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ASYMPTOTIC STABILITY OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract. In this paper, we study the asymptotic stability of a generalized nonlinear neutral differential equation with variable delays by using the fixed point theory. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some results due to Burton [11], Zhang [25], Dib, Maroun and Raffoul [16], and Ardjouni and Djoudi [3]. Two examples are also given to illustrate our results.

Key words: Fixed points, Stability, Neutral differential equations, Variable delays.

1. Introduction

Lyapunov's direct method has been, for more than 100 years, the most efficient tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [9]–[11]. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[23], [25], [26]). The fixed point theory does not only solve the problem on stability but has other significant advantages over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [9]). Moreover, the fixed point method has been successfully used to conclude stability results to delay problems which are perturbed by stochastic terms (see for example [22]). This is another important feature for applications to real-world problems.

In this paper, we consider the nonlinear neutral differential equation with vari-

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able delays

$$\begin{aligned}\frac{d}{dt}x(t) &= -\sum_{j=1}^{N} b_j(t)\,x\left(t-\tau_j(t)\right) + \frac{d}{dt}Q\left(t, x\left(t-\tau_1(t)\right), ..., x\left(t-\tau_N(t)\right)\right) \\ &\quad + G\left(t, x\left(t-\tau_1(t)\right), ..., x\left(t-\tau_N(t)\right)\right), \end{aligned} \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \ \ for\ t \in [m(t_0), t_0], \tag{1.2}$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$ and for each $t_0 \geq 0$,

$$m_j(t_0) = \inf\left\{t - \tau_j(t),\ t \geq t_0\right\},\ m(t_0) = \min\left\{m_j(t_0),\ 1 \leq j \leq N\right\}. \tag{1.3}$$

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $b_j \in C(\mathbb{R}^+, \mathbb{R})$, and $\tau_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_j(t) \to \infty$ as $t \to \infty$. The functions $Q(t, x_1, ..., x_N)$ and $G(t, x_1, ..., x_N)$ are globally Lipschitz continuous in $x_1, ..., x_N$. That is, there are positive constants $K_1, ..., K_N$ and $L_1, ..., L_N$ such that

$$\left|Q(t, x_1, ..., x_N) - Q(t, y_1, ..., y_N)\right| \leq \sum_{j=1}^{N} K_j \left\|x_j - y_j\right\|, \tag{1.4}$$

and

$$\left|G(t, x_1, ..., x_N) - G(t, y_1, ..., y_N)\right| \leq \sum_{j=1}^{N} L_j \left\|x_j - y_j\right\|. \tag{1.5}$$

We also assume that

$$Q(t, 0, ..., 0) = G(t, 0, ..., 0) = 0. \tag{1.6}$$

Equation (1.1) and its special cases have been investigated by many authors. For example, in [11], Burton studied the equation

$$x'(t) = -b_1(t)\,x\left(t - \tau_1(t)\right), \tag{1.7}$$

and proved the following theorem.

Theorem 1.1. (Burton [11]). *Suppose that $\tau_1(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |b(s+r)|\,ds + \int_0^t |b(s+r)|\,e^{-\int_s^t b(u+r)du}\left(\int_{s-r}^{s} |b(u+r)|\,du\right)ds \leq \alpha, \tag{1.8}$$

for all $t \geq 0$ and $\int_0^\infty b(s)\,ds = \infty$. Then, for every continuous initial function $\psi : [-r, 0] \to \mathbb{R}$, the solution $x(t) = x(t, 0, \psi)$ of (1.7) is bounded and tends to zero as $t \to \infty$.

Zhang in [25] and Ardjouni and Djoudi in [1] have studied the generalization of *(1.7)* as follows

$$x'(t) = -\sum_{j=1}^{N} b_j(t)\, x\left(t-\tau_j(t)\right) + \sum_{j=1}^{N} c_j(t)\, x'\left(t-\tau_j(t)\right), \tag{1.9}$$

where c_j is differentiable and obtained the following theorems.

Theorem 1.2. (Zhang [25]). *Suppose that $c_j = 0$, τ_j is differentiable, the inverse function g_j of $t-\tau_j(t)$ exists, and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$,*

$$\liminf_{t\to\infty} \int_0^t q(s)\, ds > -\infty, \tag{1.10}$$

and

$$\sum_{j=1}^{N} \left[\int_{t-\tau_j(t)}^{t} \left| b_j\left(g_j(s)\right)\right| ds + \int_0^t e^{-\int_s^t q(u)du} \left|b_j(s)\right| \left|\tau_j'(s)\right| ds \right.$$
$$\left. + \int_0^t e^{-\int_s^t q(u)du} \left|q(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|b_j\left(g_j(u)\right)\right| du \right) ds \right] \leq \alpha, \tag{1.11}$$

where $q(t) = \sum_{j=1}^{N} b_j\left(g_j(t)\right)$. Then the zero solution of (1.9) is asymptotically stable if and only if $\int_0^t q(s)\, ds \to \infty$ as $t \to \infty$.

