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ON CLASSICAL WEAKLY PRIME SUBMODULES

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Abstract. The aim of this paper is to introduce the concept of classical weakly prime submodules, which is the generalization of the notion of weakly classical prime submodules to modules over arbitrary noncommutative rings. We study some properties of classical weakly prime submodules and investigate their structure in different classes of modules. Also, the structure of such submodules of modules over duo rings is completely described. We investigate some properties of classical weakly prime submodules of multiplication modules.
Key words: weakly prime ideal, weakly prime submodule, classical prime submodule, weakly classical prime submodule, duo ring, multiplication module.

1. Introduction

Throughout this article, all rings are associative with identity element and all modules are unital. Anderson and Smith [1] studied weakly prime ideals for a commutative ring with identity. They defined a proper ideal P of a commutative ring R to be *weakly prime ideal* if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$; and then it is proved [1, Theorem 3] that the following statements are equivalent for an ideal P of a commutative ring R,

(a) P is weakly prime.

(b) for ideals A and B of R, $0 \neq AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

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For rings that are not necessarily commutative, it is clear that (b) does not imply (a). In [7], Hirano et al. said that a proper ideal P of a ring R is *weakly prime ideal* provided that $0 \neq IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, for any ideals I and J of R. Equivalently, P is weakly prime if $0 \neq aRb \subseteq P$, for some $a, b \in R$, then $a \in P$ or $b \in P$, see [7, Proposition 2].

Weakly prime submodules of modules over commutative rings were introduced by Ebrahimi Atani and Farzalipour in [6]. A proper submodule N of M is called a *weakly prime submodule* if $0 \neq am \in N$, for some $a \in R$ and $m \in M$, then $m \in N$ or $aM \subseteq N$.

Behbboodi and Koohi introduced weakly prime submodules in [5]. A proper submodule P of M is called a *weakly prime submodule* if whenever $K \subseteq M$ and $rRsK \subseteq P$, where $r, s \in R$, then either $rK \subseteq P$ or $sK \subseteq P$. If R is a commutative ring, then a proper submodule P of R-module M is a weakly prime submodule if and only if for any elements $a, b \in R$ and $m \in M$, $abm \in P$ implies $am \in P$ or $bm \in P$. It is also clear that each prime submodule is weakly prime but not conversely, see [5, Example 1]. This notion of weakly prime submodules has been extensively studied by Behboodi in [2, 3, 4], although in [2, 3], the notion of weakly prime submodules is named *"Classical prime submodule"*.

The concept of weakly classical prime submodules of modules over commutative rings were introduced by Mostafanasab, Tekir and Oral in [8]. A proper submodule N of an R-module M is called a *weakly classical prime submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.

For all submodules N and K of an R-module M, denote by $(N :_R K)$ the subset $\{a \in R \mid aK \subseteq N\}$ of R, which is an ideal of R. The annihilator of K, which is denoted by $Ann_R(K)$, is $(0 :_R K)$. If $Ann_R(K) = 0$, then K is called a *faithful submodule* of M. In particular, if $Ann_R(M) = 0$, then M is called a *faithful module.* We know that R is a domain if R has no left or right zero divisors and R is a right (left) duo ring if every right (left) ideal of R is an ideal.

In this paper, we introduce the concept of classical weakly prime submodules, which is the generalization of the notion of weakly classical prime submodules and weakly prime submodules. In the second section, we study some basic properties of classical weakly prime submodules. In the third section, we shall characterize the structure of classical weakly prime submodules of modules over duo rings. In the fourth section, we study some properties of these submodules of multiplication modules. Finally, we introduce the concept of fully classical weakly prime modules and study their structures.

2. Classical weakly prime submodules

Let R be a ring. If N is a submodule of an R-module M, we write $N \leq M$. Also, for each element $a \in R$, $\langle a \rangle$ denotes the principal ideal of R generated by a.

Definition 2.1. A proper submodule N of an R-module M is called a classical weakly prime submodule if whenever $r, s \in R$ and $K \leq M$ with $0 \neq rRsK \subseteq N$, then $rK \subseteq N$ or $sK \subseteq N$.

Theorem 2.1. *Let M be an R-module and N be a proper submodule of M. The following statements are equivalent:*

(1) *N is a classical weakly prime.*

(2) *For all ideals I and J of R and $K \leq M$, if $0 \neq IJK \subseteq N$, then either $IK \subseteq N$ or $JK \subseteq N$.*

Proof. **1** $\Rightarrow$ **2**. Let N be a classical weakly prime submodule of M. Suppose that $0 \neq IJK \subseteq N$, for some ideals I and J of R and some submodule K of M. If $IK \nsubseteq N$ and $JK \nsubseteq N$, then there are elements $a \in I$ and $b \in J$ such that $aK \nsubseteq N$ and $bK \nsubseteq N$. Since N is a classical weakly prime and $aRbK \subseteq IJK \subseteq N$, we must have $aRbK = 0$. Hence, $abK = 0$, and so $ab = 0$. In the following, we show that $IJK = 0$.

