FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 4 (2020), 1049–1057
https://doi.org/10.22190/FUMI2004049S

SOME CHARACTERIZATIONS OF THREE-DIMENSIONAL f-KENMOTSU RICCI SOLITONS

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Abstract. The aim of the present paper is to give some characterizations of f-Kenmotsu Ricci soliton with a supporting example.

Keywords: f-Kenmotsu manifold; Ricci almost soliton; gradient Ricci soliton.

1. Introduction

The revolutionary concept of Ricci flow was introduced by Hamilton [5] in order to solve Poincare conjecture. The conjecture was fully solved by Perelman [11] using Hamilton's Ricci flow technique. After the work of Perelman, the study of Ricci flow has become an important topic in differential geometry. A Ricci flow is a weak parabolic heat type partial differential equation of the following form

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij}, \tag{1.1}$$

$$g(0) = g_0. \tag{1.2}$$

Here g_{ij} denotes the components of Riemannian metric g and S_{ij} denotes the components of Ricci tensor S. A Ricci soliton is a solution of the above equation which is constant up to diffeomorphism and scaling. A Ricci soliton on a Riemannian manifold is characterized by the equation

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0.$$

Here λ is a constant, called soliton constant and the vector field V is called soliton vector field. A Ricci soliton is called expanding, shrinking or steady while λ is positive, negative or zero. A Ricci soliton is called Ricci almost soliton if λ is

Received August 25, 2020; accepted October 07, 2020
2020 *Mathematics Subject Classification.* Primary 53 C25; Secondary 53 D 15.

considered as a function instead of a constant [12]. A Ricci soliton is called gradient Ricci soliton if the soliton vector field is gradient of a potential function [13]. The study of Ricci solitons on almost contact manifolds was first initiated by Ramesh Sharma [16]. The Ricci solitons on almost contact manifolds have been studied by several authors ([4], [13], [15]). Ricci soliton on (κ, μ) contact metric manifold has been studied by the present authors in [14]

The notion of Kenmotsu manifold was introduced by K. Kenmotsu and was subsequently generalized to f-Kenmotsu manifolds. For details we refer to [8] and [9]. Ricci solitons on Kenmotsu manifold have been studied in [6]. The notion of ϕ-Ricci symmetric manifolds was introduced by U. C. De and A. Sarkar [2]. The notion of ϕ-symmetric manifolds was introduced by T. Takahashi [17]. Later several authors studied ϕ-symmetric manifolds. Three dimensional quasi-Sasakian manifolds with cyclic parallel and η-parallel Ricci tensor have been studied by U. C. De and A. Sarkar [3].

The objective of the present paper is to give some characterizations of f-Kenmotsu manifolds with Ricci solitons and hence establish the relations between such manifolds with locally ϕ-symmetric manifolds and manifolds with cyclic parallel and η-parallel Ricci tensors.

The present paper is organised as follows:
After the introduction, we give will required preliminaries in Section 2. In Section 3, we will study three dimensional f-Kenmotsu manifolds admitting Ricci soliton. Section 4 contains a supporting example.

2. Preliminaries

An odd dimensional smooth manifold M is said to be an almost contact metric manifold, if there exists a (1,1) tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g on M such that [1]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi(X)) = 0. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y \in \chi(M)$. Such a manifold of dimension (2n+1) is denoted by M^{2n+1} (ϕ, ξ, η, g) . Also M^{2n+1} (ϕ, ξ, η, g) is called an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies

$$(\nabla_X \phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.3}$$

where $f \in C^\infty(\mathrm{M})$ is such that $df \wedge \eta = 0$ ([8], [9]). If $f = \beta$ is nonzero constant, then the manifold is a β-Kenmotsu manifold [7]. If $f = 0$, then the manifold is cosymplectic [7]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$. For an f-Kenmotsu manifold, it follows from (2.3)

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{2.4}$$

The condition $df \wedge \eta = 0$ holds only for dim $M \geq 5$ [10]. In a three dimensional f-Kenmotsu manifold, we have

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z \\ &- (\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\}, \end{aligned} \tag{2.5}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.6}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{2.7}$$

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, also R, S and r are Riemannian curvature tensor, Ricci curvature tensor and scalar curvature on M respectively [9]. From (2.5) and (2.6) we get

$$R(X,Y)\xi = -(f^2 + f')(\eta(Y)X - \eta(X)Y), \tag{2.8}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{2.9}$$

$$S(\xi,\xi) = -2(f^2 + f'), \tag{2.10}$$

$$Q\xi = -2(f^2 + f')\xi. \tag{2.11}$$

As a consequence of (2.4), we also have

$$(\nabla_X \eta)(Y) = fg(\phi X, \phi Y). \tag{2.12}$$

Also from (2.9) it follows that

$$S(\phi X, \phi Y) = S(X,Y) + 2(f^2 + f')\eta(X)\eta(Y) \tag{2.13}$$

for all vector fields $X, Y \in \chi(M)$.