Theorem 1.3. (Ardjouni and djoudi [1]). *Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in \mathbb{R}^+$, and there exist continuous functions $h_j : \left[m_j(t_0), \infty\right) \to \mathbb{R}$ for $j = 1, 2, ..., N$ and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)\, ds > -\infty, \tag{1.12}$$

and

$$\sum_{j=1}^{N} \left| \frac{c_j(t)}{1-\tau_j'(t)} \right| + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| ds$$
$$+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left| -b_j(s) + h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right) - r_j(s) \right| ds$$
$$+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du \right) ds \leq \alpha, \tag{1.13}$$

where

$$H(t)=\sum_{j=1}^{N}h_j(t),\ r_j(t)=\frac{\left[c_j(t)H(t)+c_j'(t)\right]\left(1-\tau_j'(t)\right)+c_j(t)\tau_j''(t)}{\left(1-\tau_j'(t)\right)^2}. \tag{1.14}$$

Then the zero solution of (1.9) is asymptotically stable if and only if

$$\int_0^t H(s)\,ds\to\infty \text{ as } t\to\infty. \tag{1.15}$$

Obviously, Theorem 1.3 improves and generalizes Theorems 1.1 and 1.2. On the other hand, Dib, Maroun and Raffoul in [16] and Ardjouni and Djoudi in [3] considered the following nonlinear neutral differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -b_1(t)x(t-\tau_1(t))+\frac{d}{dt}Q_1(t,x(t-\tau_2(t)))\\ &\quad+G_1(t,x(t-\tau_1(t)),x(t-\tau_2(t))),\end{aligned} \tag{1.16}$$

where $Q_1(t,x)$ and $G_1(t,x,y)$ are globally Lipschitz continuous in x and in x and y, respectively. That is, there are positive constants E_1, E_2, E_3 such that

$$\begin{gathered}|Q_1(t,x)-Q_1(t,x)|\le E_1\left\|x-y\right\|,\\ \left|G_1(t,x,y)-G_1(t,z,w)\right|\le E_2\|x-z\|+E_3\left\|y-w\right\|,\end{gathered} \tag{1.17}$$

and

$$Q_1(t,0)=G_1(t,0,0)=0. \tag{1.18}$$

And obtained the following theorems.

Theorem 1.4. (Dib, Maroun and Raffoul [16]). *Suppose that* $\tau_1=0$ *and (1.17), (1.18) hold, and there exists a constant* $\alpha\in(0,1)$ *such that for* $t\ge0$, $\int_0^t b_1(s)\,ds\to\infty$ *as* $t\to\infty$, *and*

$$L_1+\int_0^t e^{-\int_s^t b_1(u)du}\left(E_1\left|b_1(s)\right|+L_2+L_3\right)ds\le\alpha, \tag{1.19}$$

Then every solution $x(t)=x(t,0,\psi)$ *of (1.16) with a small continuous initial function* ψ *is bounded and tends to zero as* $t\to\infty$.

Theorem 1.5. (Ardjouni and Djoudi [3]). *Suppose* (1.17) *and* (1.18) *hold. Let* τ_j *be differentiable, and suppose that there exists continuous functions* $h_j:\left[m_j(t_0),\infty\right)\to\mathbb{R}$ *for* $j=1,2$ *and a constant* $\alpha\in(0,1)$ *such that for* $t\ge0$,

$$\liminf_{t\to\infty}\int_0^t h(s)\,ds>-\infty, \tag{1.20}$$

and

$$E_1+\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}\left|h_j(s)\right|ds$$
$$+\int_0^t e^{-\int_s^t h(u)du}\left\{\left|-b(s)+h_1(s-\tau_1(s))\left(1-\tau_1'(s)\right)\right|\right.$$
$$\left.+\left|h_2(s-\tau_2(s))\left(1-\tau_2'(s)\right)\right|+E_1\left|h(s)\right|+E_2+E_3\right\}ds$$
$$+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t h(u)du}\left|h(s)\right|\left(\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\right)ds\ \le\ \alpha, \tag{1.21}$$

where $h(t)=\sum_{j=1}^{2}h_j(t)$*. Then the zero solution of (1.16) is asymptotically stable if and only if* $\int_0^t h(s)\,ds\to\infty$ *as* $t\to\infty$*.*

Obviously, Theorem 1.5 improves and generalizes Theorem 1.4.
Note that in our consideration the neutral term $\frac{d}{dt}Q(t,x(t-\tau_2(t)),...,x(t-\tau_N(t)))$ of *(1.1)* produces nonlinearity in the derivative term $\frac{d}{dt}x\left(t-\tau_j(t)\right)$. The neutral term $\frac{d}{dt}x\left(t-\tau_j(t)\right)$ in [1] enters linearly. So, the analysis made here is different form that in [1].

Our purpose here is to give, by using the contraction mapping principle, asymptotic stability results of a generalized nonlinear neutral differential equation with variable delays *(1.1)*. An asymptotic stability theorem with a necessary and sufficient condition is proved. Two examples are also given to illustrate our results. The results presented in this paper improve and generalize the main results in [3, 11, 16, 25].