Now, we assume that $x \in I$ and $y \in J$. If $xK \nsubseteq N$ and $yK \nsubseteq N$, then by the above argument $xy = 0$. If $xK \nsubseteq N$ and $yK \subseteq N$, then $(b+y)K \nsubseteq N$, and hence $x(b+y) = 0$. Since $xb = 0$, we have $xy = 0$. Similarly, if $xK \subseteq N$ and $yK \nsubseteq N$, then $xy = 0$. If $xK \subseteq N$ and $yK \subseteq N$, then $(a+x)K \nsubseteq N$ and $(b+y)K \nsubseteq N$. Therefore, we must have $(a+x)b = a(b+y) = (a+x)(b+y) = ab = 0$, which implies that $xy = 0$. Hence, $IJK = 0$, a contradiction, and consequently $IK \subseteq N$ or $JK \subseteq N$.
2 $\Rightarrow$ **1**. Let $r, s \in R$ and $K \leq M$ such that $0 \neq rRsK \subseteq N$. Hence,

$$0 \neq \langle r \rangle \langle s \rangle K \subseteq N.$$

By assumption, $\langle r \rangle K \subseteq N$ or $\langle s \rangle K \subseteq N$. Therefore, $rK \subseteq N$ or $sK \subseteq N$. $\square$

Corollary 2.1. *Let M be an R-module and let N be a proper submodule of M. If $(N :_R K)$ is a weakly prime ideal of R, for every submodule K of M that is not contained in N, then N is a classical weakly prime submodule.*

Corollary 2.2. *Let M be an R-module and let N be a classical weakly prime submodule of M. Suppose that K is a submodule of M. Then, for every $a \in R$ and for all ideal I of R,*

(1) *if $0 \neq aIK \subseteq N$, then $aK \subseteq N$ or $IK \subseteq N$.*
(2) *if $0 \neq IaK \subseteq N$, then $aK \subseteq N$ or $IK \subseteq N$.*

However, by definition, the zero submodule is always classical weakly prime. Furthermore, it is clear that every weakly prime submodule is classical weakly prime, but a classical weakly prime submodule need not be weakly prime. The following gives a conterexample.

Example 2.1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$, where p and q are two distinct prime integers. Notice that $pq(\bar{1}, \bar{1}, 0) = (\bar{0}, \bar{0}, 0)$, but $p(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$ and $q(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$. Then the zero submodule of M is not weakly prime.

Definition 2.1 is the generalization of the notion of weakly classical prime submodules to modules over noncommutative rings. In fact, if M is a module over a commutative ring and if N is a proper submodule of M, then it follows from Theorem 2.1 and [8, Theorem 2.25] that N is a classical weakly prime submodule if and only if N is a weakly classical prime submodule.

It is clear that whenever $\{P_i\}_{i\in I}$ is a chain of classical weakly prime submodules of an R-module M, then $\cap_{i\in I}P_i$ is always a classical weakly prime submodule. Also, it is evidence that if $\cup_{i\in I}P_i \neq M$, then $\cup_{i\in I}P_i$ is a classical weakly prime submodule.

Theorem 2.2. *Let M be an R-module and let N be a classical weakly prime submodule of M. Then the following statements hold.*

(1) *If K is a faithful submodule of M that is not contained in N, then $(N :_R K)$ is a weakly prime ideal of R.*

(2) *If $Ann_R(M)$ is a weakly prime ideal of R, then $(N :_R M)$ is a weakly prime ideal of R.*

Proof. **1**. Let K be a faithful submodule of M such that $K \nsubseteq N$. It is clear that $(N :_R K)$ is a proper ideal of R. Now, we assume that $0 \neq IJ \subseteq (N :_R K)$, for some ideals I and J of R. Then $0 \neq IJK \subseteq N$ because K is faithful. Since N is a classical weakly prime submodule of M, we have $IK \subseteq N$ or $JK \subseteq N$ by Theorem 2.1. Therefore, $I \subseteq (N :_R K)$ or $J \subseteq (N :_R K)$, and consequently $(N :_R K)$ is a weakly prime ideal of R.
2. Since N is a proper submodule of M, $(N :_R M)$ is obviously a proper ideal of R. Let $Ann_R(M)$ be a weakly prime ideal of R and $0 \neq IJ \subseteq (N :_R M)$, for some ideals I and J of R. Hence, $IJM \subseteq N$. If $IJM = 0$, then $0 \neq IJ \subseteq Ann_R(M)$, and so

$$I \subseteq Ann_R(M) \subseteq (N :_R M) \qquad \textit{or} \qquad J \subseteq Ann_R(M) \subseteq (N :_R M)$$

because $Ann_R(M)$ is a weakly prime ideal of R. If $IJM \neq 0$, then it follows from Theorem 2.1 that $IM \subseteq N$ or $JM \subseteq N$, and so

$$I \subseteq (N :_R M) \qquad \textit{or} \qquad J \subseteq (N :_R M).$$

Therefore, $(N :_R M)$ is a weakly prime ideal of R. □

The following result is obtained immediately from Theorem 2.2.

