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COMMON FIXED POINT THEOREMS FOR HYBRID PAIRS OF MAPPINGS USING IMPLICIT RELATIONS IN FUZZY METRIC SPACES

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Abstract. The purpose of this paper is to introduce the notion of a new type of common limit range property for a hybrid pair of single and multivalued mappings in fuzzy metric space and establish some common fixed point theorems satisfying the same property using implicit relations. Some related results are also derived besides furnishing illustrative examples. Further, we also present an integral type common fixed point theorem in fuzzy metric space. Our results improve and extend some previously known results.
Key words: Fuzzy metric spaces (FMS), weakly compatible mapping, new type of common limit in the range property.

1. Introduction

An inspiration of a fuzzy set is born as natural expansion of the notion of set, which plays a vital role in topology and analysis. The idea of a fuzzy set was introduced by Zadeh [36] in his paper. In the last two decades, there has been a remarkable progress and development in fuzzy mathematics and this idea has been used in mathematics and its applications in applied sciences such as mathematical programming, modelling theory, engineering sciences, image processing, control theory, communication, neural network theory, stability theory, medical sciences (medical genetics, nervous system), etc.

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In particular several authors have expansively developed the theory of fuzzy metric spaces in special directions. In 1975, Kramosil and Michalek [18] gave the concept of FMS, which opened an avenue for further growth of analysis in such spaces. Further, George and Veeramani [9] modified the notion of FMS introduced by [18] with a view to obtain a Hausdorff topology on it. On the other hand, fixed point theory is one of the most renowned theories with its application in several branches of science. A number of significant fixed point theorems have been obtained by different authors in fuzzy metric spaces using a variety of various mappings, for instance, [1, 2, 6, 7, 9, 11, 15, 16, 19, 20, 25, 31, 32, 33, 34, 35] and references therein.

In 2002, Aamri and Moutawakil [1] defined the concept of property (E.A.) for a pair of self mappings which contains the class of non-compatible mappings. Consequently, the number of results have proved for contractive conditions satisfying property (E.A) in fuzzy metric spaces (see [6, 11, 19]). In 2011, Sintunavarat and Kumam [31] introduced the idea of the common limit range property for single-valued mappings as a generalization of (E.A) property and established some common fixed point theorems. Recently, Imdad et al. [12] gave the notion of the common limit range property for two pairs of self-mappings. A number of noteworthy and motivating fixed point theorems using CLR property proved by various researchers in the framework of metric and fuzzy metric spaces, for instance, [2, 5, 8, 16, 33, 35].

On the other hand, the study of fixed points for multivalued contraction mappings with the Hausdorff metric was initiated by Nadler [23] and Markin [21]. Further, Singh et al. [30] and Khan et al. [17] studied the contraction types involving single-valued and multivalued mappings. Recently, Imdad et al. [13] have proved fixed point theorems for a hybrid pair of mappings in symmetric spaces. Imdad et al. [14] defined the concept of joint common limit range property for two pairs of hybrid mappings. Some results related to multivalued mappings are [2, 10, 13, 17, 32]. Quite recently, Popa[25] introduced the notion of a new type of common limit range property for hybrid pair of single and multivalued mapping and proved fixed point theorem for two hybrid pairs of mappings satisfying implicit relations.

The aim of the present paper is to define a new type of common limit in the range property for a hybrid pair of single and multivalued mappings in FMS and prove some common fixed point theorems for the same property in FMS using implicit relations. Some illustrative examples are furnished which demonstrate the utility of our results. Later, we also prove an integral type fixed point theorem in FMS. Our results improve and extend many previously known results [3, 19, 29].

2. Preliminaries

The following definitions and results will be needed in the sequel.

Definition 2.1. [28] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if * satisfies the following conditions:
(i) * is commutative and associative,
(ii) * is continuous,

(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $\forall a,b,c,d \in [0,1]$.

Definition 2.2. [9] A 3-tuple $(X, M, *)$ is called a FMS if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions $\forall x, y \in X$ and $t > 0$:
(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$, $\forall t > 0$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,y,t)*M(y,z,s) \leq M(x,z,t+s)$,
(5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Example 2.1. Let (X,d) be a metric space and define $a * b = min\{a,b\}, \forall a,b \in [0,1]$,

$$M(x,y,t) = \tfrac{t}{t+d(x,y)}, \forall t > 0,$$

Then $(X,M,*)$ is a FMS. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.3. [15] f and g are said to be weakly compatible if they commute at their coincidence points; i.e, $fx = gx$, for some $x \in X$ implies that $fgx = gfx$.

