, Z)$then $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ is said to be a *paracontact pseudo-metric manifold* (c.f., [13–18]).

Definition 2.1. Let $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ be a $(2n+1)$-dimensional almost paracontact pseudo-metric manifold, then it is called:

- *paracosympletic* if Φ and η are parallel, that is $\nabla\Phi = 0$ and $\nabla\eta = 0$.
- *paraSasakian* if and only if

$$(\nabla_Z\varphi)Y = \eta(Y)Z - \tilde{\mathfrak{g}}(Z,Y)\xi. \tag{2.7}$$

From equation (2.7), we can deduce that

$$\nabla_Z\xi = -\varphi Z, \tag{2.8}$$

$$\Phi(Z,Y) = (\nabla_Z\eta)Y. \tag{2.9}$$

Let $\mathcal{L}$ denotes *Lie-derivative* then for every paraSasakian manifold we have $\mathcal{L}_\xi\tilde{\mathfrak{g}} = \mathcal{L}_\xi\varphi = 0$ (see also, [15, 18–20]).

3. $\mathcal{T}$-hypersurfaces

Let $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ be an almost paracontact pseudo-metric manifold, P be a smooth connected $2n$-manifold and $\iota : P \to M$ be an *immersion*. Then $i(P)$ is known as an *immersed hypersurface* of M. Let ι induces a symmetric tensor field $\mathfrak{g}$ on the immersed hypersurface $\iota(P)$ which satisfies $\mathfrak{g}(Y,Z)|_p = \tilde{\mathfrak{g}}(\iota_*Y,\iota_*Z)|_{\iota(p)}$, $\forall Y,Z \in T_pP$, where ι_* is the *pushforward map* (or *differential map*) of ι defined by $\iota_* : T_pP \to T_{\iota(p)}M$ and $(\iota_*Z)(\beta) = Z(\beta\circ\iota)$ for any smooth function β in a vicinity of $\iota(p)$ of $\iota(P)$. Hereafter, we put p and P in place of $\iota(p)$ and $\iota(P)$. In view of *causal character* of vector fields of manifold, we have three types of hypersurface P, specifically, *pseudo-Riemannian*, *Riemannian* and *null* (or *lightlike*) and metric $\mathfrak{g}$ is a *non-degenerate* or a *degenerate* according as P is pseudo-Riemannian (Riemannian) hypersurface and lightlike hypersurface respectively [12, p. 42].

Let us suppose that $(P,\mathfrak{g})$ be a pseudo-Riemannian hypersurface of M. Then normal bundle of P is given by $TP^\perp = \{Y \in \Gamma(TM)|\mathfrak{g}(Y,Z) = 0,\ \forall Z \in \Gamma(TM)\}$. Here $\dim(\mathrm{T_pP^\perp}) = 1$, due to the fact that P is a hypersurface. The orthogonal complementary decomposition is given by $TM = TP^\perp \perp TP$, $TP^\perp \cap TP = \{0\}$.

The hypersurface P is said to be a $\mathcal{T}$-hypersurface of M if the characteristic vector field ξ is never tangent to the hyperplane. Here, ξ can be considered as affine normal to P. Now, ξ and $Y \in \Gamma(TP)$ are linearly independent, therefore $\varphi(Y)$ can be written as

$$\varphi Y = JY + \alpha(Y)\xi, \tag{3.1}$$

where J is a tensor field of type $(1,1)$ and α is a 1-form on P. Operating φ on (3.1) and using equation (2.2), we have $\varphi^2Y = \varphi JY$. Employing equations (2.1) and (3.1), this expression yields

$$Y - \eta(Y)\xi = J^2Y + (\alpha\circ J)(Y)\xi.$$

Considering normal and tangential parts from above relation, we obtain

$$J^2 = \mathcal{I},\ \alpha \circ J = -\eta. \tag{3.2}$$

From above equation, we can deduce that

$$\eta \circ J = -\alpha. \tag{3.3}$$

Therefore, we have a paracomplex structure J on $\mathcal{T}$-hypersurface P. From equation (3.1), $\forall\ Y, Z \in \Gamma(TP)$ we have