Corollary 2.3. *Let M be an R-module and let N be a classical weakly prime submodule of M. For every $m \in M \setminus N$, if $Ann_R(Rm) = 0$, then $(N :_R Rm)$ is a weakly prime ideal of R.*

Theorem 2.3. *Let $f : M \to M'$ be a homomorphism of R-modules. Then the following statements hold.*

(1) *If f is a monomorphism and if N' is a classical weakly prime submodule of M' for which $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a classical weakly prime submodule of M.*

(2) *If f is an epimorphism and if N is a classical weakly prime submodule of M containing* $\ker f$*, then $f(N)$ is a classical weakly prime submodule of M'.*

Proof. **1**. Suppose that N' is a classical weakly prime submodule of M' such that $f^{-1}(N') \neq M$. If $0 \neq rRsK \subseteq f^{-1}(N')$, for some $r, s \in R$ and some submodule K of M, then $0 \neq f(rRsK) \subseteq N'$ because f is a monomorphism. Thus, $0 \neq rRsf(K) = f(rRsK) \subseteq N'$. Since N' is a classical weakly prime submodule of M', we have

$$f(rK) = rf(K) \subseteq N' \qquad \text{or} \qquad f(sK) = sf(K) \subseteq N'.$$

Therefore, $rK \subseteq f^{-1}(N')$ or $sK \subseteq f^{-1}(N')$, and so $f^{-1}(N')$ is a classical weakly prime submodule of M.
2. Assume that N is a classical weakly prime submodule of M. Let K' be a submodule of M' such that $0 \neq rRsK' \subseteq f(N)$, for some $r, s \in R$. If $K = f^{-1}(K')$, then $f(K) = K'$ because f is onto, and so

$$f(rRsK) = rRsK' \subseteq f(N).$$

Since $\ker f \subseteq N$, we have $0 \neq rRsK \subseteq N$, and consequently $rK \subseteq N$ or $sK \subseteq N$ because N is a classical weakly prime submodule. Hence, $rK' = f(rK) \subseteq f(N)$ or $sK' = f(sK) \subseteq f(N)$. Therefore, $f(N)$ is a classical weakly prime submodule of M'. $\square$

As an immediate consequence of Theorem 2.3(2), we have the following result.

Corollary 2.4. *Let M be an R-module and let $L \subset N$ be submodules of M. If N is a classical weakly prime submodule of M, then N/L is a classical weakly prime submodule of M/L.*

Theorem 2.4. *Let M be an R-module and let K and N be proper submodules of M with $K \subset N$. If K is a classical weakly prime submodule of M and if N/K is a classical weakly prime submodule of M/K, then N is a classical weakly prime submodule of M.*

Proof. Suppose that $0 \neq rRsL \subseteq N$, for some $r, s \in R$ and $L \leq M$. If $rRsL \subseteq K$, then $rL \subseteq K \subset N$ or $sL \subseteq K \subset N$ as desired. Thus, we assume that $rRsL \nsubseteq K$. Then $0 \neq (rRs)((L + K)/K) \subseteq N/K$, and so $r((L + K)/K) \subseteq N/K$ or $s((L + K)/K) \subseteq N/K$. It follows that $rL \subseteq N$ or $sL \subseteq N$. Therefore, N is a classical weakly prime submodule of M. $\square$

Definition 2.2. Let N be a proper submodule of an R-module M. Then, N is said to be a *weakly 2-absorbing submodule* of M if whenever $0 \neq rRsK \subseteq N$, where $r, s \in R$ and $K \leq M$, we have $rK \subseteq N$ or $sK \subseteq N$ or $rRs \subseteq (N :_R M)$.

It is evidence that every classical weakly prime submodule is weakly 2-absorbing submodule.

Proposition 2.1. *Let N be a proper submodule of an R-module M. If N is a weakly 2-absorbing submodule of M and if $(N :_R M)$ is a weakly prime ideal of R, then N is a classical weakly prime submodule.*

Proof. Suppose that N is a weakly 2-absorbing submodule of M and $(N :_R M)$ is a weakly prime ideal of R. Now, we assume that $0 \neq rRsK \subseteq N$, for some $r, s \in R$ and some submodule K of M. If $rK \nsubseteq N$ and $sK \nsubseteq N$, then $0 \neq rRs \subseteq (N :_R M)$ because N is a weakly 2-absorbing submodule. Since $(N :_R M)$ is a weakly prime ideal, we have $r \in (N :_R M)$ or $s \in (N :_R M)$. Hence, $rM \subseteq N$ or $sM \subseteq N$, a contradiction. This contradiction shows that N is a classical weakly prime submodule of M. $\square$

Definition 2.3. Let N be a classical weakly prime submodule of an R-module M. If there exist a submodule K of M and elements $a, b \in R$ such that $aRbK = 0$, $aK \nsubseteq N$ and $bK \nsubseteq N$, then (a, b, K) is called a *classical triple-zero of* N.