Definition 2.4. [1] Let f and g be two selfmappings of a FMS $(X,M,*)$. We say that f and g satisfy the property (E.A) if there exists a sequence x_n such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X.$$

Definition 2.5. [31] A pair (f,g) of self-mappings of a FMS $(X,M,*)$ is said to satisfy the common limit in the range property with respect to mapping g (briefly, (CLR_g) property), if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gz, \text{ for some } z \in X.$$

Definition 2.6. [2] Let $CB(X)$ be the set of all nonempty closed bounded subsets of FMS $(X,M,*)$. Then for every $A,B,C \in CB(X)$ and $t > 0$;

$$M(A,B,t) = min\{min_{a\in A} M(a,B,t), min_{b\in B} M(A,b,t)\}$$

where $M(C,y,t) = max\{M(z,y,t) : z \in C\}$. Obviously, $M(A,B,t) \leq M(a,B,t)$, whenever $a \in A$ and $M(A,B,t) = 1$ if and only if $A = B$.

Definition 2.7. [32] A point in X is a coincidence point (fixed point) of f and g if $fx = gx$ $(gx = fx = x)$.

Definition 2.8. [32] A point x in X is a coincidence point of $f : X \to X$ and $F : X \to CB(X)$, if $fx \in Fx$.

Definition 2.9. A point $x \in X$ is a coincidence point (fixed point) of a hybrid pair (f, G) of single valued mapping $f : X \to X$ and multivalued mapping $G : X \to CB(X)$ if $fx \in Tx$ ($x = fx \in Gx$). We denote the set of all coincidence points of f and F by $C(G, f)$

Definition 2.10. [32] Let $F : X \to CB(X)$. The map $f : X \to X$ is said to be F-weakly commuting at $x \in X$, if $ffx \in Ffx$.

Definition 2.11. [2] Let $(X, M, *)$ be a FMS. Two mappings $f : X \to X$ and $F : X \to CL(X)$, where $CL(X)$ is the set of all nonempty closed subsets, are said to be satisfy the (CLR_g) property, if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = u \in A = \lim_{n\to\infty} Fx_n,$$

with $u = fv$, for some $u, v \in X$.

Popa [25] introduced a new type of common limit range property for a hybrid pair of single and multivalued mappings as follows:

Definition 2.12. Let (X, d) be a metric Space, $F : X \to CL(X)$ and $f, g : X \to X$. Then (F, f) satisfy a common limit in the range property with respect to g, (denoted $CLR_{(F,f)g}$-property), if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = z, \lim_{n\to\infty} Fx_n = D, D \in CL(X)$$

and $z \in D \cap f(X) \cap g(X)$.

Lemma 2.1. [22] *Let $(X, M, *)$ be a FMS such that $\lim_{t\to\infty} M(x, y, t) = 1$, $\forall x, y \in X$. If for two point x, y of X and for a constant $k \in (0, 1)$,*

$$M(x, y, kt) \geq M(x, y, t),$$

then $x = y$.

We consider the following relations in our results:
(δ_1) Let Φ_6 be the set of all real valued continuous functions $\psi : (0, 1]^6 \to R$ such that

$(\delta_{1.1})$ ψ non increasing in $2^{nd}, 3^{rd}, 4^{th}, 5^{th}$ and 6^{th} coordinate variables and

$(\delta_{1.2})$ if

$$\psi(u, 1, v, 1, v, 1) \geq 1,$$
$$\psi(u, v, 1, 1, v, v) \geq 1,$$
$$\psi(u, 1, 1, v, 1, v) \geq 1,$$

implies that $u \geq v$, for all $u, v \in (0, 1]$.

Example 2.2. We define real valued continuous function $\psi_1 : (0,1]^6 \to R$ such that

$$\psi_1(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 + 1 - min\{t_2, t_3, t_4, t_5, t_6\}max\{t_2, t_3, t_4, t_5, t_6\},$$

for all $t_1, t_2, t_3, t_4, t_5, t_6 \in (0,1]$.

It is easy to see that $\psi_1 \in \Phi_6$, since
for $t_1 = u$, $t_3 = t_5 = v$, $t_2 = t_4 = t_6 = 1$ if

$$\psi_1(u, 1, v, 1, v, 1) \geq 1 \text{ implies that } u \geq v,$$

for $t_1 = u$, $t_2 = t_5 = t_6 = v$, $t_3 = t_4 = 1$ if

$$\psi_1(u, v, 1, 1, v, v) \geq 1 \text{ implies that } u \geq v,$$

for $t_1 = u$, $t_4 = t_6 = v$, $t_2 = t_3 = t_5 = 1$ if

$$\psi_1(u, 1, 1, v, 1, v) \geq 1 \text{ implies that } u \geq v.$$

Example 2.3. A real valued continuous function $\psi_2 : (0,1]^6 \to R$ such that

$$\psi_2(t_1, t_2, t_3, t_4, t_5, t_6) = \frac{t_1}{min(t_2, t_3, t_4, t_5, t_6)},$$

for all $t_i \in (0,1]$, where i=1,2,3,4,5,6.