$$\mathfrak{g}(\varphi Y, \varphi Z) = \mathfrak{g}(JY, JZ) + \alpha(Y)\mathfrak{g}(\xi, JZ) + \alpha(Z)\mathfrak{g}(JY, \xi) + \alpha(Y)\alpha(Z)\mathfrak{g}(\xi, \xi).$$

Employing equations (2.1)-(2.3) and (3.3) in the above expression, we attain that

$$\mathfrak{g}(JY, JZ) + \mathfrak{g}(Y, Z) = \eta(Y)\eta(Z) + \alpha(Y)\alpha(Z). \tag{3.4}$$

Let us define

$$H(Y, Z) = \mathfrak{g}(\varphi Y, \varphi Z). \tag{3.5}$$

We claim that H is paraHermitian metric. From equation (3.5), we find

$$H(JY, JZ) = \mathfrak{g}(\varphi JY, \varphi JZ).$$

In light of (2.3), above expression can be written as

$$H(JY, JZ) + \mathfrak{g}(JY, JZ) = \eta(JY)\eta(JZ)$$

using equations (3.3) and (3.4) in the above relation, we have

$$H(JY, JZ) = \mathfrak{g}(Y, Z) - \eta(Y)\eta(Z) = -H(Y, Z).$$

This shows that H is a paraHermitian metric. Thus, we are in position to give the following result:

Proposition 3.1. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold. Then P admits an almost paraHermitian structure.*

Let P be a orientable $\mathcal{T}$-hypersurface of M, D denotes the induced Levi-Civita connection on P and N be a unit normal vector field to the hypersurface P. Then the formulas of *Gauss* and *Weingarten* formulas are given respectively by

$$\nabla_Y N = -A_N Y, \tag{3.6}$$

$$\nabla_Y Z = D_Y Z + h(Y, Z)N, \tag{3.7}$$

where

$$h(Y, Z) = \mathfrak{g}(A_N Y, Z) \tag{3.8}$$

is a *second fundamental form* and A_N is the *shape operator* allied with the normal section N. The hypersurface P is *totally geodesic* in M if second fundamental form vanishes identically. A point p of P is called *umbilical* if $h(Y,Z)|_p = \rho\mathfrak{g}(Y,Z)|_p$, $\forall\ Y, Z \in T_pM$, where $\rho \in \mathbb{R}$ and depends on p. The hypersurface P is said to be *totally umbilical* if every point of P is umbilical, that is, $h = \zeta\mathfrak{g}$, where ζ is a smooth function (see, [1, 15, 21]).

Given $Y \in \Gamma(TP)$, the vector field φY does not belong to $\Gamma(TP)$. Therefore, φY can be decomposed as follows

$$\varphi Y = \mathfrak{f}Y + \mu(Y)N, \tag{3.9}$$

where $\mathfrak{f}$ is a $(1,1)$-type tensor field and μ is a non-zero 1-form.
Next, we define

$$\varphi N = -U,\ \xi = V + \delta N,\ \eta(Y) = \upsilon(Y),\ \eta(N) = \delta, \tag{3.10}$$

where $U, V \in \Gamma(TP)$, υ is a 1-form and δ is a smooth function on P. Clearly $\delta \neq 0$ because if $\delta = 0$ then $\mathfrak{g}(\xi, N) = 0$, this implies that ξ is perpendicular to N so we have $\xi \in \Gamma(TP)$, which contradicts the fact that P is a $\mathcal{T}$-hypersurface. Substituting U in place of Y in (3.9), we get