Proposition 2.2. *Let N be a classical weakly prime submodule of an R-module M and $aRbK \subseteq N$, for some $a, b \in R$ and some submodule K of M. If (a, b, K) is not a classical triple-zero of N, then $aK \subseteq N$ or $bK \subseteq N$.*

Proof. We assume that (a, b, K) is not a classical triple-zero of N. Therefore, if $aRbK = 0$, then $aK \subseteq N$ or $bK \subseteq N$. If $aRbK \neq 0$, then $aK \subseteq N$ or $bK \subseteq N$ because N is a classical weakly prime submodule. $\square$

Corollary 2.5. *Let N be a classical weakly prime submodule of an R-module M. If (a, b, K) is not a classical triple-zero of N, for all $a, b \in R$ and every submodule K of M, then N is a weakly prime submodule of M.*

Definition 2.4. Let N be a classical weakly prime submodule of an R-module M. Suppose that $IJK \subseteq N$, for some ideals I and J of R and some submodule K of M. We say that N is a *free classical triple-zero with respect to IJK* if (a, b, K) is not a classical triple-zero of N for all $a \in I$ and $b \in J$.

Remark 2.1. Let N be a classical weakly prime submodule of M and $IJK \subseteq N$, for some ideals I and J of R and some submodule K of M. If N is a free classical triple-zero with respect to IJK, then for all $a \in I$ and $b \in J$, we have $aK \subseteq N$ or $bK \subseteq N$.

Proposition 2.3. *Let N be a classical weakly prime submodule of an R-module M. Suppose that $IJK \subseteq N$, for some ideals I and J of R and some submodule K of M. If N is a free classical triple-zero with respect to IJK, then $IK \subseteq N$ or $JK \subseteq N$.*

Proof. Assume that N is a free classical triple-zero with respect to IJK. If $IK \nsubseteq N$ and $JK \nsubseteq N$, then there are elements $a \in I$ and $b \in J$ such that $aK \nsubseteq N$ and $bK \nsubseteq N$. Since $aRbK \subseteq IJK \subseteq N$ and N is free classical triple-zero with respect to IJK, we must have $aRbK \neq 0$. Therefore, by assumption $aK \subseteq N$ and $bK \subseteq N$ which is a contradiction. Consequently, $IK \subseteq N$ or $JK \subseteq N$. □

Theorem 2.5. *Let N be a classical weakly prime submodule of M. If (a, b, K) is a classical triple-zero of N, for some $a, b \in R$ and $K \leq M$, then the following statements are hold.*

(1) $aRbN = 0$.

(2) $a(N :_R M)K = 0$.

(3) $(N :_R M)bK = 0$.

(4) $(N :_R M)^2K = 0$.

(5) $a(N :_R M)N = 0$.

(6) $(N :_R M)bN = 0$.

Proof. Assume that (a, b, K) is a classical triple-zero of N, for some $a, b \in R$ and $K \leq M$.
1. If $aRbN \neq 0$, then there exists an element $x \in N$ such that $aRbx \neq 0$. Thus, $aRbRx \neq 0$. Set $L = K + Rx$. We may conclude that $0 \neq aRbL \subseteq N$. Hence, $aL \subseteq N$ or $bL \subseteq N$ because N is a classical weakly prime submodule, and so $aK \subseteq aL \subseteq N$ or $bK \subseteq bL \subseteq N$, a contradiction. Therefore, $aRbN = 0$.
2. If $a(N :_R M)K \neq 0$, then there exists an element $s \in (N :_R M)$ such that $asK \neq 0$. Thus, $aRsK \neq 0$. We can conclude that $0 \neq aR(b + s)K \subseteq N$ because $sM \subseteq N$. Since N is a classical weakly prime submodule, we have $aK \subseteq N$ or $(b + s)K \subseteq N$, and so $aK \subseteq N$ or $bK \subseteq N$. This contradicts the hypothesis. Therefore, $a(N :_R M)K = 0$.
3. The proof is similar to (2).
4. If $(N :_R M)^2K \neq 0$, then there exist elements $r, s \in (N :_R M)$ such that $rsK \neq 0$. Hence, by the assumption, (2) and (3), we have

$$0 \neq (a + r)R(b + s)K = rRsK \subseteq rM \subseteq N.$$

Therefore, $(a + r)K \subseteq N$ or $(b + s)K \subseteq N$ because N is a classical weakly prime submodule, and so $aK \subseteq N$ or $bK \subseteq N$, which is a contradiction. Consequently, $(N :_R M)^2K = 0$.
5. If $a(N :_R M)N \neq 0$, then there exists an element $s \in (N :_R M)$ such that $asN \neq 0$. It follows from (1) that $aR(b + s)N = aRsN \neq 0$. Thus, $aR(b + s)x \neq 0$, for some $x \in N$. Set $L = K + Rx$. We conclude that $0 \neq aR(b + s)L \subseteq N$ because $sM \subseteq N$. Since N is a classical weakly prime submodule, we have $aL \subseteq N$ or $(b + s)L \subseteq N$, and so $aK \subseteq aL \subseteq N$ or $bK \subseteq bL \subseteq N$. This contradicts the hypothesis. Hence, $a(N :_R M)N = 0$.
6. The proof is similar to (5). □