Now, we consider a positive real number p such that

$$\psi^p(t_1, t_2, t_3, t_4, t_5, t_6) = [\psi(t_1, t_2, t_3, t_4, t_5, t_6)]^p \text{ for every } t_i \in (0,1]$$

where $i = 1, 2, 3, 4, 5, 6$ and $\psi \in \Phi_6$.

3. Main Results

Firstly, motivated by above Definition 2.12, we define a new type of common limit in the range property for a hybrid pair of single and multivalued mappings in fuzzy metric space as follows:

Definition 3.1. Let $(X, M, *)$ be a FMS, $F : X \to CL(X)$ and $f, g : X \to X$. Then (F, f) satisfy a common limit in the range property with respect to g, (denoted $CLR_{(F,f)g}$-property), if there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = z, \lim_{n\to\infty} Fx_n = D, D \in CL(X)$$

and $z \in D \cap f(X) \cap g(X)$.

Theorem 3.1. *Let f and g be mappings from a FMS $(X, M, *)$ into itself and $F, G : X \to CL(X)$ satisfy the following condition:*
(a) *if there exists a constant $k \in (0, 1)$ and $\psi \in \Phi_6$ such that*

$$\psi^p \left(\begin{array}{c} M(Fx, Gy, kt), M(fx, gy, t), M(Fx, fx, t), \\ M(Gy, gy, t), M(Fx, gy, t), M(Gy, fx, t) \end{array} \right) \geq 1, \tag{3.1}$$

$\forall x, y \in X, t > 0$ & $p > 0$ and pair (F, f) and g satisfy the $CLR_{(F,f)g}$-property then $C(F, f) \neq \phi$ and $C(G, g) \neq \phi$.
Moreover,

(i) hybrid pair (F, f) have a common fixed point in X, provided that f is F-weakly commuting at $v \in C(F, f)$,

(ii) hybrid pair (G, g) have a common fixed point in X, provided that g is G-weakly commuting at $u \in C(G, g)$,

(iii) f, g, F and G have a common fixed point in X, provided that both (i) and (ii) are true.

Proof. Since (F, f) and g satisfy the $CLR_{(F,f)g}$-property therefore there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = z, \lim_{n\to\infty} Fx_n = D, D \in CL(X)$$

and $z \in D \cap f(X) \cap g(X)$.
Since, $z \in g(X)$ there exists $u \in X$ such that $z = gu$.
Using (3.1), we have

$$\psi^p \left(\begin{array}{c} M(Fx_n, Gu, kt), M(fx_n, gu, t), M(Fx_n, fx_n, t), \\ M(Gu, gu, t), M(Fx_n, gu, t), M(Gu, fx_n, t) \end{array} \right) \geq 1.$$

Take the limit as $n \to \infty$, we get

$$\psi^p \left(\begin{array}{c} M(D, Gu, kt), 1, M(D, z, t), \\ M(Gu, z, t), M(D, z, t), M(Gu, z, t) \end{array} \right) \geq 1.$$

Since, $z \in D$ and $M(x, y, \cdot)$ is nondecreasing then we have

$$\psi^p \left(\; M(z, Gu, kt), 1, 1, M(Gu, z, t), 1, M(Gu, z, t) \; \right) \geq 1.$$

Using (δ_1), we have

$$M(z, Gu, kt) \geq M(z, Gu, t).$$

By Lemma 2.1, we have,

$$gu = z \in Gu, \text{ i.e. } u \in C(G, g).$$

Therefore, $C(G,g) \neq \phi$.

On the other hand, Since $z \in f(X)$ there exists $v \in X$ such that $z = fv$. Using (3.1), we have

$$\psi^p \begin{pmatrix} M(Fv,Gu,kt), M(fv,gu,t), M(Fv,fv,t), \\ M(Gu,gu,t), M(Fv,gu,t), M(Gu,fv,t) \end{pmatrix} \geq 1.$$