$$\varphi U = \mathfrak{f}U + \mu(U)N,$$

in the light of (3.10), we obtain

$$-\varphi^2 N = \mathfrak{f}U + \mu(U)N.$$

Now employing (2.2) in above expression, we have

$$-N + \eta(N)\xi = \mathfrak{f}U + \mu(U)N,$$

applying (3.10) in above relation, we arrive at

$$-N + \delta V + \delta^2 N = \mathfrak{f}U + \mu(U)N,$$

considering normal and tangential parts of above expression, we obtain

$$\mathfrak{f}U = \delta V,\ \mu(U) = \delta^2 - 1. \tag{3.11}$$

On the other hand, substituting $X = V$ in (3.9), we get

$$\varphi V = \mathfrak{f}V + \mu(V)N.$$

Using (3.10), above equation takes the form

$$\varphi(\xi - \delta N) = \mathfrak{f}V + \mu(V)N,$$

comparing normal and tangential parts from the above equality, we find

$$\mathfrak{f}V = \delta U,\ \mu(V) = 0. \tag{3.12}$$

By the consequences of equations (3.9) and (3.10), we get $\mu(Y) = \mathfrak{g}(U, Y)$ and

$$\mu(\mathfrak{f}Y) = \mathfrak{g}(\mathfrak{f}Y, U) = \mathfrak{g}(\varphi(Y) - \mu(Y)N, -\varphi(N)),$$

employing (2.3) in above relation, we achieve that

$$\mu \circ \mathfrak{f} = -\delta\upsilon. \tag{3.13}$$

Similarly, we can find

$$\upsilon \circ f = -\delta\mu, \tag{3.14}$$
$$\upsilon(U) = 0, \ \upsilon(V) = 1 - \delta^2. \tag{3.15}$$

Replacing Y by fY in (3.9), we have

$$\varphi(\mathfrak{f}Y) = \mathfrak{f}(\mathfrak{f}Y) + \mu(\mathfrak{f}Y)N,$$

again using (3.9) in above equation, we obtain

$$\varphi^2(Y) - \mu(Y)\varphi N = \mathfrak{f}^2(Y) - \delta\upsilon(Y)N.$$

Employing (2.2) and (3.10) in the above relation, we conclude that

$$Y - \eta(Y)\xi + \mu(Y)U = \mathfrak{f}^2(Y) - \delta\upsilon(Y)N,$$

reusing (3.10) in above expression, we have

$$\mathfrak{f}^2 = I - \upsilon \otimes V + \mu \otimes U. \tag{3.16}$$

With the help of (2.3) and (3.9), we find that g satisfying

$$\mathfrak{g}(\mathfrak{f}Y, \mathfrak{f}Z) + \mathfrak{g}(Y, Z) = \upsilon(Y)\upsilon(Z) - \mu(Y)\mu(Z), \tag{3.17}$$

and

$$\mathfrak{g}(Y, \mathfrak{f}Z) + \mathfrak{g}(\mathfrak{f}Y, Z) = 0, \tag{3.18}$$

$\forall Y, Z \in \Gamma(TP)$. The above computations lead to the following result:

Proposition 3.2. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold M. Then P admits a $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure.*

Example 3.1. *Let $M = (\mathbb{R} - \{0, 1\}) \times \mathbb{R}^4_2 \subset \mathbb{R}^5_2$ with standard Cartesian coordinates $(x_1, x_2, x_3, x_4, x_5)$. Define φ, ξ, η and $\tilde{\mathfrak{g}}$ on M by*

$$\varphi\partial_{x_1} = \partial_{x_2},\ \varphi\partial_{x_2} = \partial_{x_1},\ \varphi\partial_{x_3} = \partial_{x_4},\ \varphi\partial_{x_4} = \partial_{x_3},\ \varphi\partial_{x_5} = 0,$$
$$\xi = \partial_{x_5},\ \eta = dx_5 \text{ and } \tilde{\mathfrak{g}} = x_1^2(dx_2^2 - dx_1^2) + x_1(dx_4^2 - dx_3^2) + \eta \otimes \eta,$$