2. Main Results

For each $(t_0,\psi)\in\mathbb{R}^+\times C([m(t_0),t_0],\mathbb{R})$, a solution of *(1.1)* through (t_0,ψ) is a continuous function $x:[m(t_0),t_0+\alpha)\to\mathbb{R}$ for some positive constant $\alpha>0$ such that x satisfies *(1.1)* on $[t_0,t_0+\alpha)$ and $x=\psi$ on $[m(t_0),t_0]$. We denote such a solution by $x(t)=x(t,t_0,\psi)$. For each $(t_0,\psi)\in\mathbb{R}^+\times C([m(t_0),t_0],\mathbb{R})$, there exists a unique solution $x(t)=x(t,t_0,\psi)$ of *(1.1)* defined on $[t_0,\infty)$. For fixed t_0, we define $\|\psi\|=\max\left\{\left|\psi(t)\right|:m(t_0)\le t\le t_0\right\}$. Stability definitions may be found in [9], for example.

Our aim here is to generalize Theorems 1.1, 1.2, 1.4, 1.5 to *(1.1)*.

Theorem 2.1. *Suppose that* τ_j *is twice differentiable and* $\tau_j'(t)\neq 1$ *for all* $t\in\mathbb{R}^+$*, and there exist continuous functions* $h_j:\left[m_j(t_0),\infty\right)\to\mathbb{R}$ *for* $j=1,2,...,N$ *and a constant*

$\alpha \in (0,1)$ such that for $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty, \tag{2.1}$$

and

$$\begin{aligned} &\sum_{j=1}^{N} K_j + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| ds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left\{ \left| -b_j(s) + h_j(s-\tau_1(s)) \left(1-\tau_j'(s)\right) \right| \right. \\ &\left. + K_j \left|H(s)\right| + L_j \right\} ds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du \right) ds \ \leq \ \alpha, \end{aligned} \tag{2.2}$$

where $H(t) = \sum\limits_{j=1}^{N} h_j(t)$. Then the zero solution of (1.1) is asymptotically stable if and only if

$$\int_0^t H(s)\,ds \to \infty \text{ as } t \to \infty. \tag{2.3}$$

Proof. First, suppose that (2.3) holds. For each $t_0 \geq 0$, we set

$$K = \sup_{t\geq 0} \left\{ e^{-\int_0^t H(s)ds} \right\}. \tag{2.4}$$

Let $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$\begin{aligned} S \;=\; &\{\varphi \in C([m(t_0), \infty), \mathbb{R}) : \varphi(t) \to 0 \text{ as } t \to \infty, \\ &\varphi(t) = \psi(t) \ \text{ for } t \in [m(t_0), t_0]\}. \end{aligned}$$

This S is a complete metric space with metric $\rho(x,y) = \sup_{t\geq m(t_0)} \left\{ \left| x(t) - y(t) \right| \right\}$.

Multiply both sides of (1.1) by $e^{\int_{t_0}^{t} H(u)du}$ and then integrate from t_0 to t to obtain

$$\begin{aligned} x(t) &= \left(\psi(t_0) - Q(t_0, \psi(t_0 - \tau_1(t_0)), ..., \psi(t_0 - \tau_N(t_0)))\right) e^{-\int_{t_0}^{t} H(u)du} \\ &+ Q(t, x(t-\tau_1(t)), ..., x(t-\tau_N(t))) + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} h_j(s)\, x(s)\, ds \\ &- \int_{t_0}^{t} e^{-\int_s^t H(u)du} \sum_{j=1}^{N} b_j(s)\, x\left(s - \tau_j(s)\right) ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{G(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s))) \\ &- H(s)\, Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s)))\}\, ds \end{aligned}$$

Performing an integration by parts, we have

$$\begin{aligned} & x(t) \\ = \; & \left(\psi(t_0) - Q(t_0, \psi(t_0 - \tau_1(t_0)), ..., \psi(t_0 - \tau_N(t_0)))\right) e^{-\int_{t_0}^{t} H(u)du} \\ & + Q(t, x(t-\tau_1(t)), ..., x(t-\tau_N(t))) \\ & + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} d\left(\int_{s-\tau_j(s)}^{s} h_j(u)\, x(u)\, du\right) \\ & + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right\} x\left(s - \tau_j(s)\right) ds \\ & + \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{G(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s))) \\ & - H(s)\, Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s)))\}\, ds \end{aligned}$$

Thus,

$$
\begin{aligned}
& x(t) \\
= \; & \Big\{ \psi(t_0) - Q(t_0, \psi(t_0 - \tau_1(t_0)), ..., \psi(t_0 - \tau_N(t_0))) \\
& - \sum_{j=1}^{N} \int_{t_0 - \tau_j(t_0)}^{t_0} h_j(s) \psi(s)\, ds \Big\} e^{-\int_{t_0}^{t} H(u)du} \\
& + Q(t, x(t - \tau_1(t)), ..., x(t - \tau_N(t))) + \sum_{j=1}^{N} \int_{t - \tau_j(t)}^{t} h_j(s) x(s)\, ds \\
& + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{ -b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right) \right\} x\left(s - \tau_j(s)\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{ G(s, x(s - \tau_1(s)), ..., x(s - \tau_N(s))) \\
& - H(s) Q(s, x(s - \tau_1(s)), ..., x(s - \tau_N(s))) \} ds \\
& - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s) \left(\int_{s - \tau_j(s)}^{s} h_j(u) x(u)\, du \right) ds.
\end{aligned}
\tag{2.5}
$$