Theorem 2.6. *Let N be a classical weakly prime submodule of an R-module M. If N is not weakly prime, then $(N :_R M)^2 N = 0$.*

Proof. Suppose that N is not a weakly prime submodule of M. Thus, there exists a classical triple-zero (a, b, K) of N, for some $a, b \in R$ and $K \leq M$. Now, if $(N :_R M)^2 N \neq 0$, then there are elements $r, s \in (N :_R M)$ and $x \in N$ such that $rsx \neq 0$. Set $L = K + Rx$. Hence, by the assumption, (2) and (3) of Theorem 2.5, we conclude that

$$0 \neq (a + r)R(b + s)L \subseteq N.$$

Therefore, $(a + r)L \subseteq N$ or $(b + s)L \subseteq N$ because N is a classical weakly prime submodule, and so $aK \subseteq aL \subseteq N$ or $bK \subseteq bL \subseteq N$, which contradicts the fact that (a, b, K) is a classical triple-zero of N. Consequently, $(N :_R M)^2 N = 0$. □

Proposition 2.4. *Let R be a ring and let I be a proper ideal of R. Then the following conditions are equivalent:*

(1) *I is a weakly prime ideal of R.*

(2) *I is a classical weakly prime submodule of ${}_RR$.*

Proof. **1** $\Rightarrow$ **2**. It is clear.
2 $\Rightarrow$ **1**. Suppose that I is a classical weakly prime submodule of ${}_RR$. Then by Corollary 2.3, $(I :_R R1) = I$ is a weakly prime ideal of R. □

Theorem 2.7. *Let M_1 and M_2 be R-modules and $M = M_1 \times M_2$. If $N = N_1 \times M_2$ is a classical weakly prime submodule of M, for some submodule N_1 of M_1, then N_1 is a classical weakly prime submodule of M_1. Furthermore, for all $r, s \in R$ and $K_1 \leq M_1$, if $rRsK_1 = 0$, $rK_1 \nsubseteq N_1$ and $sK_1 \nsubseteq N_1$, then $rRs \subseteq Ann(M_2)$.*

Proof. Suppose that $N = N_1 \times M_2$ is a classical weakly prime submodule of M and $0 \neq rRsK_1 \subseteq N_1$, for some $r, s \in R$ and $K_1 \leq M_1$. Then, $(0, 0) \neq rRs(K_1 \times 0) \subseteq N$. Thus, $r(K_1 \times 0) \subseteq N$ or $s(K_1 \times 0) \subseteq N$, by assumption, and so $rK_1 \subseteq N_1$ or $sK_1 \subseteq N_1$. Consequently, N_1 is a classical weakly prime submodule of M_1.

We now assume that $rRsK_1 = 0$, for some $r, s \in R$ and submodule K_1 of M_1 such that $rK_1 \nsubseteq N_1$ and $sK_1 \nsubseteq N_1$. If $rRs \nsubseteq Ann_R(M_2)$, then there exists an element $x \in M_2$ such that $rRsx \neq 0$. Hence,

$$(0, 0) \neq rRs(K_1 \times Rx) \subseteq N.$$

Therefore, $r(K_1 \times Rx) \subseteq N$ or $s(K_1 \times Rx) \subseteq N$, by assumption, and so $rK_1 \subseteq N_1$ or $sK_1 \subseteq N_1$, which is a contradiction. Consequently, $rRs \subseteq Ann_R(M_2)$. □

3. Duo rings and classical weakly prime submodules

Let M be an R-module and let N be a submodule of M. For every $a \in R$, the subset $\{m \in M \mid am \in N\}$ of M is denoted by $(N :_M a)$. We recall that a ring

R is called a *left duo ring* if all left ideals of R is two sided ideal. It is easy to see that if R is a left duo ring, then $xR \subseteq Rx$, for each $x \in R$. Therefore, if M is a module over a left duo ring R, then for every submodule N of M and $a \in R$, the subset $(N :_M a)$ is a submodule of M containing N.

Theorem 3.1. *Let R be a left duo ring, let M be an R-module, and let N be a classical weakly prime submodule of M. If $0 \neq abm \in N$, for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$.*

Proof. Suppose that $0 \neq abm \in N$, for some $a, b \in R$ and $m \in M$. Thus, $Rabm \subseteq N$. Since R is a left duo ring, $Rab = RaRbR$, and so $0 \neq aRbRm \subseteq N$. Hence, $aRm \subseteq N$ or $bRm \subseteq N$ because N is a classical weakly prime submodule. Therefore, $am \in N$ or $bm \in N$. □

The following result follows from Theorem 3.1.

Corollary 3.1. *Let R be a left duo ring and let N be a classical weakly prime submodule of an R-module M. If $abm \in N$ and if (a, b, Rm) is not a classical triple-zero of N, for some $a, b \in R$ and some $m \in M$, then $am \in N$ or $bm \in N$.*

A submodule N of an R-module M is called *u-submodule* of M, provided that N contained in a finite union of submodules must be contained in one of them. M is called *u-module* if every submodule of M is a u-submodule, see [9].