where $\partial_{x_j} = \frac{\partial}{\partial x_j}(j \in \{1,2,3,4,5\})$*. Then from simple computations, we find that* $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ *is an almost paracontact pseudo-metric 5-manifold. Consider* $(P,\mathfrak{g})$ *be a pseudo-Riemannian hypersurface of* M *which is given by*

$$\mathfrak{F}(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4,x_1).$$

Then the local basis of tangent hyperplane of P *is given by*

$$X_1 = \partial_{x_1} + \partial_{x_5},\ X_2 = \partial_{x_2},\ X_3 = \partial_{x_3},\ X_4 = \partial_{x_4}$$

and normal vector field N *of the hypersurface is given by* $N = \partial_{x_1} + x_1^2\partial_{x_5}$*. Here, it is clear that* ξ_p*,* $p \in P$ *is not tangent to the hypersurface. Therefore,* P *is a* $\mathcal{T}$*-hypersurface of* M*. Here, we find*

$$\eta(N) = x_1^2 = \delta,\ V = -x_1^2\partial_{x_1} + (1-x_1^4)\partial_{x_5}\,\text{and}\,U = -\partial_{x_2}.$$

Further, any tangent vector field of the hypersurface P *can be expressed as* $X = \sum_{i=1}^{4} a_i X_i$*, where* a_1,a_2,a_3 *and* a_4 *are smooth functions. Operating* φ *on both the sides, we have*

$$\begin{aligned}\varphi X &= a_2(1+x_1^2)\partial_{x_1} + a_1\partial_{x_2} + a_4\partial_{x_3} + a_3\partial_{x_4} + a_2x_1^4\partial_{x_5} - x_1^2 a_2 N\\ &= \mathfrak{f}X + \mu(X)N,\end{aligned}$$

where $\mu(X) = -x_1^2 a_2$ *and* $\mathfrak{f}$ *is given by*

$$\mathfrak{f} = \begin{pmatrix} 0 & 1+x_1^2 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & x_1^4 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, P *is a* $\mathcal{T}$*-hypersurface of* M *which admits a* $(\mathfrak{f},\mathfrak{g},\mu,\upsilon,\delta)$*-structure.*

Lemma 3.1. *If* P *be a* $\mathcal{T}$*-hypersurface of an almost paracontact pseudo-metric manifold* M*. Then, we have*

$$\delta\alpha = \mu, \tag{3.19}$$

$$J = \mathfrak{f} - \tfrac{1}{\delta}\mu \otimes V, \tag{3.20}$$

$$H(\cdot, J\cdot) = -\mathfrak{g}(\cdot, \mathfrak{f}\cdot), \tag{3.21}$$

$$JU = \tfrac{1}{\delta}V, \tag{3.22}$$

$$\mu \circ J = \mu \circ \mathfrak{f} = -\delta\upsilon, \tag{3.23}$$

$$JV = \mathfrak{f}V = \delta U. \tag{3.24}$$

Proof. Using (3.10) in equation (3.1), we obtain $\varphi Y = JY + \alpha(Y)V + \delta\alpha(Y)N$. Now with the help of (3.9), we achieve that $fY + \mu(Y)N = JY + \alpha(Y)V + \delta\alpha(Y)N$. Comparing tangential and normal parts from above relation, we find (3.19) and

$$\mathfrak{f} = J + \alpha \otimes V.$$

In view of (3.19), the above expression yields (3.20). By the virtue of equations (3.3) and (3.5), we have

$$H(Y,JZ)+\mathfrak{g}(Y,JZ)+\alpha(Z)\eta(Y)=0. \tag{3.25}$$

Using equations (3.19) and (3.20) in (3.25), we get (3.21). Now from (3.20), we conclude

$$JU=\mathfrak{f}U-\frac{1}{\delta}\mu(U)V.$$

Employing (3.11) in above equality, we achieve (3.22). Now, we have $\mu(JY)=\mu(\mathfrak{f}Y)-\alpha(Y)\mu(V)$, by the consequences of equations (3.12) and (3.13), we derive (3.23). Further, (3.24) follows from equations (3.12) and (3.20). These completes the proof. □