Since, $z \in Gu$ and $M(x,y,\cdot)$ is nondecreasing then we have

$$\psi^p \left(\ M(Fv,z,kt), 1, M(Fv,z,t), 1, M(Fv,z,t), 1 \ \right) \geq 1.$$

Using (δ_1), we have

$$M(Fv,z,kt) \geq M(Fv,z,t).$$

By Lemma 2.1, we have,

$$fv = z \in Fv, \text{ i.e. } v \in C(F,f).$$

Therefore, $C(F,f) \neq \phi$.
Moreover,

(i) Since, f is F-weakly commutativity at $v \in C(F,f)$ and $fv = z$ then we get

$$ffv \in Ffv \text{ i.e. } fz \in Fz.$$

Using (3.1), we get

$$\psi^p \begin{pmatrix} M(Ffv,Gu,kt), M(ffv,gu,t), M(Ffv,ffv,t), \\ M(Gu,gu,t), M(Ffv,gu,t), M(Gu,ffv,t) \end{pmatrix} \geq 1,$$
$$\psi^p \begin{pmatrix} M(ffv,Gu,kt), M(ffv,gu,t), M(ffv,ffv,t), \\ M(Gu,gu,t), M(ffv,gu,t), M(Gu,ffv,t) \end{pmatrix} \geq 1,$$
$$\psi^p \left(\ M(fz,z,kt), M(fz,z,t), 1, 1, M(fz,z,t), M(z,fz,t) \ \right) \geq 1.$$

Using (δ_1), we have

$$M(fz,z,kt) \geq M(fz,z,t).$$

By Lemma 2.1, we have, $fz = z$.
Hence, $z = fz \in Fz$, i.e. z is a common fixed point of f and F.
This proves (i).

(ii) Since, g is G-weakly commutativity at $u \in C(G,g)$ and $gu = z$ then we get

$$ggv \in Ggu \text{ i.e. } gz \in Gz.$$

Using (3.1), we get

$$\psi^p \begin{pmatrix} M(Fv, Ggu, kt), M(fv, ggu, t), M(Fv, fv, t), \\ M(Ggu, ggu, t), M(Fv, ggu, t), M(Ggu, fv, t) \end{pmatrix} \geq 1,$$
$$\psi^p \begin{pmatrix} M(fv, Gz, kt), M(fv, gz, t), M(fv, fv, t), \\ M(Gz, gz, t), M(fv, gz, t), M(Gz, fv, t) \end{pmatrix} \geq 1,$$
$$\psi^p \begin{pmatrix} M(z, gz, kt), M(z, gz, t), 1, 1, M(z, gz, t), M(gz, z, t) \end{pmatrix} \geq 1.$$

Using (δ_1), we have

$$M(gz, z, kt) \geq M(gz, z, t).$$

By Lemma 2.1, we have, $gz = z$.
Hence, $z = gz \in Gz$, i.e. z is a common fixed point of g and G.
This proves (ii). Then (iii) follows immediately. □

Example 3.1. Let $X = [0, 1]$ and we define,

$$M(x, y, t) = \tfrac{t}{t+|x-y|},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FMS, where $*$ is continuous t-norm.
We define $f, g: X \to X$ and $F, G: X \to CL(X)$ as follows:

$$fx = gx = 1,\ Fx = [\tfrac{x}{4}, 1].\ \text{and}\ Gx = [\tfrac{x}{6}, 1].$$

Clearly,

1. There exists a sequence $x_n = \frac{1}{2n} \in X$ such that

   $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\frac{1}{2n}) = 1$, $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} F(\frac{1}{2n}) = [0, 1] = D$,

   and $z = 1 \in D \cap f(X) \cap g(X)$ where $f(X) = 1 = g(X)$. Therefore, the hybrid pair (F, f) and T enjoy the $CLR_{(F,f)g}$-property.

2. Also, f is F-weakly commuting at $1 \in C(F, f)$, Since, $ff(1) \in Ff(1)$ and g is G-weakly commuting at $1 \in C(G, g)$, Since, $gg(1) \in Gg(1)$.

   One can easily verify that the mappings f, g, F and G satisfy condition (3.1) with with ψ as defined in Example 2.2 for $p > 0$.

Thus, the hybrid pair (F, f) and $g\&G$ satisfy all the conditions of the Theorem 3.1 so the mappings have a common fixed point $1 \in X$.

Example 3.2. Let $X = [0, 2]$ and we define,

$$M(x, y, t) = \tfrac{t}{t+|x-y|},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FMS, where $*$ is continuous t-norm.
We define $f, g: X \to X$ and $F, G: X \to CL(X)$ as follows:

$$fx = \begin{cases} 1, & \text{if } x \in [0,2); \\ \frac{3}{4}, & \text{if } x = 2, \end{cases} \qquad gx = \begin{cases} 1, & \text{if } x \in [0,2); \\ \frac{4}{5}, & \text{if } x = 2, \end{cases}$$

$$Fx = [1 - \tfrac{x}{3}, 2] \qquad \text{and} \qquad Gx = [\tfrac{x}{4}, 2], \forall x \in X.$$

Clearly,

1. There exists a sequence $x_n = 1 - \frac{1}{n} \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(1 - \tfrac{1}{n}) = 1 = z, (say)$$
$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} F(1 - \tfrac{1}{n}) = [\tfrac{2}{3}, 2] = D \in CL(X),$$

   and $z = 1 \in D \cap f(X) \cap g(X)$ where $f(X) = \{\frac{3}{4}, 1\}$ and $g(X) = \{\frac{4}{5}, 1\}$. Therefore, the hybrid pair (F, f) and g enjoy the $CLR_{(F,f)g}$-property.