Use *(2.5)* to define the operator $P : S \to S$ by $(P\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ and

$$
\begin{aligned}
(P\varphi)(t) = \; & \Big\{ \psi(t_0) - Q(t_0, \psi(t_0 - \tau_1(t_0)), ..., \psi(t_0 - \tau_N(t_0))) \\
& - \sum_{j=1}^{N} \int_{t_0 - \tau_j(t_0)}^{t_0} h_j(s) \psi(s)\, ds \Big\} e^{-\int_{t_0}^{t} H(u)du} \\
& + Q(t, \varphi(t - \tau_1(t)), ..., \varphi(t - \tau_N(t))) + \sum_{j=1}^{N} \int_{t - \tau_j(t)}^{t} h_j(s) \varphi(s)\, ds \\
& + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{ -b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right) \right. \\
& \left. - r_j(s) \right\} \varphi\left(s - \tau_j(s)\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t H(u)du} \{ G(s, \varphi(s - \tau_1(s)), ..., \varphi(s - \tau_N(s))) \\
& - H(s) Q(s, \varphi(s - \tau_1(s)), ..., \varphi(s - \tau_N(s))) \} ds \\
& - \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s) \left(\int_{s - \tau_j(s)}^{s} h_j(u) \varphi(u)\, du \right) ds,
\end{aligned}
\tag{2.6}
$$

for $t \geq t_0$. It is clear that $(P\varphi) \in C([m(t_0), \infty), \mathbb{R})$. We now show that $(P\varphi)(t) \to 0$ as $t \to \infty$. To this end, denote the six terms on th right hand side of *(2.6)* by $I_1, I_2, ..., I_6$, respectively. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (2.3). Also, due to the facts that $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ for $j = 1, 2, ..., N$ as $t \to \infty$, the second term I_2 in (2.6) tends to zero as $t \to \infty$. What is left to show is each of the remaining terms in (2.6) go to zero at infinity.

Let $\varphi \in S_\psi$ be fixed. For a given $\varepsilon > 0$, we choose $T_0 > 0$ large enough such that $t - \tau_j(t) \geq T_0$, $j = 1, 2, ..., N$, implies $\left|\varphi(s)\right| < \varepsilon$ if $s \geq t - \tau_j(t)$. Therefore, the third term I_3 in (2.6) satisfies

$$\begin{aligned}
|I_3| &= \left|\sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} h_j(s)\varphi(s)\,ds\right| \\
&\leq \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| \left|\varphi(s)\right| ds \\
&\leq \varepsilon \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| ds \leq \alpha\varepsilon < \varepsilon.
\end{aligned}$$

Thus $I_3 \to 0$ as $t \to \infty$. Now consider I_4. For the given $\varepsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $\left|\varphi\left(s - \tau_j(s)\right)\right| < \varepsilon$ for $j = 1, 2, ..., N$. Thus, for $t \geq T_1$, the term I_4 in (2.6) satisfies

$$\begin{aligned}
&|I_4| \\
&= \left|\sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right\} \varphi\left(s - \tau_j(s)\right) ds\right| \\
&\leq \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| \left|\varphi\left(s - \tau_j(s)\right)\right| ds \\
&\quad + \sum_{j=1}^{N} \int_{T_1}^{t} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| \left|\varphi\left(s - \tau_j(s)\right)\right| ds \\
&\leq \sup_{\sigma \geq m(t_0)} \left|\varphi(\sigma)\right| \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| ds \\
&\quad + \epsilon \sum_{j=1}^{N} \int_{T_1}^{t} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| ds.
\end{aligned}$$

By (2.6), we can find $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned}
& \sup_{\sigma\geq m(t_0)} \left|\varphi(\sigma)\right| \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right)\right| ds \\
= & \sup_{\sigma\geq m(t_0)} \left|\varphi(\sigma)\right| e^{-\int_{T_2}^t H(u)du} \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^{T_2} H(u)du} \\
& \times \left|-b_j(s) + h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right)\right| ds \ < \ \epsilon.
\end{aligned}$$