Lemma 3.2. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold. Then, we have*

$$(\nabla_Y\varphi)Z=(D_Y\mathfrak{f})Z-\mu(Z)A_NY+h(Y,Z)U+\{(D_Y\mu)Z+h(Y,\mathfrak{f}Z)\}N, \tag{3.26}$$

$$\nabla_Y\xi=D_YV-\delta A_NY+\{h(Y,V)+Y.\delta\}N, \tag{3.27}$$

$$(\nabla_Y\varphi)N=-D_YU+\mathfrak{f}A_NY+(u(A_NY)-h(U,Y))N, \tag{3.28}$$

$$(\nabla_Y\eta)Z=(D_Y\upsilon)Z-\delta h(Y,Z), \tag{3.29}$$

for any $Y,Z\in\Gamma(TP)$.

Proof. We have $(\nabla_Y\varphi)Z=\nabla_Y\varphi Z-\varphi\nabla_YZ$, by the consequence of (3.7) this expression reduces to

$$(\nabla_Y\varphi)Z=D_Y\varphi Z+h(Y,\varphi Z)N-\varphi(D_YZ+h(Y,Z)N).$$

Employing equations (3.9) and (3.10) in the above relation, we find

$$\begin{aligned}(\nabla_Y\varphi)Z= \quad &(D_Y\mathfrak{f})Z+\mu(Z)D_YN+(Y.\mu(Z))N\\ &+h(Y,\mathfrak{f}Z)N-\mu(D_YZ)N+h(Y,Z)U.\end{aligned}$$

In view of (3.6), the above equation leads to (3.26). From (3.10), we get

$$\nabla_Y\xi=\nabla_Y(V+\delta N)=\nabla_YV+Y.\delta N+\delta\nabla_YN.$$

Now employing (3.6) and (3.7), we find (3.27). We have $(\nabla_Y\varphi)N=\nabla_Y\varphi N-\varphi(\nabla_YN)$, by the virtue of (3.7) and (3.9), this expression yields (3.28). Since $(\nabla_Y\eta)Z=\mathfrak{g}(\nabla_Y\xi,Z)$, therefore using equations (3.6), (3.7) and (3.10) we obtain (3.29). This completes the proof of lemma. □

As a direct consequence of above lemma, we obtain the following result:

Proposition 3.3. *Let P be a $\mathcal{T}$-hypersurface of a paracosympletic manifold, then we have*

$$(D_Y\mathfrak{f})Z = \mu(Z)A_NY - h(Y,Z)U, \tag{3.30}$$
$$(D_Y\mu)Z = -h(Y,\mathfrak{f}Z), \tag{3.31}$$
$$D_YV = \delta A_NY, \tag{3.32}$$
$$(D_Y\upsilon)Z = \delta h(Y,Z), \tag{3.33}$$
$$Y.\delta = h(Y,V), \tag{3.34}$$
$$D_YU = \mathfrak{f}A_NY. \tag{3.35}$$

Remark 3.1. *Let the vector field U be parallel on $\mathcal{T}$-hypersurface P of a paracosympletic manifold M, then from (3.35) we receive that $\mathfrak{f}A_NY = 0$, which shows that 0 is an eigen value of $\mathfrak{f}$.*

Remark 3.2. *(a) If $\mathfrak{f}$ is parallel that is, $(D_Y\mathfrak{f})Z = 0$, then by equation (3.30) we obtain that $h(Y,Z)U = \mu(A_NY)\mu(Z)$.*

(b) If υ is parallel then from equation (3.33), we have $h(X,Y) = 0$ that is, P is a totally geodesic, since $\delta \neq 0$.