2. Also, f is F-weakly commuting at $1 \in C(F, f)$, Since

$$ff(1) \in Ff(1) = [\tfrac{2}{3}, 2]$$

   and g is G-weakly commuting at $1 \in C(G, g)$, Since

$$gg(1) \in Gg(1) = [\tfrac{1}{4}, 2].$$

One can easily verify that the mappings f, g, F and G satisfy condition (3.1) with ψ as defined in Example 2.2 for $p > 0$.
Thus, the hybrid pair (F, f) and $g\&G$ satisfy all the conditions of the Theorem 3.1 so the mappings have a common fixed point $1 \in X$.

If take $p = 1$ in Theorem 3.1 then we have following:

Corollary 3.1. *Let f and g be two self mappings from a FMS $(X, M, *)$ and $F, G : X \to CL(X)$ satisfies the following condition:*
(a*) *if there exists a constant $k \in (0, 1)$ and $\psi \in \Phi_6$ such that*

$$\psi \begin{pmatrix} M(Fx, Gy, kt), M(fx, gy, t), M(Fx, fx, t), \\ M(Gy, gy, t), M(Fx, gy, t), M(Gy, fx, t) \end{pmatrix} \geq 1, \tag{3.2}$$

$\forall x, y \in X, t > 0$ and pair (F, f) and g satisfy the $CLR_{(F,f)g}$-property then $C(F, f) \neq \phi$ and $C(G, g) \neq \phi$.
Moreover,

(i) hybrid pair (F, f) have a common fixed point in X, provided that f is F-weakly commuting at $v \in C(F, f)$,

(ii) hybrid pair (G, g) have a common fixed point in X, provided that g is G-weakly commuting at $u \in C(G, g)$,

(iii) f, g, F and G have a common fixed point in X, provided that both (i) and (ii) are true.

Proof. Proof follows from Theorem 3.1 by taking $p = 1$. $\square$

Remark 3.1. Corollary 3.1 is improved result of [3, 19, 29] for a hybrid pair of single and multivalued mappings in the light of new type of common limit in the range property.

Now inspired by [2, 5], we consider the following relations:
(δ_1) Let Φ'_6 be the set of all real valued continuous functions $\psi' : (0,1]^6 \to R$ such that

($\delta_{1.1}$) ψ' non increasing in $2^{nd}, 3^{rd}, 4^{th}, 5^{th}$ and 6^{th} coordinate variables and

($\delta_{1.2}$) if

$$\begin{aligned}\psi'(u,1,v,1,v,1) \geq 0,\\ \psi'(u,v,1,1,v,v) \geq 0,\\ \psi'(u,1,1,v,1,v) \geq 0,\end{aligned}$$

implies that $u \geq v$, for all $u, v \in (0,1]$.

Example 3.3. We define real valued continuous relation $\psi'_1 : (0,1]^6 \to R$ such that

$$\psi'_1(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - min(t_2,t_3,t_4,t_5,t_6),$$

for all $t_1,t_2,t_3,t_4,t_5,t_6 \in (0,1]$.

It is easy to see that $\psi'_1 \in \Phi'_6$, since
for $t_1 = u$, $t_3 = t_5 = v$, $t_2 = t_4 = t_6 = 1$ if

$$\psi'_1(u,1,v,1,v,1) \geq 0 \text{ implies that } u \geq v,$$

for $t_1 = u$, $t_2 = t_5 = t_6 = v$, $t_3 = t_4 = 1$ if

$$\psi'_1(u,v,1,1,v,v) \geq 0 \text{ implies that } u \geq v,$$

for $t_1 = u$, $t_4 = t_6 = v$, $t_2 = t_3 = t_5 = 1$ if

$$\psi'_1(u,1,1,v,1,v) \geq 0 \text{ implies that } u \geq v.$$

The study of fixed points for mappings satisfying a contractive condition of integral type is introduced by Branciari [4]. Some fixed point results for mappings satisfying contractive conditions of integral type have been obtained in [5, 24, 26, 27] and in other papers. Before proving our next theorem for integral type contractive condition, Following the approach of [5], we suppose that there exists a function $\psi' \in \Phi'_6$ satisfying

$$\int_0^{\psi'(u,1,v,1,v,1)} \mu(s)\,ds \geq 0, \tag{3.3}$$

$$\int_0^{\psi'(u,v,1,1,v,v)} \mu(s)\,ds \geq 0, \tag{3.4}$$

$$\int_0^{\psi'(u,1,1,v,1,v)} \mu(s)\,ds \geq 0, \tag{3.5}$$

where $\mu : [0,\infty) \to [0,\infty)$ is a summable non-negative Lebesgue integrable function such that for each $\epsilon \in [0,1)$

$$\int_\epsilon^1 \mu(s)\,ds > 0,$$

implies $u \geq v$, $\forall u, v \in (0,1]$.