Now, apply (2.5) to have $|I_4| < \epsilon + \alpha\epsilon < 2\epsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, by using (1.4)–(1.6) and (2.6), then, if $t \geq T_2$ then the terms I_5 and I_6 in (2.6) satisfy

$$\begin{aligned}
|I_5| &= \left|\int_{t_0}^{t} e^{-\int_s^t H(u)du} G\left(s, \varphi\left(s-\tau_1(s)\right), ..., \varphi\left(s-\tau_N(s)\right)\right)\right. \\
&\quad \left. - H(s)\, Q\left(s, x\left(s-\tau_1(s)\right), ..., x\left(s-\tau_N(s)\right)\right) ds\right| \\
&\leq \sup_{\sigma\geq m(t_0)} \left|\varphi(\sigma)\right| e^{-\int_{T_2}^t H(u)du} \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^{T_2} H(u)du} \left(K_j\left|H(s)\right| + L_j\right) ds \\
&\quad + \epsilon \sum_{j=1}^{N} \int_{T_1}^{t} e^{-\int_s^t H(u)du} \left(K_j\left|H(s)\right| + L_j\right) ds \\
&< \epsilon + \alpha\epsilon < 2\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
|I_6| &= \left|\sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u)\, \varphi(u)\, du\right) ds\right| \\
&\leq \sup_{\sigma\geq m(t_0)} \left|\varphi(\sigma)\right| \sum_{j=1}^{N} \int_{t_0}^{T_1} e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du\right) ds \\
&\quad + \varepsilon \sum_{j=1}^{N} \int_{T_1}^{t} e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du\right) ds \\
&< \epsilon + \alpha\epsilon < 2\epsilon.
\end{aligned}$$

Thus, $I_5, I_6 \to 0$ as $t \to \infty$. In conclusion $(P\varphi)(t) \to 0$ as $t \to \infty$, as required. Hence P maps S into S. Also, by (2.2), P is a contraction mapping with contraction constant α. Indeed, for $\varphi, \eta \in S$ and $t \geq t_0$

$$\begin{aligned}
& \left|(P\varphi)(t) - (P\eta)(t)\right| \\
\leq \; & \left|Q\left(t, \varphi\left(t - \tau_1(t)\right), ..., \varphi\left(t - \tau_N(t)\right)\right) - Q\left(t, \eta\left(t - \tau_1(t)\right), ..., \eta\left(t - \tau_N(t)\right)\right)\right| \\
& + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} h_j(s) \left|\varphi(s) - \eta(s)\right| ds \\
& + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| \\
& \times \left|\varphi\left(s - \tau_j(s)\right) - \eta\left(s - \tau_j(s)\right)\right| ds \\
& + \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left\{\left|G\left(s, \varphi\left(s - \tau_1(s)\right), ..., \varphi\left(s - \tau_N(s)\right)\right)\right.\right. \\
& \left. - G\left(s, \eta\left(s - \tau_1(s)\right), ..., \eta\left(s - \tau_N(s)\right)\right)\right| \\
& + \left|H(s)\right| \left|Q\left(s, \varphi\left(s - \tau_1(s)\right), ..., \varphi\left(s - \tau_N(s)\right)\right)\right. \\
& \left.\left. - Q\left(s, \eta\left(s - \tau_1(s)\right), ..., \eta\left(s - \tau_N(s)\right)\right)\right|\right\} ds \\
& + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| \left|\varphi(u) - \eta(u)\right| du\right) ds. \\
\leq \; & \left(\sum_{j=1}^{N} K_j + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| ds\right. \\
& + \sum_{j=1}^{N} \int_{0}^{t} e^{-\int_s^t H(u)du} \left\{\left|-b_j(s) + h_j\left(s - \tau_1(s)\right)\left(1 - \tau_j'(s)\right)\right|\right. \\
& \left. + K_j \left|H(s)\right| + L_j\right\} ds \\
& \left. + \sum_{j=1}^{N} \int_{0}^{t} e^{-\int_s^t H(u)du} \left|H(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du\right) ds\right) \left\|\varphi - \eta\right\|.
\end{aligned}$$

By condition *(2.5)*, P is a contraction mapping with constant α. By the Contraction Mapping Principle (Smart [24], p. 2), P has a unique fixed point x in S which is a solution of *(1.1)* with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of *(1.1)* is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ $(\delta < \varepsilon)$ satisfying $2\delta K e^{\int_0^{t_0} H(u)du} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of *(1.1)* with $\left\|\psi\right\| < \delta$, then $x(t) = (Px)(t)$ defined in *(2.6)*. We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $m(t_0) \leq s < t^*$, then it follows from *(2.6)* that

$$\begin{aligned}
|x(t^*)| &\le \|\psi\|\left(1+\sum_{j=1}^{N}K_j+\sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}\left|h_j(s)\right|ds\right)e^{-\int_{t_0}^{t^*}H(u)du}\\
&\quad+\epsilon\sum_{j=1}^{N}K_j+\epsilon\sum_{j=1}^{N}\int_{t^*-\tau_j(t^*)}^{t^*}\left|h_j(s)\right|ds\\
&\quad+\epsilon\sum_{j=1}^{N}\int_{t_0}^{t^*}e^{-\int_s^{t^*}H(u)du}\left|-b_j(s)+h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right)\right|ds\\
&\quad+\epsilon\sum_{j=1}^{N}\int_{t_0}^{t^*}e^{-\int_s^{t^*}H(u)du}\left(K_j|H(s)|+L_j\right)ds\\
&\quad+\epsilon\sum_{j=1}^{N}\int_{t_0}^{t^*}e^{-\int_s^{t^*}H(u)du}|H(s)|\left(\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\right)ds\\
&\le 2\delta Ke^{\int_0^{t_0}H(u)du}+\alpha\varepsilon<\epsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, $|x(t)|<\varepsilon$ for all $t\ge t_0$, and the zero solution of *(1.1)* is stable. This shows that the zero solution of *(1.1)* is asymptotically stable if *(2.3)* holds.