An f-Kenmotsu manifold $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be ϕ-symmetric if its curvature tensor R bears the condition

$$\phi^2(\nabla_X R)(Y,Z)W = 0, \tag{2.14}$$

for all vector fields $X, Y, Z, W \in \chi(M)$ [17]. In particular, if X, Y, Z, W are orthogonal to ξ, then $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be locally ϕ-symmetric. An f-Kenmotsu manifold $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be ϕ-Ricci symmetric if its Ricci operator Q bears the condition

$$\phi^2(\nabla_X Q)Y = 0 \tag{2.15}$$

for all vector fields $X, Y \in \chi(M)$. If X and Y are orthogonal to ξ, then $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be locally ϕ-Ricci symmetric. It may be noted that ϕ-symmetric implies ϕ-Ricci symmetric, but the converse is not valid in general.

Ricci tensor S of a Riemannian manifold (M, g) is called η-parallel if

$$g((\nabla_X S)Y, Z) = 0$$

for all vector fields X, Y, Z tangent to M and orthogonal to ξ where g and ∇ denote Riemannian metric and Riemannian connection respectively.

Ricci tensor S of a Riemannian manifold (M, g) is called cyclic-parallel if

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{2.16}$$

for all vector fields X, Y, Z tangent to M. Here ∇ denotes Riemannian connection.

3. Three-dimensional f-Kenmotsu manifolds with Ricci soliton

In this section we prove the following:

Theorem 3.1. *In a three-dimensional f Kenmotsu Ricci soliton, if f is constant and the soliton vector field is Killing, then the soliton is expanding.*

Proof. For a three-dimensional f-Kenmotsu manifold, from (2.7), we get

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi. \tag{3.1}$$

Differentiating covariantly along Y and using (2.4) and (2.12) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= (\frac{dr(Y)}{2} + 2fdf(Y) + df'(Y))X + (\frac{r}{2} + f^2 + f')\nabla_Y X \\ &- (\frac{dr(Y)}{2} + 6fdf(Y) + 3df'(Y))\eta(X)\xi \\ &- (\frac{r}{2} + 3f^2 + 3f')fg(\phi X, \phi Y)\xi - (\frac{r}{2} + 3f^2 + 3f') \\ & \quad \eta(X)f(Y - \eta(Y)\xi).\end{aligned} \tag{3.2}$$

Taking inner product of (3.2) with Y we have

$$\begin{aligned}g((\nabla_Y Q)X, Y) &= (\frac{dr(Y)}{2} + 2fdf(Y) + df'(Y))g(X,Y) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_Y X, Y) \\ &- (\frac{dr(Y)}{2} + 6fdf(Y) + 3df'(Y))\eta(X)\eta(Y) \\ &- (\frac{r}{2} + 3f^2 + 3f')fg(\phi X, \phi Y)\eta(Y) \\ &- (\frac{r}{2} + 3f^2 + 3f')\eta(X)g(Y,Y)f \\ &+ (\frac{r}{2} + 3f^2 + 3f')\eta(X)(\eta(Y))^2 f.\end{aligned} \tag{3.3}$$

Let $\{e_1, e_2, \xi\}$ be an orthonormal ϕ-basis at any point of a tangent space. It is known that

$$div(Q)X = g((\nabla_{e_1} Q)X, e_1) + g((\nabla_{e_2} Q)X, e_2) + g((\nabla_{e_3} Q)X, e_3). \tag{3.4}$$

Using (3.3) in (3.4) we get

$$\begin{aligned} div(Q)X &= (\frac{dr(e_1)}{2} + 2f df(e_1) + df'(e_1))g(X, e_1) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_{e_1} X, e_1) \\ &- (\frac{dr(e_2)}{2} + 6f df(e_2) + 3df'(e_2))g(X, e_2) \\ &+ (\frac{r}{2} + 3f^2 + 3f')g(\nabla_{e_2} X, e_2) \\ &+ (\frac{dr(\xi)}{2} + 2f df(\xi) + df')\eta(X) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_{\xi} X, \xi) \\ &- (\frac{dr(\xi)}{2} + 2f df(\xi) + df')\eta(X). \end{aligned} \tag{3.5}$$

We know that $div(Q)X = \frac{1}{2}dr(X)$. Putting $X = \xi$ in (3.5) we obtain

$$\frac{1}{2}dr\xi = 2(\frac{r}{2} + f^2 + f')f - 4f df(\xi) - 2df'(\xi). \tag{3.6}$$

If f-Kenmotsu manifold admits Ricci soliton then

$$S(X, Y) = -\frac{1}{2}((\mathcal{L}_V g)(X, Y) - \lambda g(X, Y)). \tag{3.7}$$

If V is a Killing vector field, from (3.7) we get $r = -3\lambda =$ constant. Therefore, from (3.6)

$$(\frac{r}{2} + f^2 + f')f = 2f df(\xi) - df'(\xi). \tag{3.8}$$

If f is a non-zero constant then

$$r = -2f^2. \tag{3.9}$$

Consequently, $\lambda = \frac{2}{3}f^2$. This completes the proof. □

We know from [6] that a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, is locally ϕ-Ricci symmetric if and only if the scalar curvature is constant. So we get the following corollary

