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NEW ASPECTS OF STRONGLY *Log*-PREINVEX FUNCTIONS

Muhammad Aslam Noor and Khilda Inayat Noor

**Department of Mathematics, COMSATS University Islamabad
Islamabad, Pakistan**

Abstract. In this paper, we consider some new classes of log-preinvex functions. Several properties of the log-preinvex functions are studied. We also discuss their relations with convex functions. Several interesting results characterizing the log-convex functions are obtained. Optimality conditions of differentiable strongly log-preinvex are characterized by a class of variational-like inequalities. Results obtained in this paper can be viewed as significant improvement of previously known results.
Key words: Preinvex functions, variational inequalities, log-convex functions

1. Introduction

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. Hanson [5] introduced the notion of invex functions in mathematical programming, which inspired a great interest. Invex sets and preinvex functions were introduced by Ben-Israel and Mond [3]. They proved that the differentiable preinvex functions are invex functions and the converse is also true under certain conditions. Noor [15] proved that the minimum of the differentiable preinvex functions are characterized by variational-like inequalities. For the applications, numerical methods, variational-like inequalities and other aspects of preinvex functions,

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Corresponding Author: Muhammad Aslam Noor, Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan | E-mail: noormaslam@gmail.com
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see [1, 2, 3, 5, 8, 14, 15, 18, 19, 20, 21, 23, 26, 24, 25, 30] and the references therein. It is known that more accurate and inequalities can be obtained using the log-convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave), the origin of exponentially convex functions can be traced back to Bernstein [4]. Noor and Noor [20, 21] introduced and discussed various aspects of exponentially preinvex functions and their variant forms. The exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See, for example, [1, 2, 3, 4, 5, 6, 13, 14, 20, 21, 22, 23, 26, 25] and the references therein.

Recently, Noor et al [23]considered the equivalent formulation of log-convex functions and proved that the log-convex functions have similar properties as the convex functions enjoy. For example. the function e^x is a log-convex function, but not convex. Hypergeometric functions including Gamma and Beta functions are log-convex functions, which have important applications in several branches of pure and applied sciences. Noor and Noor [22] introduced the concept of strongly log-biconvex functions and studied their characterization. It is shown that the optimality conditions of the biconvex functions can be characterized by the bivariational inequalities, which can be viewed as novel generalization of the variational inequalities.

Inspired and motivated by the ongoing research in this interesting, applicable and dynamic field, we reconsider the concept of strongly log-preinvex functions. We discuss the basic properties of the log-preinvex functions. It is has been shown that the log-preinvex(preincave) have nice properties. Several new concepts of strongly log-preinvex functions have been introduced and investigated. We show that the local minimum of the log-convex functions is the global minimum. The difference (sum) of the strongly log-preinvex function and affine strongly log-preinvex function is again a log-preinvex function. The optimal conditions of the differentiable strongly log-preinvex functions can be characterized by a class of variational-like inequalities, which is itself an interesting outcome of our main results. The ideas and techniques of this paper may be a starting point for further research in these different areas of mathematical programming, machine learning and related optimization problems.

2. Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K \to R$ be a continuous function.

Definition 2.1. [10] The set K in H is said to be convex set, if

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2.2. [7, 8, 9] A function F is said to be convex, if

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

Polyak [27] introduced the concept of strongly convex functions in optimization and mathematical programming.

Definition 2.3. A function F is said to be a strongly convex, if there exists a constant $\mu \geq 0$ such that

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v)-\mu t(1-t)\|v-u\|^2, \ \forall u,v \in K, \ t \in [0,1].$$

Clearly every strongly convex function is a convex function, but the converse is not true. For the applications of strongly convex functions in variational inequalities, differential equations and equilibrium problems, see [6, 7, 9, 10, 11, 17, 18, 19, 21, 27, 31] and the references therein.

In many problems, the underlying set may not a convex set. To overcome this deficiency, Ben-Israel and Mond [3] introduced the invex and preinvex functions with respect to an arbitrary bifunction, which can be viewed as important generalization of the convexity and inspired a great interest in nonlinear mathematical programming.

Definition 2.4. [3] The set K_η in H is said to be invex set with respect to an arbitrary bifunction $\eta(.,.)$, if

$$u+t\eta(v,u) \in K, \qquad \forall u,v \in K_\eta, t \in [0,1].$$

Note that, if $\eta(v,u)=v-u$, then the invex set becomes convex set. In particular, it follows that the set $K_\eta \subset K$.