Conversely, suppose *(2.3)* fails. Then by *(2.1)* there exists a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n}H(u)\,du=l$ for some $l\in\mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J\le\int_0^{t_n}H(u)\,du\le J, \tag{2.7}$$

for all $n\ge 1$. To simplify our expressions, we define

$$\begin{aligned}
\omega(s) &= \sum_{j=1}^{N}\Bigg[\left|-b_j(s)+h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right)\right|\\
&\qquad+K_j|H(s)|+L_j+|H(s)|\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\Bigg],
\end{aligned}$$

for all $s\ge 0$. By *(2.2)*, we have

$$\int_0^{t_n}e^{-\int_s^{t_n}H(u)du}\omega(s)\,ds\le\alpha. \tag{2.8}$$

This yields

$$\int_0^{t_n}e^{\int_0^{s}H(u)du}\omega(s)\,ds\le\alpha e^{\int_0^{t_n}H(u)du}\le e^J. \tag{2.9}$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds = \gamma, \tag{2.10}$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds < \delta_0/4K, \tag{2.11}$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \leq 1$.

By *(2.1)*, K in *(2.4)* is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of *(1.1)* with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\begin{aligned}&\psi(t_m) - Q(t_m, \psi(t_m - \tau_1(t_m)), ..., \psi(t_m - \tau_N(t_m)))\\ &-\sum_{j=1}^{N}\int_{t_m-\tau_j(t_m)}^{t_m} h_j(s)\psi(s)\,ds \;\geq\; \frac{1}{2}\delta_0.\end{aligned}$$

It follows from *(2.6)* with $x(t) = (Px)(t)$ that for $n \geq m$

$$\begin{aligned}&\left|x(t_n) - Q(t_n, x(t_n - \tau_1(t_n)), ..., x(t_n - \tau_N(t_n)))\right.\\ &\left.-\sum_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds\right|\\ \geq\;& \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)\,ds\\ =\;& \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\\ =\;& e^{-\int_{t_m}^{t_n} H(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right)\\ \geq\;& e^{-\int_{t_m}^{t_n} H(u)du}\left(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)\,ds\right)\\ \geq\;& \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)du} \geq \frac{1}{4}\delta_0 e^{-2J} > 0.\end{aligned} \tag{2.12}$$

On the other hand, if the zero solution of *(1.1)* is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and *(2.2)* holds, we have

$$x(t_n) - Q(t_n, x(t_n - \tau_1(t_n)), ..., x(t_n - \tau_N(t_n))) - \sum_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \to 0, \tag{2.13}$$

as $n \to \infty$, which contradicts *(2.12)*. Hence condition *(2.3)* is necessary for the asymptotic stability of the zero solution of *(1.1)*. The proof is complete. □

Remark 2.1. It follows from the first part of the proof of Theorem 2.1 that the zero solution of *(1.1)* is stable under *(2.1)* and *(2.2)*. Moreover, Theorem 2.1 still holds if *(2.2)* is satisfied for $t \geq t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

For the special case $Q(t, x_1, ..., x_N) = G(t, x_1, ..., x_N) = 0$, we can get

Corollary 2.1. *Suppose that τ_j is differentiable, and there exist continuous functions $h_j : \left[m_j(t_0), \infty\right) \to \mathbb{R}$ for $j = 1, 2, ..., N$ and a constant $\alpha \in (0, 1)$ such that for $t \geq 0$*

$$\liminf_{t \to \infty} \int_0^t H(s)\, ds > -\infty, \tag{2.14}$$

and

$$\begin{aligned} &\sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|h_j(s)\right| ds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} \left|-b_j(s) + h_j\left(s - \tau_j(s)\right)\left(1 - \tau_j'(s)\right)\right| ds \\ &+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} \left|h_j(u)\right| du\right) ds \leq \alpha, \end{aligned} \tag{2.15}$$

where $H(t) = \sum_{j=1}^{N} h_j(t)$. Then the zero solution of (1.9) is asymptotically stable if and only if

$$\int_0^t H(s)\, ds \to \infty \text{ as } t \to \infty. \tag{2.16}$$

Remark 2.2. When $h_j(s) = b_j\left(g_j(s)\right)$ for $j = 1, 2, ..., N$, Corollary 2.1 reduces to Theorem 1.2. When $N = 2$, $b_2(t) = 0$, $Q(t, x_1, x_2) = Q_1(t, x_2)$ and $G(t, x_1, x_2) = G_1(t, x_1, x_2)$, Theorem 2.1 reduces to Theorem 1.5. Therefore, Theorem 2.1 is a generalization of Theorem 1.5.