Theorem 3.2. *Let R be a left duo ring and let M be a u-module over R. The following statements are equivalent for every proper submodule N of M.*

(1) *N is a classical weakly prime submodule.*

(2) *For each $m \in M$ and all $a, b \in R$, if $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.*

(3) *For all $a, b \in R$, one of the following holds:*

 i - $(N :_M ab) = (0 :_M ab)$

 ii - $(N :_M ab) = (N :_M a)$

 iii - $(N :_M ab) = (N :_M b)$.

(4) *For all $a, b \in R$ and every $K \leq M$, if $0 \neq abK \subseteq N$, then $aK \subseteq N$ or $bK \subseteq N$.*

(5) *For every $a \in R$ and every submodule K of M, if $aK \nsubseteq N$, then*

$$(N :_R aK) = (0 :_R aK) \quad \text{or} \quad (N :_R aK) = (N :_R K).$$

(6) *For every $a \in R$, every ideal I of R and every submodule K of M, if $0 \neq IaK \subseteq N$, then $aK \subseteq N$ or $IK \subseteq N$.*

(7) *For every ideal I of R and every submodule K of M, if $IK \nsubseteq N$, then*

$$(N :_R IK) = (0 :_R IK) \qquad \textit{or} \qquad (N :_R IK) = (N :_R K).$$

Proof. **1** $\Rightarrow$ **2**. It proved in Theorem 3.1.
2 $\Rightarrow$ **3**. Let $a, b \in R$. For every $m \in (N :_M ab)$, we have $abm \in N$. If $abm = 0$, then $m \in (0 :_M ab)$. If $abm \neq 0$, then by assumption $am \in N$ or $bm \in N$, and so $m \in (N :_M a)$ or $m \in (N :_M b)$. Hence,

$$(N :_M ab) \subseteq (0 :_M ab) \cup (N :_M a) \cup (N :_M b).$$

Therefore, one of

i- $(N :_M ab) \subseteq (0 :_M ab)$

ii- $(N :_M ab) \subseteq (N :_M a)$

iii- $(N :_M ab) \subseteq (N :_M b)$

holds because M is a u-module. Since R is a left duo ring, the conclusion follows.
3 $\Rightarrow$ **4**. Let $0 \neq abK \subseteq N$, for some $a, b \in R$ and $K \leq M$. Hence, $K \subseteq (N :_M ab)$ and $K \nsubseteq (0 :_M ab)$. Consequently, by assumption $K \subseteq (N :_M a)$ or $K \subseteq (N :_M b)$, and so $aK \subseteq N$ or $bK \subseteq N$.
4 $\Rightarrow$ **5**. Let $aK \nsubseteq N$, for some $a \in R$ and $K \leq M$. For every $r \in (N :_R aK)$, we have $raK \subseteq N$. If $raK = 0$, then $r \in (0 :_R aK)$. If $raK \neq 0$, then by assumption $rK \subseteq N$, and so $r \in (N :_R K)$. Hence, $(N :_R aK) \subseteq (0 :_R aK) \cup (N :_R K)$. Therefore, by hypothesis the conclusion is true.
5 $\Rightarrow$ **6**. Suppose that I is an ideal of R and K is a submodule of M such that $0 \neq IaK \subseteq N$, for some $a \in R$. Then, $I \subseteq (N :_R aK)$ and $I \nsubseteq (0 :_R aK)$. Hence, by assumption $aK \subseteq N$ or $I \subseteq (N :_R K)$, and so $aK \subseteq N$ or $IK \subseteq N$, as desired.
6 $\Rightarrow$ **1**. Let I and J be ideals of R and let K be a submodule of M such that $0 \neq IJK \subseteq N$. Assume that $JK \nsubseteq N$. Then there exists an element $a \in J$ such that $aK \nsubseteq N$. Furthermore, $IaK \subseteq IJK \subseteq N$. Now, if $IaK \neq 0$, then by assumption $IK \subseteq N$. If $IaK = 0$, then there exists an element $b \in J$ such that $IbK \neq 0$ because $IJK \neq 0$. Now, we assume that $bK \subseteq N$. Thus, $(a + b)K \nsubseteq N$ and $0 \neq I(a + b)K \subseteq IJK \subseteq N$. Hence, $IK \subseteq N$ by assumption. If $bK \nsubseteq N$, then $0 \neq IbK \subseteq N$ implies that $IK \subseteq N$. Therefore, N is a classical weakly prime submodule.
1 $\Rightarrow$ **7**. Let I be an ideal of R and let K be a submodule of M such that $IK \nsubseteq N$. For every $a \in (N :_R IK)$, we have $aIK \subseteq N$. If $aIK = 0$, then $a \in (0 :_R IK)$. If $aIK \neq 0$, then $0 \neq \langle a \rangle IK \subseteq N$. Since N is a classical weakly prime submodule and $IK \nsubseteq N$, we have $aK \subseteq \langle a \rangle K \subseteq N$. Hence, $a \in (N :_R K)$. Therefore,

$$(N :_R IK) \subseteq (0 :_R IK) \cup (N :_R K).$$

Consequently, $(N :_R IK) \subseteq (0 :_R IK)$ or $(N :_R IK) \subseteq (N :_R K)$. Obviously, the conclusion is true.