Definition 2.5. A strictly positive function F is said to be preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(u+t\eta(v,u)) \leq (1-t)F(u)+tF(v), \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

It is known that the differentiable preinvex functions is an invex function, that is

Definition 2.6. A function F is said to be an invex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(v)-F(u) \geq \langle F'(u), \eta(v,u)\rangle, \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

The converse is also true under certain conditions, see [8].

Noor [15] has proved that $u \in K_\eta$ is a minimum of a differentiable preinvex functions F, if and only if, $u \in K_\eta$ satisfies the inequality

$$\langle F'(u), \eta(v,u)\rangle \geq 0, \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

which is known as the variational-like inequality. For the formulation, applications, numerical methods and other aspects of variational-like inequalities and related optimization problems, see [2, 3, 5, 8, 15, 16, 28, 29] and the references therein.

Noor [14] has also proved that a function F is a preinvex function, if and only if, F satisfies the inequality

$$F\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{2}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)dx \leq \frac{F(a)+F(b)}{2},$$

which is known as the Hermite-Hadamard-Noor inequality. Such type of inequalities are used to find the upper and lower estimates of the integrals and have important applications in physical and material sciences.

Definition 2.7. A strictly positive function F is said to be log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(u+t\eta(v,u)) \leq (F(u))^{1-t}(F(v))^{-t}, \quad \forall u,v \in K_{\eta}, \quad t \in [0,1]. \tag{2.1}$$

We can rewrite the Definition 2.7 in the following equivalent form as

Definition 2.8. [14] A strictly positive function F is said to be log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\log F(u+t\eta(v-u)) \leq (1-t)\log F(u) + t\log F(v), \qquad \forall u,v \in K_{\eta}, t \in [0,1]. \tag{2.2}$$

We use this equivalent Definition 2.8 to discuss some new aspects of log-preinvex functions.

If $\log F = e^{f(u)}$, then we recover the concepts of the exponentially preinvex function, which are mainly due to Noor and Noor [19, 21] as:

Definition 2.9. [19, 21] A positive function f is said to be exponentially preinvex function, if

$$e^{f(u+t\eta(v,u))} \leq (1-t)e^{f(u)} + te^{f(v)}, \quad \forall u,v \in K_{\eta}, \quad t \in [0,1].$$

We remark that Definition 2.9 can be rewritten in the following equivalent way, which is mainly due to Antczak [2].

Definition 2.10. A function f is said to be exponentially preinvex function, if

$$f(u+t\eta(v,u)) \leq \log[(1-t)e^{f(u)} + te^{f(v)}], \quad \forall u,v \in K_{\eta}, t \in [0,1]. \tag{2.3}$$

A function is called the exponentially preincave function f, if $-f$ is exponentially preinvex function. For the applications and properties of exponentially preinvex functions, see [1, 2, 3, 17, 18].

We now introduce the concept of strongly log-preinvex functions and study their basic properties.

Definition 2.11. A strictly positive function F is said to be strongly log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if there exists a constant $\mu \geq 0$, such that

$$\begin{aligned} \log F(u+t\eta(v-u)) &\leq (1-t)\log F(u) + t\log F(v) \\ &\quad -\mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, \quad t \in [0,1]. \end{aligned} \tag{2.4}$$

Definition 2.12. A strictly positive function F on the invex set K_η is said to be strongly log-quasi preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u)) &\leq \max\{\log F(u), \log F(v)\} - \mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v \in K_\eta, t \in [0,1]. \end{aligned}$$

Definition 2.13. A strictly positive function F on the invex set K is said to be first kind of strongly log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u))) &\leq (\log(F(u))^{1-t}(\log F(v))^t - \mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v \in K_\eta, t \in [0,1], \end{aligned}$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} \log F(u+t\eta(v,u)) &\leq (\log(F(u))^{1-t}(\log F(v))^t - \mu t(1-t)\|\eta(v,u)\|^2 \\ &\leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2 \\ &\leq \max\{\log F(u), \log F(v)\} - \mu t(1-t)\|\eta(v,u)\|^2. \end{aligned}$$

This shows that every fist kind of strongly log-preinvex function is a strongly log-preinvex function and strongly log-preinvex function is a strongly log-quasip reinvex function. However, the converse is not true.