4. $\mathcal{T}$-hypersurface of a paraSasakian manifold

Here, we consider a $\mathcal{T}$-hypersurface P of a paraSasakian manifold M.

Theorem 4.1. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold, then we have*

$$(D_Y\mathfrak{f})Z = \mu(Z)A_NY + \upsilon(Z)Y - h(Z,Y)U - \mathfrak{g}(Z,Y)V, \tag{4.1}$$
$$(D_Y\mu)Z = -\delta\mathfrak{g}(Y,Z) - h(Y,\mathfrak{f}Z), \tag{4.2}$$
$$D_YV - \delta A_NY + \mathfrak{f}Y = 0, \tag{4.3}$$
$$h(Y,V) + \mu(Y) + Y.\delta = 0, \tag{4.4}$$
$$D_YU + \delta Y - \mathfrak{f}A_NY = 0, \tag{4.5}$$
$$(D_Y\eta)Z - \delta h(Z,Y) - g(\mathfrak{f}Z,Y) = 0, \tag{4.6}$$

for any $Z, Y \in \Gamma(TP)$.

Proof. Using equation (2.7) in (3.26), we get

$$\begin{aligned} -\mathfrak{g}(Z,Y)\xi + \eta(Z)Y = \quad & (D_Y\mathfrak{f})Z - \mu(Z)A_NY + h(Z,Y)U \\ & +\{(D_Y\mu)Z + h(Y,\mathfrak{f}Z)\}N. \end{aligned}$$

In view of (3.10) above equation reduces to the following form

$$\begin{aligned} -\mathfrak{g}(Z,Y)V - \delta\mathfrak{g}(Z,Y)N + \upsilon(Z)Y = \quad & (D_Y\mathfrak{f})Z - \mu(Z)A_NY + h(Z,Y)U \\ & +\{(D_Y\mu)Z + h(\mathfrak{f}Z,Y)\}N. \end{aligned}$$

Considering normal and tangential parts from above expression, we receive (4.1) and (4.2). By the virtue of equations (2.8), (3.9) and (3.27), we obtain (4.3) and (4.4). In view of equations (2.7) and (3.28), we have (4.5). equation (4.6) follows from (2.9) and (3.29). Hence this completes the proof of the theorem. □

Using $h(Z,Y) = \zeta\mathfrak{g}(Z,Y)$ in equation (4.4), we obtain following result:

Corollary 4.1. *If P be a totally umbilical $\mathcal{T}$-hypersurface of a paraSasakian manifold, then necessary and sufficient condition for P to be a totally geodesic is that*

$$\mu(Z) + Z.\delta = 0. \tag{4.7}$$

equation (4.6) leads to the following remark:

Remark 4.1. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold M. Then P is a totally geodesic $\iff$ $(D_Y\eta)Z = \mathfrak{g}(\mathfrak{f}Z,Y)$, $\forall\ Y,Z \in \Gamma(TP)$.*

Let us consider the fundamental 2-form $\mathfrak{F}$ on P, given by $\mathfrak{F}(Y,Z) = H(Y,JZ)$. Using the equation (3.21), this reduces to $\mathfrak{F}(Y,Z) = \mathfrak{g}(Y,\mathfrak{f}Z)$. From equation (4.1), we have

$$(D_Y\mathfrak{F})(Z,W) = \mu(W)h(Z,Y) + \upsilon(W)\mathfrak{g}(Y,Z) - \mu(Z)h(Y,W) - \upsilon(Z)\mathfrak{g}(Y,W).$$

In view of the above equation, we find

$$(D_W\mathfrak{F})(Y,Z) + (D_Y\mathfrak{F})(Z,W) + (D_Z\mathfrak{F})(W,Y) = 0.$$