7 ⇒ 1. Let $0 \neq IJK \subseteq N$, for some ideals I and J of R and $K \leq M$. Then, $I \subseteq (N :_R JK)$ and $I \nsubseteq (0 :_R JK)$. Assume that $JK \nsubseteq N$. By assumption, $(N :_R JK) = (N :_R K)$, and so $I \subseteq (N :_R K)$. Hence, $IK \subseteq N$. Consequently, N is a classical weakly prime submodule. □

Remark 3.1. Let R be a left duo ring and let I be an ideal of R. It is easily seen that the subset $\{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$ of R is an ideal of R containing I, denoted by $\sqrt{I}$.

Proposition 3.1. *Let N be a classical weakly prime submodule of an R-module M that is not weakly prime. Then the following statements hold.*

(1) $(N :_R M)^3 \subseteq Ann_R(M)$.

(2) *If R is a left duo ring, then* $\sqrt{Ann_R(M)} = \sqrt{(N :_R M)}$.

Proof. **1**. By Theorem 2.6, $(N :_R M)^2 N = 0$. Then

$$\begin{aligned} (N :_R M)^3 &= (N :_R M)^2 (N :_R M) \\ &\subseteq ((N :_R M)^2 N :_R M) \\ &= (0 :_R M) = Ann_R(M). \end{aligned}$$

2. By the above remark,

$$\sqrt{Ann_R(M)} = \{r \in R \mid r^n \in \sqrt{Ann_R(M)} \text{ for some } n \in \mathbb{N}\}.$$

Hence, it follows from (1) that $(N :_R M) \subseteq \sqrt{Ann_R(M)}$. Since $Ann_R(M) \subseteq (N :_R M)$, we have $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$. □

4. Classical weakly prime submodules of a multiplication module

Let R be a ring. An R-module M is called a *multiplication module* if every submodule N of M has the form IM, for some ideal I of R, see [10]. We know that M is a multiplication R-module if and only if $N = (N :_R M)M$, for every submodule N of M.

Proposition 4.1. *Let M be a multiplication R-module and let N be a proper submodule of M. If $(N :_R M)$ is a weakly prime ideal of R, then N is a classical weakly prime submodule.*

Proof. Let $(N :_R M)$ be a weakly prime ideal of R. Suppose that $0 \neq IJK \subseteq N$, for some ideals I and J of R and $K \leq M$. Since M is a multiplication module, there exists an ideal L of R such that $K = LM$. Then, $0 \neq IJLM \subseteq N$, and so $0 \neq IJL \subseteq (N :_R M)$. Since $(N :_R M)$ is a weakly prime ideal, $I \subseteq (N :_R M)$ or $JL \subseteq (N :_R M)$, and hence $IK \subseteq IM \subseteq N$ or $JK = JLM \subseteq N$. Therefore, N is a classical weakly prime submodule of M. □

The following result is a direct consequence of Theorem 2.2 and Proposition 4.1.

Corollary 4.1. *Let M be a faithful multiplication R-module and let N be a proper submodule of M. Then, N is a classical weakly prime submodule if and only if $(N :_R M)$ is a weakly prime ideal of R.*

Also, the following result is obtained from Theorem 2.6.

Corollary 4.2. *Let M be a faithful multiplication R-module and let N be a classical weakly prime submodule of M. If N is not a weakly prime submodule of M, then $(N :_R M)^3 = 0$.*

Proposition 4.2. *Let M be a faithful multiplication R-module and let N be a proper submodule of M. Then the following conditions are equivalent:*

(1) *N is a classical weakly prime submodule.*

(2) *$(N :_R M)$ is a weakly prime ideal of R.*

(3) *$N = PM$, where P is a weakly prime ideal and it is maximal with respect to this property (i.e., $IM \subseteq N$ implies that $I \subseteq P$).*

Proof. **1** $\Leftrightarrow$ **2**. It follows from Corollary 4.1.
2 $\Rightarrow$ **3**. Since M is multiplication, $N = (N :_R M)M$. By hypothesis, $P := (N :_R M)$ is a weakly prime ideal. We now assume that $N = IM$, for some ideal I of R. It is clear that $I \subseteq (N :_R M) = P$.
3 $\Rightarrow$ **2**. Let $N = PM$, where P is a weakly prime ideal of R. Then, $P \subseteq (N :_R M)$. Since M is a multiplication module, we have $N = (N :_R M)M$. It follows from maximality of P that $(N :_R M) \subseteq P$. Therefore, $(N :_R M) = P$ is a weakly prime ideal of R. □

Proposition 4.3. *Let M be a multiplication R-module and let N be a proper submodule of M. Then the following conditions are equivalent:*

(1) *N is a classical weakly prime submodule of M.*

(2) *If $0 \neq IK \subseteq N$, for some ideal I of R and submodule K of M, then $K \subseteq N$ or $IM \subseteq N$.*