If $t = 1$, then Definitions 2.13 and 2.14, we have:

Condition A. $\log F(u+\eta(v,u) \leq F(v)), \quad \forall u,v \in K_\eta.$

Condition A plays an important part in the derivation of the main results.

Definition 2.14. A strictly positive function F is said to be strongly affine log-preinvex function with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u)) &= (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v \in K_\eta, t \in [0,1]. \end{aligned}$$

Let $K_\eta = I_\eta = [a, a+\eta(b,a)]$ be the interval. We now define the log-preinvex functions on the interval I_η.

Definition 2.15. Let $I_\eta = [a, a + \eta(b, a)]$. Then F is log-convex function, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\eta(b,a) \\ \log F(a) & \log F(x) & \log F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following are equivalent:

1. F is a log-preinvex function.
2. $\log F(x) \leq \log F(a) + \frac{\log F(b) - \log F(a)}{\eta(b,a)}(x - a)$.
3. $\frac{\log F(x) - \log F(a)}{x - a} \leq \frac{\log F(b) - \log F(a)}{\eta(b,a)}$.
4. $(a + \eta(b, a) - x) \log F(a) + \eta(a, b) \log F(x) + (x - a) \log F(b)) \geq 0$.
5. $\frac{\log F(a)}{\eta(b,a)(a-x)} + \frac{\log F(x)}{(x-a-\eta(b,a))(a-x)} + \frac{\log F(b)}{\eta(b,a)(x-b)} \leq 0$,

where $x = a + t\eta(b, a) \in [0, 1]$.

We also need the following assumption regarding the bifunction $\eta(\cdot, \cdot)$, which played a crucial part in the field of variational and integral inequalities,

Condition C [8]. Let $\eta(\cdot, \cdot) : K_\eta \times K_\eta \to H$ satisfy assumptions

$$\begin{aligned} &\eta(u, u + \lambda\eta(v, u)) = -\lambda\eta(v, u) \\ &\eta(v, u + \lambda\eta(v, u)) = (1 - \lambda)\eta(v, u), \quad \forall u, v \in K_\eta, \lambda \in [0, 1]. \end{aligned}$$

Clearly for $\lambda = 0$, we have $\eta(u, v) = 0$, if and only if $u = v, \forall u, v \in K_\eta$. One can easily show that $\eta(u + \lambda\eta(v, u), u) = \lambda\eta(v, u), \forall u, v \in K_\eta$.

3. Properties of log-preinvex functions

In this section, we consider some basic properties of log-preinvex functions.

Theorem 3.1. *Let F be a strictly* log*-preinvex function. Then any local minimum of F is a global minimum.*

Proof. Let the log-preinvex function F have a local minimum at $u \in K_\eta$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K$. Since F is a log-preinvex function, so

$$\log F(u + t\eta(v, u)) < t \log F(v) + (1 - t) \log F(u), \quad \text{for} \quad 0 < t < 1.$$

Thus

$$\log F(u + t\eta(v, u)) - \log F(u) < t[\log F(v) - \log F(u)] < 0,$$

from which it follows that

$$\log F(u + t\eta(v, u)) < \log F(u),$$

for arbitrary small $t > 0$, contradicting the local minimum. $\square$

Theorem 3.2. *If the function F on the invex set K_η is* log*-preinvex, then the level set*

$$L_\alpha = \{u \in K : \log F(u) \leq \alpha, \quad \alpha \in R\}$$

is an invex set.

Proof. Let $u, v \in L_\alpha$. Then $\log F(u) \leq \alpha$ and $\log F(v) \leq \alpha$.
Now, $\forall t \in (0,1), \quad w = v + t\eta(u,v) \in K_\eta$, since K_η is an invex set. Thus, by the log-preinvexity of F, we have

$$\begin{aligned} \log F(v + t\eta(u,v)) &\leq (1-t)\log F(v) + t\log F(u) \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which, it follows that $v + t\eta(u,v) \in L_\alpha$ Hence L_α is an invex set. $\square$

Theorem 3.3. *A positive function F is a* log*-preinvex, if and only if*

$$epi(F) = \{(u,\alpha) : u \in K : \log F(u) \leq \alpha, \alpha \in R\}$$

is an invex set.