Theorem 3.2. *Let $(X, M, *)$ be a FMS. Let $f, g : X \to X$ and $F, G : X \to CL(X)$ are single and multivalued mappings respectively; satisfy the following condition:*
(a*) *there exists a constant $k \in (0,1)$ and $\psi' \in \Phi'_6$ such that*

$$\int_0^{\psi'\begin{pmatrix} M(Fx,Gy,kt), M(fx,gy,t), M(Fx,fx,t), \\ M(Gy,gy,t), M(Fx,gy,t), M(Gy,fx,t) \end{pmatrix}} \mu(s)\,ds \geq 0, \tag{3.6}$$

$\forall x, y \in X, t > 0$ and pair (F, f) and g satisfy the $CLR_{(F,f)g}$-property then $C(F, f) \neq \phi$ and $C(G, g) \neq \phi$.
Moreover,

(i) hybrid pair (F, f) have a common fixed point in X, provided that f is F-weakly commuting at $v \in C(F, f)$,

(ii) hybrid pair (G, g) have a common fixed point in X, provided that g is G-weakly commuting at $u \in C(G, g)$,

(iii) f, g, F and G have a common fixed point in X, provided that both (i) and (ii) are true.

Proof. Since (F, f) and g satisfy the $CLR_{(F,f)g}$-property therefore there exists a sequence x_n in X such that

$$\lim_{n\to\infty} fx_n = z, \lim_{n\to\infty} Fx_n = D, D \in CL(X)$$

and $z \in D \cap f(X) \cap g(X)$.
Since, $z \in g(X)$ there exists $u \in X$ such that $z = gu$.
Using (a^*), we have

$$\int_0^{M'_{xy}(t)} \mu(s)\,ds \geq 0,$$

where

$$M'_{xy}(t) = \psi'\begin{pmatrix} M(Fx_n,Gu,kt), M(fx_n,gu,t), M(Fx_n,fx_n,t), \\ M(Gu,gu,t), M(Fx_n,gu,t), M(Gu,fx_n,t) \end{pmatrix}.$$

Take the limit as $n \to \infty$, we get

$$M'_{xy}(t) = \psi'\begin{pmatrix} M(D,Gu,kt), 1, M(D,z,t), \\ M(Gu,z,t), M(D,z,t), M(Gu,z,t) \end{pmatrix}.$$

It implies that

$$\int_0^{\psi'(M(D,Gu,kt),1,M(D,z,t),M(Gu,z,t),M(D,z,t),M(Gu,z,t))} \mu(s)\,ds \geq 0.$$

Since, $z \in D$ and $M(x, y, \cdot)$ is nondecreasing then we have

$$\int_0^{\psi'(M(z,Gu,kt),1,1,M(Gu,z,t),1,M(Guz,t))} \mu(s)\,ds \geq 0,$$

Using (3.5), we have

$$M(z, Gu, kt) \geq M(z, Gu, t).$$

By Lemma 2.1, we have,

$$gu = z \in Gu, \text{ i.e. } u \in C(G, g).$$

Therefore, $C(G, g) \neq \phi$.

On the other hand, Since $z \in f(X)$ there exists $v \in X$ such that $z = fv$. Using (a^*), we have

$$\int_0^{M'_{xy}(t)} \mu(s)\, ds \geq 0,$$

where

$$M'_{xy}(t) = \psi' \begin{pmatrix} M(Fv, Gu, kt), M(fv, gu, t), M(Fv, fv, t), \\ M(Gu, gu, t), M(Fv, gu, t), M(Gu, fv, t) \end{pmatrix}.$$