Corollary 3.1. *If a three-dimensional f-Kenmotsu manifold with constant f admits a Ricci soliton with Killing soliton vector field, then it is ϕ-Ricci symmetric, and hence ϕ-symmetric.*

Again we know from [6] that in a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, the Ricci tensor is η-parallel if and only if the scalar curvature is constant. Hence we get

Corollary 3.2. *If a three-dimensional f-Kenmotsu manifold with constant f admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is η-parallel.*

From [6] we know that a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, satisfies cyclic parallel Ricci tensor if and only if the scalar curvature is constant. So, we can state the following:

Corollary 3.3. *If a three-dimensional f-Kenmotsu manifold with constant f admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is cyclic parallel.*

4. Example

Example 4.1. Let $M = \{(u, v, w) \in R^3 : u, v, w(\neq 0) \in R\}$ be a Riemannian manifold, where (u, v, w) denotes the standard coordinates of a point in R^3. Let us suppose that

$$e_1 = 3w\frac{\partial}{\partial u}, \quad e_2 = 3w\frac{\partial}{\partial v}, \quad e_3 = -3w\frac{\partial}{\partial w} \tag{4.1}$$

are three linearly independent vector fields at each point of M and therefore it forms a basis for the tangent space $\chi(M)$. We also define the Riemannian metric g of the manifold M given by

$$g = \frac{1}{w^2}[du \odot du + dv \odot dv + dw \odot dw]. \tag{4.2}$$

Let η be the one form satisfying

$$\eta(U) = g(U, e_3) \tag{4.3}$$

for any $U \in \chi(M)$ and let ϕ be the (1, 1) tensor field defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$. By the linear properties of ϕ and g, we can easily verify the following relations

$$\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3 \tag{4.4}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{4.5}$$

for arbitrary vector fields $U, V \in \chi(M)$. This shows that $\xi = e_3$ the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M. If ∇ is the Livi-Civita connection with respect to the Riemannian metric g, then with the help of above, we can easily calculate that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 3e_1, \quad [e_2, e_3] = 3e_2. \tag{4.6}$$

Now we recall Koszul's formula as

$$\begin{aligned} 2g(\nabla_U V, W) &= U(g(V, W)) + V(g(W, X)) - W(g(U, V)) \\ &- g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]) \end{aligned}$$

for arbitrary vector fields $U, V, W \in \chi(M)$. Making use of Koszul's formula, we get the following:

$$\nabla_{e_2} e_3 = 3e_2 \quad \nabla_{e_2} e_2 = 3e_3 \quad \nabla_{e_2} e_1 = 0 \tag{4.7}$$

$$\nabla_{e_3} e_3 = 0 \quad \nabla_{e_3} e_2 = 0 \quad \nabla_{e_3} e_1 = 0 \tag{4.8}$$

$$\nabla_{e_1} e_3 = 3e_1 \quad \nabla_{e_1} e_2 = 0 \quad \nabla_{e_1} e_1 = 3e_3. \tag{4.9}$$

From the above calculation, it is clear that M satisies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = 3$ is a non-zero constant. Thus we conclude that M leads to an f-Kenmotsu manifold. Also $f^2 + f'$ is non-zero. This implies that M is a three-dimensional regular f-Kenmotsu manifold. We find the components of curvature tensor and Ricci tensor as follows:

$$R(e_2, e_3)e_3 = -3e_2, \qquad R(e_3, e_2)e_2 = -3e_3, \tag{4.10}$$

$$R(e_1, e_3)e_3 = -3e_1, \qquad R(e_3, e_1)e_1 = -3e_3, \tag{4.11}$$

$$R(e_1, e_2)e_2 = -3e_1, \qquad R(e_1, e_2)e_3 = 0, \tag{4.12}$$

$$R(e_2, e_1)e_1 = -3e_2, \qquad R(e_3, e_1)e_2 = 0, \tag{4.13}$$

$$S(e_1, e_1) = -6, \quad S(e_2, e_2) = -6, \quad S(e_3, e_3) = -6, \tag{4.14}$$

$$S(\phi e_1, \phi e_1) = -6, \quad S(\phi e_2, \phi e_2) = -6, \quad S(\phi e_3, \phi e_3) = -0, \tag{4.15}$$

$S(\phi e_i, \phi e_j) = 0$ for all $i, j = 1, 2, 3 (i \neq j)$. From the above consequence, it is clear that $\phi^2\{(\nabla_U Q)(V)\} = 0$ for all vector fields $U, V \in \chi(M)$. Hence M is locally ϕ-Ricci symmetric. From above we get $r = -18$, this implies the scalar curvature is constant. Moreover, $(\nabla_X S)(\phi e_i, \phi e_j) = 0$ for $X \in \chi(M) i, j = 1, 2, 3$. So M is η-parallel, cyclic parallel. This example is also satisfying the Ricci soliton equation if $\lambda = 6$. Hence $\lambda = \frac{2}{3} f^2$ is verified. So the soliton is expanding. Thus, Theorem 3.1 and the associated corollaries are verified by this example.

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