Proof. Assume that$F$ is log-preinvex function. Let $(u,\alpha), (v,\beta) \in epi(F)$. Then it follows that $\log F(u) \leq \alpha$ and $\log F(v) \leq \beta$. Thus, $\forall t \in [0,1], \quad u, v \in K_\eta$, we have

$$\begin{aligned} \log F(u + t\eta(v,u)) &\leq (1-t)\log F(u) + t\log F(v) \\ &\leq (1-t)\alpha + t\beta, \end{aligned}$$

which implies that

$$(u + t\eta(v,u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is an invex set. Conversely, let $epi(F)$ be an invex set. Let $u, v \in K_\eta$. Then $(u, \log F(u)) \in epi(F)$ and $(v, \log F(v)) \in epi(F)$. Since $epi(F)$ is an invex set, we must have

$$(u + t\eta(v,u), (1-t)\log F(u) + t\log F(v)) \in epi(F),$$

which implies that

$$\log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(u).$$

This shows that F is a log-preinvex function. $\square$

Theorem 3.4. *A positive function F is quasi* log*-preinvex, if and only if, the level set*

$$L_\alpha = \{u \in K_\eta, \alpha \in R : \log F(u) \leq \alpha\}$$

is an invex set.

Proof. Let $u, v \in L_\alpha$. Then $u, v \in K_\eta$ and $\max(\log F(u), \log F(v)) \leq \alpha$.
Now for $t \in (0,1), w = u + t\eta(v-u) \in K_\eta$, We have to prove that $u + t\eta(v,u) \in L_\alpha$. By the quasi log-preinvexity of F, we have

$$\log F(u + t(v-u)) \leq \max(\log F(u), \log F(v)) \leq \alpha,$$

which implies that $u + t\eta(v,u) \in L_\alpha$, showing that the level set L_α is indeed an invex set.
Conversely, assume that L_α is an invex set. Then $\forall u, v \in L_\alpha, t \in [0,1]$, $u + t(v-u) \in L_\alpha$. Let $u, v \in L_\alpha$ for

$$\alpha = max(\log F(u), \log F(v) \quad \text{and} \quad \log F(v) \leq \log F(u).$$

From the definition of the level set L_α, it follows that

$$\log F(u + t\ (v,u)) \leq \max(\log F(u), \log F(v)) \leq \alpha.$$

Thus F is a quasi log-preinvex function. This completes the proof. $\square$

Theorem 3.5. *Let F be a* log*-preinvex function.. Let $\mu = \inf_{u \in K} F(u)$. Then the set $E = \{u \in K : \log F(u) = \mu\}$ is an invex set of K_η. If F is strictly* log*-preinvex, then E is a singleton.*

Proof. Let $u, v\ \in E$. For $0 < t < 1$, let $w = u + t\eta(v,u)$. Since F is a log-preinvex function,

$$F(w) = \log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) = t\mu + (1-t)\mu = \mu,$$

which implies that to $w \in E$. and hence E is an invex set. For the second part, assume to the contrary that $F(u) = F(v) = \mu$. Since K is an invex set, for $0 < t < 1, u + t\eta(v,u) \in K_\eta$. Further, since F is strictly log-preinvex,

$$\begin{aligned} \log F(u + t(v-u)) &< (1-t)\log F(u) + t\log F(v) \\ &= (1-t)\mu + t\mu = \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{u \in K} F(u)$ and hence the result follows. $\square$

Theorem 3.6. *If F is a* log*-preinvex function such that*

$$\log F(v) < \log F(u), \forall u, v \in K,$$

then F is a strictly quasi log*-preinvex function.*

Proof. By the log-convexity of the function F, $\forall u, v \in K, t \in [0,1]$, we have

$$\log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) < \log F(u),$$

since $\log F(v) < \log F(u)$, which shows that the function F is strictly quasi log-preinvex. $\square$

4. Strongly log-preinvex functions

In this section, we now discuss some properties of the strongly log-preinvex functions.

Theorem 4.1. *Let F be a differentiable function on the invex set K_η and Condition C hold. Then the function F is* log*-preinvex function, if and only if,*

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall v,u \in K_\eta. \tag{4.1}$$

Proof. Let F be a strongly log-preinvex function. Then, $\forall u, v \in K_\eta$,

$$\log F(u+t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2,$$

which can be written as

$$\begin{aligned}\log F(v) - \log F(u) &\geq \{\frac{\log F(u+t\eta(v,u)) - \log F(u)}{t}\} \\ &\quad +\mu(1-t)\|\eta(v,u)\|^2.\end{aligned}$$