Since, $z \in Gu$ and $M(x, y, \cdot)$ is nondecreasing then we have

$$\int_0^{\psi'(M(Fv,z,kt),1,M(Fv,z,t),1,M(Fv,z,t),1)} \mu(s)\, ds \geq 0,$$

Using (3.3), we have

$$M(Fv, z, kt) \geq M(Fv, z, t).$$

By Lemma 2.1, we have,

$$fv = z \in Fv, \text{ i.e. } v \in C(F, f).$$

Therefore, $C(F, f) \neq \phi$.
Moreover,

(i) Since, f is F-weakly commutativity at $v \in C(F, f)$ and $fv = z$ then we get

$$ffv \in Ffv \text{ i.e. } fz \in Fz.$$

Using (a^*), we get

$$\int_0^{\psi' \begin{pmatrix} M(Ffv, Gu, kt), M(ffv, gu, t), M(Ffv, ffv, t), \\ M(Gu, gu, t), M(Ffv, gu, t), M(Gu, ffv, t) \end{pmatrix}} \mu(s)\, ds \geq 0,$$

$$\int_0^{\psi' \begin{pmatrix} M(ffv, Gu, kt), M(ffv, gu, t), M(ffv, ffv, t), \\ M(Gu, gu, t), M(ffv, gu, t), M(Gu, ffv, t) \end{pmatrix}} \mu(s)\, ds \geq 0,$$

$$\int_0^{\psi'(M(fz,z,kt),M(fz,z,t),1,1,M(fz,z,t),M(z,fz,t)} \mu(s)\, ds \geq 0,$$

Using (3.4), we have

$$M(fz, z, kt) \geq M(fz, z, t).$$

By Lemma 2.1, we have, $fz = z$.
Hence, $z = fz \in Fz$, i.e. z is a common fixed point of f and F.
This proves (i).

(ii) Since, g is G-weakly commutativity at $u \in C(G, g)$ and $gu = z$ then we get

$$ggv \in Ggu \text{ i.e. } gz \in Gz.$$

Using (a^*), we get

$$\int_0^{\psi'\begin{pmatrix} M(Fv, Ggu, kt), M(fv, ggu, t), M(Fv, fv, t), \\ M(Ggu, ggu, t), M(Fv, ggu, t), M(Ggu, fv, t) \end{pmatrix}} \mu(s)\, ds \geq 0,$$

$$\int_0^{\psi'\begin{pmatrix} M(fv, Gz, kt), M(fv, gz, t), M(fv, fv, t), \\ M(Gz, gz, t), M(fv, gz, t), M(Gz, ffv, t) \end{pmatrix}} \mu(s)\, ds \geq 0,$$

$$\int_0^{\psi'(M(z, gz, kt), M(z, gz, t), 1, 1, M(z, gz, t), M(gz, z, t)} \mu(s)\, ds \geq 0,$$

Using (3.4), we have

$$M(gz, z, kt) \geq M(gz, z, t).$$

By Lemma 2.1, we have, $gz = z$.
Hence, $z = gz \in Gz$, i.e. z is a common fixed point of g and G.
This proves (ii). Then (iii) follows immediately. $\square$

4. Conclusion

In the present paper, we established some common fixed point theorems for a hybrid pair of single and multivalued mappings satisfying a new type of common limit in the range property using implicit relations in FMS. Some related results are also derived along with illustrative examples. Further, we also presented an integral type common fixed point theorem in FMS. Our results improve and extend some previous known results [3, 19, 29].