3. Two examples

In this section, we give two examples to illustrate the applications of Corollary 2.1 and Theorem 2.1.

Example 3.1. Consider the following linear delay differential equation

$$x'(t) = -b_1(t)\,x(t-\tau_1(t)) - b_2(t)\,x(t-\tau_2(t)), \tag{3.1}$$

where $\tau_1(t) = 0.271t$, $\tau_2(t) = 0.287t$, $b_1(t) = 1/(1.458t+2)$ and $b_2(t) = 1/(1.426t+2)$. Then the zero solution of *(3.1)* is asymptotically stable.

Proof. Choosing $h_1(t) = h_2(t) = 0.61/(t+1)$ in Corollary 2.1, we have $H(t) = 1.22/(t+1)$ and

$$\begin{aligned}\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}\left|h_j(s)\right|ds &= \int_{0.729t}^{t}\frac{0.61}{s+1}ds + \int_{0.713t}^{t}\frac{0.61}{s+1}ds\\ &= 0.61\ln\frac{t+1}{0.729t+1} + 0.61\ln\frac{t+1}{0.713t+1} < 0.3992,\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_s^t H(u)du}\,|H(s)|\left(\int_{s-\tau_j(s)}^{s}\left|h_j(u)\right|du\right)ds\\ &< \int_0^t e^{-\int_s^t (1.22/(u+1))du}\frac{1.22}{1+s}\times 0.3992ds < 0.3992,\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_s^t H(u)du}\left|-b_j(s) + h_j\left(s-\tau_j(s)\right)\left(1-\tau_j'(s)\right) - r_j(s)\right|ds\\ =\;& \frac{1}{2}\int_0^t e^{-\int_s^t (1.22/(u+1))du}\frac{1-1.22\times 0.729}{0.729s+1}ds\\ &+\frac{1}{2}\int_0^t e^{-\int_s^t (1.22/(u+1))du}\frac{1-1.22\times 0.713}{0.713s+1}ds\\ <\;& \frac{1}{2}\left(\frac{1-1.22\times 0.729}{1.22\times 0.729} + \frac{1-1.22\times 0.713}{1.22\times 0.713}\right)\\ &\times\int_0^t e^{-\int_s^t (1.22/(u+1))du}\frac{1.22}{s+1}ds \;<\; 0.137.\end{aligned}$$

It is easy to see that all the conditions of Corollary 2.1 hold for $\alpha = 0.3992 + 0.3992 + 0.137 = 0.9354 < 1$. Thus, Corollary 2.1 implies that the zero solution of *(3.1)* is asymptotically stable.

However, Theorem 1.2 cannot be used to verify that the zero solution of *(3.1)* is asymptotically stable. In fact, $b_1(g_1(t)) = 1/(2t+2)$, $b_2(g_2(t)) = 1/(2t+2)$, and

$q(t) = 1/(1+t)$. As $t \to \infty$,

$$\begin{aligned}\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} \left| b_j\left(g_j(s)\right)\right| ds &= \int_{0.729t}^{t} \frac{1}{2s+2} ds + \int_{0.713t}^{t} \frac{1}{2s+2} ds \\ &= \frac{1}{2}\ln\frac{t+1}{0.729t+1} + \frac{1}{2}\ln\frac{t+1}{0.713t+1} \\ &\to -\frac{1}{2}\ln(0.729 \times 0.713),\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^{2} \int_{0}^{t} e^{-\int_s^t q(u)du} \left|q(s)\right| \left(\int_{s-\tau_j(s)}^{s} \left| b_j\left(g_j(u)\right)\right| du\right) ds \\ = \;& \int_0^t e^{-\int_s^t (1/(u+1))du} \frac{1}{1+s} \left(\int_{0.729s}^{s} \frac{1}{2u+2} du + \int_{0.713s}^{s} \frac{1}{2u+2} du\right) ds \\ = \;& \frac{1}{2(t+1)} \int_0^t \left[2\ln(s+1) - \ln(0.729s+1) - \ln(0.713s+1)\right] ds \\ = \;& \ln(t+1) - \frac{t+1/0.729}{2(t+1)}\ln(0.729t+1) \\ & - \frac{t+1/0.713}{2(t+1)}\ln(0.713t+1) \\ \to \;& -\frac{1}{2}\ln(0.729 \times 0.713),\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^{2} \int_{0}^{t} e^{-\int_s^t q(u)du} \left|b_j(s)\right| \left|\tau_j'(s)\right| ds \\ = \;& \frac{1}{2(t+1)} \left[0.271 \int_0^t \frac{s+1}{0.729s+1} ds + 0.287 \int_0^t \frac{s+1}{0.713s+1} ds\right] \\ = \;& \frac{1}{2(t+1)} \left[\frac{0.271t}{0.729} - \left(\frac{0.271}{0.729}\right)^2 \ln(0.729t+1) \right. \\ & \left. + \frac{0.287t}{0.713} - \left(\frac{0.287}{0.713}\right)^2 \ln(0.713t+1)\right] \\ \to \;& \frac{1}{2}\left(\frac{0.271}{0.729} + \frac{0.287}{0.713}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\limsup_{t\geq 0}\left\{\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}\left|b_j\left(g_j(s)\right)\right|ds+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t q(u)du}\left|b_j(s)\right|\left|\tau_j'(s)\right|ds\right.\\&\left.+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t q(u)du}\left|q(s)\right|\left(\int_{s-\tau_j(s)}^{s}\left|b_j\left(g_j(u)\right)\right|du\right)ds\right\}\\=\;&-\ln(0.729\times 0.713)+\frac{1}{2}\left(\frac{0.271}{0.729}+\frac{0.287}{0.713}\right)\simeq 1.0415.\end{aligned}$$