Taking the limit in the above inequality as $t \to 0$, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

which is (4.1), the required result.
Conversely, let (4.1) hold. Then $\forall u, v \in K_\eta, t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$ and using Condition C, we have

$$\begin{aligned} logF(v) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,v_t))\rangle + \mu\|\eta(v,v_t)\|^2 \\ &= (1-t)\langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle + (1-t)^2\mu\|\eta(v,u)\|^2. \end{aligned} \tag{4.2}$$

In a similar way, we have

$$\begin{aligned} \log F(u) - \log F(v_t) &\geq \langle \frac{F'(v_t)}{F(v_t)}, \eta(u,v_t)\rangle + \mu\|\eta(u.v_t)\|^2 \\ &= -t\langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle + \mu t^2\|\eta(v,u)\|^2. \end{aligned} \tag{4.3}$$

Multiplying (4.2) by t and (4.3) by $(1-t)$ and adding the resultant, we have

$$\log F(u+t(v-u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t \in [0.1],$$

showing that F is a strongly log-preinvex function. □

Remark 4.1. From (4.1), we have

$$F(v) \geq F(u)exp\{\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2\}, \quad \forall u, v \in K_\eta.$$

Changing the role of u and v in the above inequality, we also have

$$F(u) \geq F(v)exp\{\langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle + \mu\|\eta(u,v)\|^2\}, \quad \forall u, v \in K_\eta.$$

Thus, we can obtain the following inequality

$$\begin{aligned} F(u) + F(v) \quad &\geq \quad F(v)exp\{\langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle + \mu\|\eta(u,v)\|^2\}, \\ &\quad +F(u)exp\{\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2\}, \quad \forall u, v \in K. \end{aligned}$$

Theorem 4.1 enables us to introduce the concept of the log-monotone operators, which appears to be new ones.

Definition 4.1. The differential $F'(.)$ is said to be strongly log-monotone, if

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u, v \in H.$$

Definition 4.2. The differential $F'(.)$ is said to be log-monotone, if

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in H.$$

Definition 4.3. The differential $F'(.)$ is said to be log-pseudo-monotone, if

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle \geq 0, \quad \Rightarrow -\langle \frac{F'(v)}{F(v)}, \eta(u-v)\rangle \geq 0, \quad \forall u, v \in H.$$

From these definitions, it follows that strongly log-monotonicity implies log-monotonicity implies log-pseudo-monotonicity, but the converse is not true.

Theorem 4.2. *Let F be differentiable strongly* log-*preinvex function on the invex set K_η. Let Condition C and Condition A hold. Then (4.1) holds, if and only if, $F'(.)$ satisfies*

$$\begin{aligned} \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle \quad &+ \quad \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \\ &\leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u, v \in K_\eta. \end{aligned} \tag{4.4}$$

Proof. Let F be a strongly log-preinvex function on the invex set K_η. Then, from Theorem 4.1, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K_\eta. \tag{4.5}$$

Changing the role of u and v in (4.5), we have

$$\log F(u) - \log F(v) \geq \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle + \|\eta(u,v)\|^2, \quad \forall u, v \in K_\eta. \tag{4.6}$$

Adding (4.5) and (4.6), we have

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u, v \in K_\eta.$$

which shows that F' is a strongly log-monotone.
Conversely, from (4.4) and Condition C, we have

$$\langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\} - \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle, \quad \forall u, v \in K_\eta. \tag{4.7}$$

Since K is an invex set, $\forall u, v \in K_\eta, \quad t \in [0,1]$ $v_t = u + t\eta(v,u) \in K_\eta$.
Taking $v = v_t$ in (4.7), we have

$$\langle \frac{F'(v_t)}{F(v_t)}, \eta(u,v_t)\rangle \leq -\mu\{\|\eta(v_t,u)\|^2 + \|\eta(u,v_t)\|^2\} - \langle \frac{F'(u)}{F(u)}, \eta(v_t,u)\rangle, \quad \forall u, v \in K_\eta.$$

Using Condition C, we obtain

$$\langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2\rangle. \tag{4.8}$$

Consider the auxiliary function

$$\xi(t) = \log F(u + t\eta(v,u)),$$

from which, we have

$$\xi(1) = \log F(u + \eta(v,u)), \quad \xi(0) = \log F(u).$$

Then, from (4.8), we have

$$\xi'(t) = \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2. \tag{4.9}$$

Integrating (4.9) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u) \rangle + \mu \|\eta(v,u)\|^2.$$

Thus it follows, using Condition A, that

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u) \rangle + \mu \|\eta(v,u)\|^2,$$

which is the required (4.1). □

We now give a necessary condition for log-pseudoconvex function.