This implies that $\mathfrak{F}$ is closed. Now differentiating (3.20) covariantly along X and using equations (4.1)-(4.4), we get

$$(D_YJ)Z = \upsilon(Z)Y - h(Z,Y)U + \frac{1}{\delta}(h(JZ,Y) + \mu(Z)JY). \tag{4.8}$$

In view of the above equation, we find that the Nijenhuis tensor N_J formed with J satisfies $N_J(Y,Z) = 0$. These lead to the following proposition:

Proposition 4.1. *Every $\mathcal{T}$-hypersurface of a paraSasakian manifold admits paraKäehlerian structure.*

Let the tensor field $\mathfrak{f}$ be parallel then from (4.1), we have

$$h(Z,Y)U = \mu(Y)A_NZ + \upsilon(Y)Z - \mathfrak{g}(Z,Y)V. \tag{4.9}$$

Operating μ on (4.9) and using (3.11), we find

$$(\delta^2 - 1)h(Z,Y) = \mu(A_NZ)\mu(Y) + \upsilon(Y)\mu(Z). \tag{4.10}$$

Replacing Z by V and employing (3.11), the above equation reduces to

$$h(Y,V) + \mu(Y) = 0. \tag{4.11}$$

In view of equations (4.4) and (4.11), we obtain that $Y.\delta = 0$. This leads to the following proposition:

Proposition 4.2. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold M and the tensor field $\mathfrak{f}$ be parallel. Then δ is a non-zero constant.*

Let $S_{\mathfrak{f}}$ denotes the torsion tensor of $\mathfrak{f}$ defined by

$$S_{\mathfrak{f}}(Z,Y) = N_{\mathfrak{f}}(Z,Y) + d\mu(Z,Y)U + d\upsilon(Z,Y)V, \tag{4.12}$$

where $N_{\mathfrak{f}}$ is the Nijenhuis torsion of $\mathfrak{f}$, and

$$\begin{aligned} d\mu(Z,Y) &= (D_Z\mu)Y - (D_Y\mu)Z, \\ d\upsilon(Z,Y) &= (D_Z\upsilon)Y - (D_Y\upsilon)Z. \end{aligned}$$

If $S_{\mathfrak{f}}$ vanishes identically, then the structure $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$ is said to be *normal*. Let P be a $\mathcal{T}$-hypersurface of paraSasakian manifold and the structure $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$ be normal. Then, we find

$$\eta\left(N_{\mathfrak{f}}(Z,Y)\right) + \left(1-\delta^2\right) d\eta(Z,Y) = 0, \ \forall\ Z,Y \in \Gamma(TP). \tag{4.13}$$

Theorem 4.2. *If P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold. Then the structure $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$ is normal if and only if the shape operator A_N of P satisfies $A_N\mathfrak{f} = \mathfrak{f}A_N$.*

Proof. Employing equations (3.18) and (4.1), we have

$$\begin{aligned} N_{\mathfrak{f}}(Z,Y) = \ & \mu(Y)(A_N\mathfrak{f}Z - \mathfrak{f}A_NZ) - \mu(Z)(A_N\mathfrak{f}Y - \mathfrak{f}A_NY) \\ & +(h(Z,\mathfrak{f}Y) - h(\mathfrak{f}Z,Y))U - 2\mathfrak{g}(Z,\mathfrak{f}Y)V. \end{aligned} \tag{4.14}$$

In light of equations (4.2) and (4.3), we get

$$d\mu(Z,Y) = h(\mathfrak{f}Z,Y) - h(Z,\mathfrak{f}Y), \tag{4.15}$$

$$d\upsilon(Z,Y) = 2\mathfrak{g}(Z,\mathfrak{f}Y). \tag{4.16}$$

Using equations (4.14)-(4.16) in (4.12), we obtain

$$S_{\mathfrak{f}}(Z,Y) = \mu(Y)(A_N\mathfrak{f}Z - \mathfrak{f}A_NZ) - \mu(Z)(A_N\mathfrak{f}Y - \mathfrak{f}A_NY). \tag{4.17}$$