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REFERENCES

1. M. Aamri and D. El Moutawakil: *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), 181-188.
2. M.A. Ahmed and H.A. Nafadi: *Common fixed point theorems for hybrid pairs of maps in fuzzy metric spaces*, J. Egyptian Math. Soc. **22** (2014), 453-458.
3. Altun and D. Turkoglu: *Some fixed point theorems on fuzzy metric spaces with implicit relations*, Commun. Korean Math. Soc. **23**(1)(2008), 111-124.
4. Branciari: *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (9) (2002), 531-536.
5. S. Chauhan, M.A. Khan and S. Kumar: *Unified fixed point theorems in fuzzy metric spaces via common limit range property*, J. Inequal. Appl. **2013:182**, 2013.
6. S. Chauhan and S. Kumar: *Coincidence and fixed points in fuzzy metrtc spaces using common property (E.A)*, Kochi J. Math. **8** (2013), 135-154.
7. S. Chauhan, S. Kutukcu, N. Dhiman and S. Kumar: *Variants of R-weakly commuting map-pings and common fixed point theorems in intuitionistic fuzzy metric spaces*, Kuwait J. Sci. **41**(2) (2014), 49-64.
8. B.S. Choudhary, P. Das and P. Saha: *Some φ-contraction results using CLRg property in probabilistic and fuzzy metric spaces*, Ann. Fuzzy Math. Inform. **12** (3) (2016), 387-395.
9. A. George and P. Veeramani: *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994), 395-399.
10. M. Imdad and M.A. Ahmed: *Some common fixed point theorems for hybrid pairs of maps without the completeness assumption*, Math. Slovaca **62** (2012), 301-314.
11. M. Imdad, J. Ali and M. Hasan: *Common fixed point theorems in fuzzy metric spaces employing common property (E.A.)*, Math. Comput. Model. **55**(3-4) (2012), 770-778.
12. M. Imdad, S. Chauhan and Z. Kadelburg: *Fixed point theorem for mappings with common limit range property satisfying generalized (ψ,φ)-weak contractive conditions*, Math. Sci. (Springer) **7**:16 (2013), 8 pp.
13. M. Imdad, S. Chauhan, A.H. Soliman and M.A. Ahmed: *Hybrid fixed point theorems in symmetric spaces via common limit range property*, Demonstr. Math. **47** (4) (2014), 949-962.
14. V. Imdad, C. Chauhan and P. Kumam: *Fixed point theorems for two hybrid pairs of non-self mappings under joint common limit range property in metric spaces*, J. Nonlinear Convex Anal. **16** (2) (2015), 243-254.
15. G. Jungck: *Common fixed points for noncontinuous non-self maps on non-metric spaces*, Far East J. Math Sci. **4**(2) (1996), 199-215,.
16. E. Karapinar, D.K. Patel, M. Imdad and D. Gopal: *Some nonunique common fixed point theorems in symmetric spaces through $CLR_{(S,T)}$ property*, Int. J. Math. Math. Sci. **2013**, Art. ID 753965, 8 pp.
17. M.S. Khan, Y.J. Cho, V. Park and M. Mumtaz: *Coincidence and common fixed points of hybrid contractions*, J. Austral. Math. Son. (Series A) **55** (1993), 369-385.
18. I. Kramosil and J. Michalek: *Fuzzy metric and statistical metric spaces*, Kybernetika **11** (1975), 326-334.

19. S. Kumar and B. Fisher: *A common fixed point theorem in fuzzy metric space using property (E.A.) and implicit relation*, Thai J. Math. **8**(3) (2010), 439-446.

20. R. Lashkaripour, H. Baghani and Z. Ahmadi: *A new approach to some fixed point theorems for multivalued nonlinear f-contractive maps*, Mat. Vesnik **71** (4) (2019), 368-377.

21. J.T. Markin: *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc. **38** (1973) 545-547.

22. S.N. Mishra, S.N. Sharma and S.L. Singh: *Common fixed points of maps in fuzzy metric spaces,* Int. J. Math. Math. Sci. **17** (1994) 253–258.

23. S. Nadler: *Multi-valued contraction mappings*, Pacific J. Math. **20** (2) (1969) 475-488.

24. H.K. Pathak, R. Rodríguez-López and R.K. Verma: *A common fixed point theorem of integral type using implicit relation*, Nonlinear Funct. Anal. Appl. **15** (2010), pages 12.

25. V. Popa: *A general fixed point theorem for two hybrid pairs of mappings satisfying a mixed implicit relation and applications*, Math. Morav. **21** 2 (2017), 103-114.

26. V. Popa and M. Mocanu: *Altering distance and common fixed points under implicit relations*, Hacet. J. Math. Stat. **38** 3 (2009), 329-337.

27. B.E. Rhoades: *Two fixed point theorems for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **2003** 63 (2003), 4007-4013.

28. B. Schweizer and A. Sklar: *Statistical metric spaces*, Pacific J. of Math. **10**(1960), 314-334.

29. P.K. Sharma, S. Choudhary and K. Wadhwa: *A common fixed point theorem in fuzzy metric space using property (S-B) and implicit relation*, Ultra Scientist **23**(1) (2011), 277-282.

30. S. Singh, K.S. Ha and Y.J. Cho: *Coincidence and fixed points of nonlinear hybrid contractions*, Int. J. Math. Math. Sci. **2** (2) (1989) 247-256.

31. W. Sintunavarat and P. Kumam: *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math. **2011** 2011, Article ID 637958, pages 14.

32. P. Vijayaraju and Z.M.I. Sajath: *Common fixed points of single and multivalued maps in fuzzy metric spaces*, Applied Mathematics **2** (2011) 595-599.

33. K. Wadhwa and V.P. Bhardwaj: *Fixed point theorems for faintly compatible mappings in intuitionistic fuzzy metric space*, J. Indian Math. Soc. **84** (1-2) (2017), 130-142.

34. K. Wadhwa and V.P. Bhardwaj: *Some remarks on results of Pant and Pant along with generalization for hybrid mappings*, Ann. Fuzzy Math. Inform. **14** (3) (2017), 303-313.

35. S. Wang: *Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using CLRg property*, J. Nonlinear Sci. Appl. **9** (2016), 1043-1051.

36. L.A. Zadeh: *Fuzzy sets*, Information and Control **8** (1) (1965), 338-353.