Theorem 4.3. *Let $F'(.)$ be a* log*-pseudomonotone and let Condition C and Condition A hold. Then F is a* log*-pseudo preinvex function.*

Proof. Let $F'(.)$ be a log-pseudomonotone. Then,

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u) \rangle \geq 0, \quad \forall u, v \in K_\eta,$$

implies that

$$-\langle \frac{F'(v)}{F(v)}, \eta(v,u) \rangle \geq 0. \tag{4.10}$$

Since K_η is an invex set, $\forall u, v \in K_\eta$, $t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$. Taking $v = v_t$ in(4.10) and using Condition C, we have

$$\langle e^{F(v_t)} F'(v_t), \eta(v,u) \rangle \geq 0. \tag{4.11}$$

Consider the auxiliary function

$$\xi(t) = \log F(u + t\eta(v,u)) = \log F(v_t), \quad \forall u, v \in K_\eta, t \in [0,1],$$

which is differentiable. Then, using (4.11), we have

$$\xi'(t) = \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u) \rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq 0.$$

Using Condition A, we have

$$\log F(v) - \log F(u) \geq 0,$$

showing that F is a log-pseudo preinvex function. □

Definition 4.4. The function F is said to be sharply log-pseudo preinvex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle &\geq 0 \\ &\Rightarrow \\ F(v) &\geq \log F(u+t\eta(v,u)), \quad \forall u,v \in K_\eta, t\in[0,1]. \end{aligned}$$

Theorem 4.4. *Let F be a sharply* log*-pseudo preinvex function on K_η. Then*

$$\langle \frac{F'(v)}{F(v)}, \eta(v,u)\rangle \geq 0, \quad \forall u,v \in K_\eta.$$

Proof. Let F be a sharply log-pseudo preinvex function on K_η. Then

$$\log F(v) \geq \log F(v+t\eta(u,v)), \quad \forall u,v \in K_\eta, t\in[0,1].$$

from which we have

$$0 \leq \lim_{t\to 0}\{\frac{\log F(v+t\eta(u,v)) - \log F(v)}{t}\} = \langle \frac{F'(v)}{F(v)}, \eta(v,u)\rangle,$$

the required result. □

Definition 4.5. A function F is said to be a log-pseudo preinvex function with respect to a strictly positive bifunction $B(.,.)$, such that

$$\begin{aligned} \log F(v) &< \log F(u) \\ &\Rightarrow \\ \log F(u+t\eta(v,u)) &< \log F(u) + t(t-1)B(v,u), \forall u,v\in K_\eta, t\in[0,1]. \end{aligned}$$

Theorem 4.5. *If the function F is strongly* log*-preinvex function such that $\log F(v) < \log F(u)$, then the function F is strongly* log*-pseudo preinvex.*

Proof. Since $\log F(v) < \log F(u)$ and F is strongly log-preinvex function, then $\forall u,v \in K_\eta, t\in[0,1]$, we have

$$\begin{aligned} & \log F(u+t\eta(v,u)) \\ \leq\ & \log F(u) + t(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ <\ & \log F(u) + t(1-t)(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ =\ & \log F(u) + t(t-1)(\log F(u) - \log F(v)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ <\ & \log F(u) + t(t-1)B(u,v) - \mu t(1-t)\|\eta(v,u)\|^2, \end{aligned}$$

where $B(u,v) = \log F(u) - \log F(v) > 0$. This shows that the function F is strongly log-preinvex function. □

We now show that the difference of strongly log-preinvex function and affine strongly log-preinvex function is again a log-preinvex function.

Theorem 4.6. *Let f be a affine strongly* log*-preinvex function. Then F is a strongly* log*-preinvex function, if and only if, $g = F - f$ is a* log*-preinvex function.*

Proof. Let f be an affine strongly log-preinvex function. Then

$$\log f((u+t\eta(v,u)) = (1-t)\log f(u) + t\log f(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t\in[0,1]. \tag{4.12}$$

From the strongly log-preinvexity of F, we have

$$\log F(u+t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t\in[0,1]. \tag{4.13}$$

From (4.12) and (4.13), we have

$$\log F((u+t\eta(v,u)) - \log f((u+t\eta(v,u)) \leq (1-t)(\log F(u) - \log f(u)) + t(\log F(v) - \log f(v)), \tag{4.14}$$

from which it follows that

$$\begin{aligned} \log g((u+t\eta(v,u)) &= \log F((u+t\eta(v,u) - \log f((u+t\eta(v,u)) \\ &\leq (1-t)(\log F(u) - \log f(u)) + t(\log F(v) - \log f(v)), \end{aligned}$$

which shows that $g = F - f$ is a log-preinvex function.
The inverse implication is obvious. □

We now discuss the optimality condition for the differentiable strongly log-preinvex functions, which is the main motivation of our next result.