Proof. **1** $\Rightarrow$ **2**. Suppose that N is a classical weakly prime submodule of M and $0 \neq IK \subseteq N$, for some ideal I of R and submodules K of M. Since M is multiplication, there is an ideal J of R such that $K = JM$. Hence, $0 \neq IJM = IK \subseteq N$. Therefore, $IM \subseteq N$ or $K = JM \subseteq N$ by Theorem 2.1.
2 $\Rightarrow$ **1**. Suppose that $0 \neq IJK \subseteq N$, for some ideals I and J of R and submodule K of M. We set $N_1 := JK$. Hence, $0 \neq IN_1 = IJK \subseteq N$, and so $JK = N_1 \subseteq N$ or $IK \subseteq IM \subseteq N$ by hypothesis. By Theorem 2.1, N is a classical weakly prime submodule of M. □

5. Fully classical weakly prime modules

Recall that R is a *fully weakly prime ring* if every proper ideal of R is weakly prime, see [7]. We call an R-module M a *fully classical weakly prime module* if every proper submodule of M is a classical weakly prime submodule. A ring R is called a *fully classical weakly prime ring* if R itself is a fully classical weakly prime left R-module. For example, every module over a simple ring R is fully classical weakly prime module.

Theorem 5.1. *Let R be a ring. An R-module is fully classical weakly prime if and only if R is a fully weakly prime ring.*

Proof. $\Rightarrow$. Suppose that I is a proper ideal of R. Then, I is classical weakly prime by Proposition 2.4.
$\Leftarrow$. Let M be an R-module and $N \leq M$. Suppose that $0 \neq IJK \subseteq N$, for some ideals I and J of R and $K \leq M$. Thus, $0 \neq IJ \subseteq Ann((K+N)/N)$. Since $Ann((K+N)/N)$ is weakly prime, $I \subseteq Ann((K+N)/N)$ or $J \subseteq Ann((K+N)/N)$. Therefore, $IK \subseteq N$ or $JK \subseteq N$, and so N is a classical weakly prime submodule of M. Consequently, M is a fully classical weakly prime module. □

Proposition 5.1. *Let M be an R-module. Then M is a fully classical weakly prime module if and only if for each submodule K of M and for all ideals I and J of R,*

$$IJK = 0 \quad \text{or} \quad IJK = JK \subseteq IK \quad \text{or} \quad IJK = IK \subseteq JK.$$

Proof. Suppose that every submodule of M is classical weakly prime. Let I and J be ideals of R and $K \leq M$. If $IJK \neq M$, then IJK is a classical weakly prime submodule by assumption. If $IJK \neq 0$, then

$$IK \subseteq IJK \subseteq IK \quad \text{or} \quad JK \subseteq IJK \subseteq JK,$$

and so $IK = IJK \subseteq JK$ or $JK = IJK \subseteq IK$. If $IJK = M$, then $IK = JK = M$.

Conversely, we assume that N is a proper submodule of M. If $0 \neq IJK \subseteq N$, for some ideals I and J of R and $K \leq M$, then

$$IK = IJK \subseteq N \quad \text{or} \quad JK = IJK \subseteq N.$$

Therefore, N is a classical weakly prime submodule of M. □

Corollary 5.1. *Let M be a fully classical weakly prime R-module. Then for each submodule K of M and each ideal I of R ,either $I^2K = 0$ or $I^2K = IK$.*

Remark 5.1. Let M be an R-module and let N be a maximal submodule of M. If K is a submodule of M such that $K \nsubseteq N$, then $N + K = M$. We now assume that $IK \subseteq N$, for some ideal I of R. Therefore, $IM = IN + IK \subseteq N$, and so N is a prime submodule of M.

Proposition 5.2. *Let M be a multiplication R-module. If M is a fully classical weakly prime module, then M has at most two maximal submodules.*

Proof. Suppose that M has more than two distinct maximal submodules. Let N_1, N_2 and N_3 be three distinct maximal submodules of M. Since M is multiplication, there is an ideal I of R such that $N_1 = IM$. If $IN_2 = 0$, then $IN_2 \subseteq N_3$. By Remark 5.1, N_3 is a prime submodule of M. Hence, $N_1 = IM \subseteq N_3$ or $N_2 \subseteq N_3$, which contradicts the maximality of N_1 or N_2. Thus, $IN_2 \neq 0$. Since $IN_2 \subseteq N_2$ and $IN_2 \subseteq IM = N_1$, we have $0 \neq IN_2 \subseteq N_1 \cap N_2$, and so by Proposition 4.3, $N_1 = IM \subseteq N_1 \cap N_2$ or $N_2 \subseteq N_1 \cap N_2$ because $N_1 \cap N_2$ is a classical weakly prime submodule by assumption. Therefore, $N_2 \subseteq N_1$ or $N_1 \subseteq N_2$, a contradiction. $\square$

Corollary 5.2. *Let M be a multiplication and fully classical weakly prime R-module. If $N_1 = IM$ and $N_2 = JM$ are two distinct submodules of M, then N_1 and N_2 are comparable by inclusion or $IN_2 = JN_1 = 0$. In particular, if N_1 and N_2 are two distinct maximal submodules, then $IN_2 = JN_1 = 0$.*

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