This completes the proof. $\square$

Example 4.1. *Let $M = \mathbb{R}^3_1$ with coordinates (x,y,z). Define φ, ξ and η on M by*

$$\varphi\partial_x = \partial_y - 2x\partial_z,\ \varphi\partial_y = \partial_x,\ \varphi\partial_z = 0, \xi = \partial_z, \text{ and } \eta = 2xdy + dz,$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. Then (φ, ξ, η) is an almost paracontact structure on M. By simple computations, it can be seen that the structure is normal. Now, we consider $\tilde{\mathfrak{g}} = -dx^2 + dy^2 + \eta \otimes \eta$. Using φ and the metric $\tilde{\mathfrak{g}}$, we find

$\tilde{\mathfrak{g}}(\varphi Y, \varphi Z) + \tilde{\mathfrak{g}}(Y, Z) = \eta(Y)\eta(Z)$ *and* $\eta(Y) = \tilde{\mathfrak{g}}(Y, \xi)$*, and thus* $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ *is a normal almost paracontact pseudo-metric 3-manifold. With respect to* $\tilde{\mathfrak{g}}$*, we have*

$$\begin{gathered}\nabla_{\partial_x}\partial_x = 0, \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 2x\partial_y + (1-4x^2)\partial_z,\\ \nabla_{\partial_x}\partial_z = \nabla_{\partial_z}\partial_x = \partial_y - 2x\partial_z, \nabla_{\partial_y}\partial_y = 4x\partial_x,\\ \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = \partial_x, \nabla_{\partial_z}\partial_z = 0.\end{gathered}$$

Using equation (2.7) and the above expressions, we find that M *is a paraSasakian manifold. Let* $(P, \mathfrak{g})$ *be a pseudo-Riemannian hypersurface of* M *which is defined by*

$$\mathfrak{F}(r, \vartheta) = (r, \sinh\vartheta, \cosh\vartheta),$$

where $r, \vartheta \in \mathbb{R}$*. Then the local basis of tangent bundle of* P *is given by the vector fields*

$$Z_1 = \partial_x, \text{ and } Z_2 = \cosh\vartheta\,\partial_y + \sinh\vartheta\,\partial_z.$$

The normal vector field N *of the hypersurface is expressed as*

$$N = \partial_y - \frac{(4r^2+1) + 2r\tanh\vartheta}{2r + \tanh\vartheta}\partial_z.$$

Here, it is clear that ξ *is never tangent to the hypersurface. Therefore,* P *is a* $\mathcal{T}$*-hypersurface of* M*. Now, we obtain that*

$$\begin{gathered}\eta(N) = -\tfrac{1}{2r+\tanh\vartheta} = \delta,\\ V = \tfrac{1}{2r+\tanh\vartheta}\partial_y + \left(\tfrac{2r\tanh\vartheta - \operatorname{sech}^2\vartheta}{(2r+\tanh\vartheta)^2}\right)\partial_z \text{ and } U = -\partial_x.\end{gathered}$$

Further, any tangent vector field of the hypersurface P *can be expressed as* $Z = b_1Z_1 + b_2Z_2$*, where* b_1 *and* b_2 *are smooth functions. Operating* φ *on both the sides, we have*

$$\varphi Z = \mathfrak{f}Z + \mu(Z)N,$$

where $\mu(Z) = b_1$ *and* $\mathfrak{f}$ *is given by*

$$\mathfrak{f} = \begin{pmatrix} 0 & \cosh\vartheta & 0\\ 0 & 0 & 0\\ \frac{1}{2r+\tanh\vartheta} & 0 & 0\end{pmatrix}.$$

Hence, P *is a* $\mathcal{T}$*-hypersurface of a paraSasakian manifold* M *and admits* $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$*-structure.*

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