Theorem 4.7. *Let F be a differentiable strongly* log*-preinvex function. Then $u \in K_\eta$ is a minimum of the function F, if and only if, $u \in K_\eta$ satisfies the inequality*

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \quad \forall u,v \in K_\eta. \tag{4.15}$$

Proof. Let $u \in K_\eta$ be a minimum of the log-preinvex function F. Then

$$F(u) \leq F(v), \forall v \in K_\eta.$$

from which, we have

$$\log F(u) \leq \log F(v), \forall v \in K_\eta. \tag{4.16}$$

Since K is an invex set, so, $\forall u, v \in K_\eta, \quad t \in [0,1]$,

$$v_t = u + t\eta(v,u) \in K_\eta.$$

Taking $v = v_t$ in (4.16), we have

$$0 \leq \lim_{t\to 0}\{\frac{\log F(u + t\eta(v,u)) - \log F(u)}{t}\} = \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle. \tag{4.17}$$

Since F is differentiable strongly log-preinvex function, so

$$\log F(u + t\eta(v,u)) \leq \log F(u) + t(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2, \\ \forall u, v \in K_\eta, t \in [0,1].$$

Using (4.17), we have

$$\begin{aligned} \log F(v) - \log F(u) \quad &\geq \quad \lim_{t\to 0}\{\frac{\log F(u + t\eta(v,u)) - \log F(u)}{t}\} + \mu\|\eta(v,u)\|^2 \\ &= \quad \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0. \end{aligned}$$

Thus, it follows that

$$\log F(v) - \log F(u) \geq \mu\|\eta(v,u)\|^2,$$

which is the required result(4.15). □

Remark 4.2. We note that, if $u \in K_\eta$ satisfies the

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \forall v \in K_\eta, \tag{4.18}$$

then $u \in K_\eta$ is a minimum of a strongly log-preinvex function F. The inequality of the type (4.18) is called the log-variational-like inequality and appears to be a new one. For the applications, formulations, numerical methods and other aspects of variational inequalities, see Noor [12, 13, 15, 16, 31].

We remark that, if a strictly positive function F is a strongly log-preinvex function, then, we have

$$\begin{aligned} \log F(u + t\eta(v,u)) \quad &+ \quad \log F(v + t\eta(u,v)) \leq \log F(u) \\ &+ \quad \log F(v) - 2\mu t(1-t)\|\eta(v,u)\|^2, \forall u, v \in K_\eta, t \in [0,1], \end{aligned} \tag{4.19}$$

which is called the Wright strongly log-preinvex function.
From (4.19), we have

$$\begin{aligned} \log F(u + t\eta(v,u))F(v + t\eta(u,v)) \quad &= \quad \log F(u + t\eta(v,u)) + \log F(v + t\eta(u,v)) \\ &\leq \quad \log F(u) + \log F(v) \\ &= \quad \log F(u)F(v), \quad \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

This implies that

$$F((u+t\eta(v,u))F(tu+(1-t)v)\leq F(u)F(v),\quad \forall u,v\in K_{\eta},t\in[0,1],$$

which shows that a strictly positive function F is a multiplicative Wright strongly log-preinvex function. It is an interesting problem to study the properties and applications of the Wright log-preinvex functions.

Conclusion

In this paper, we have studied some new aspects of log-preinvex functions. It has been shown that log-preinvex functions enjoy several properties which convex functions have. Several new classes of strongly log-preinvex functions have been introduced and their properties are investigated. We have shown that the minimum of the differentiable strongly log-preinvex functions can be characterized by a new class of variational inequalities, which is called the log-variational inequality. Using the technique of auxiliary principle technique [13, 15, 25, 31], one can discuss the existence of a solution and suggest iterative methods for solving the *log* variational-like inequalities. One can explore the applications of the log-variational inequalities in pure and applied sciences. This may stimulate further research.

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