In addition, the left-hand side of the following inequality is increasing in $t>0$, then there exists some $t_0>0$ such that for $t>t_0$,

$$\begin{aligned}&\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}\left|b_j\left(g_j(s)\right)\right|ds+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t q(u)du}\left|b_j(s)\right|\left|\tau_j'(s)\right|ds\\&+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t q(u)du}\left|q(s)\right|\left(\int_{s-\tau_j(s)}^{s}\left|b_j\left(g_j(u)\right)\right|du\right)ds\;>\;1.04.\end{aligned}$$

This implies that condition *(1.11)* does not hold. Thus, Theorem 1.2 cannot be applied to equation *(3.1)*. □

Example 3.2. Consider the following nonlinear neutral delay differential equation

$$\begin{aligned}\frac{d}{dt}x(t)\;&=\;-\sum_{j=1}^{2}b_j(t)\,x\left(t-\tau_j(t)\right)+\frac{d}{dt}Q\left(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right)\right)\\&\quad+G\left(t,x\left(t-\tau_1(t)\right),x\left(t-\tau_2(t)\right)\right),\end{aligned}\tag{3.2}$$

where $\tau_1(t)=0.221t$, $\tau_2(t)=0.217t$, $b_1(t)=1/(1.558t+2)$, $b_2(t)=1/(1.566t+2)$, $Q(t,x,y)=0.072\sin(x/2)+0.036\sin(y/3)$, $G(t,x,y)=0$. Then the zero solution of *(3.2)* is asymptotically stable.

Proof. Choosing $h_1(t)=h_2(t)=0.63/(t+1)$ in Theorem 2.1, we have $H(t)=1.26/(t+1)$ and

$$K_1=0.036,\;K_2=0.012,\;\sum_{j=1}^{2}K_j=0.048,\;L_1=L_2=0,\tag{3.3}$$

$$\begin{aligned}\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}\left|h_j(s)\right|ds\;&=\;\int_{0.779t}^{t}\frac{0.63}{s+1}ds+\int_{0.783t}^{t}\frac{0.63}{s+1}ds\\&=\;0.63\ln\frac{t+1}{0.779t+1}+0.63\ln\frac{t+1}{0.783t+1}<0.312,\end{aligned}$$

$$\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_{s}^{t}H(u)du}\left|H(s)\right|\left(\int_{s-\tau_{j}(s)}^{s}\left|h_{j}(u)\right|du\right)ds$$
$$< \int_{0}^{t}e^{-\int_{s}^{t}(1.26/(u+1))du}\frac{1.26}{s+1}\times 0.312ds<0.312,$$

and

$$\sum_{j=1}^{2}\int_{0}^{t}e^{-\int_{s}^{t}H(u)du}\left\{\left|-b_{j}(s)+h_{j}\left(s-\tau_{j}(s)\right)\left(1-\tau_{j}'(s)\right)\right|\right.$$
$$\left.+K_{j}\left|H(s)\right|+L_{j}\right\}ds$$
$$= \int_{0}^{t}e^{-\int_{s}^{t}(1.26/(u+1))du}\left\{\left|\frac{1}{2}\frac{1.26\times 0.779-1}{0.779s+1}\right|+\frac{0.036\times 1.26}{s+1}\right\}ds$$
$$+\int_{0}^{t}e^{-\int_{s}^{t}(1.26/(u+1))du}\left\{\left|\frac{1}{2}\frac{1.26\times 0.783-1}{0.783s+1}\right|+\frac{0.012\times 1.26}{s+1}\right\}ds$$
$$< \left(\frac{1}{2}\frac{1-1.26\times 0.779}{1.26\times 0.779}+\frac{1}{2}\frac{1-1.26\times 0.783}{1.26\times 0.783}+0.048\right)$$
$$\times\int_{0}^{t}e^{-\int_{s}^{t}(1.26/(u+1))du}\frac{1.26}{s+1}ds < 0.065.$$

It is easy to see that all the conditions of Theorem 2.1 hold for $\alpha = 0.048+0.312+0.312+0.065 = 0.737 < 1$. Thus, Theorem 2.1 implies that the zero solution of *(3.2)* is asymptotically stable. □

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