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# ON ESTIMATES FOR THE GENERALIZED DUNKL TRANSFORM AND TITCHMARSH'S THEOREM IN THE SPACE $L^p_{\alpha,Q}(\mathbb{R})$ $(1 < p \leq 2)$

**Akhlidj Abdellatif, Daher Radouan, Afaf Dahani and Mohamed El Hamma**

**Faculty of Science Ain Chock, Department of Mathematics and Informatics**
**Laboratoire Mathématiques Fondamentales et Appliquées**
**Hassan II University of Casablanca, B.P 5366 Mâarif, Morocco**

**Abstract.** In this paper, we study two estimates useful in applications and proved for the generalized Dunkl transform in the space $L^p_{\alpha,Q}(\mathbb{R})$ where $1 < p \leq 2$ and $\alpha > \frac{-1}{2}$, as applied to some classes of functions characterized by a generalized modulus of continuity. Also, we extend two interesting E.C. Titchmarsh's theorems with the higher order at same space.
**Key words:** Generalized Dunkl transform, differential operator, Spaces of measurable functions.

## 1. Introduction and preliminaries

Consider the first order singular differential difference operator on the real line

$$\Lambda_{\alpha,Q}f(x) = f^{'}(x) + \left(\alpha + \frac{1}{2}\right)\left(\frac{f(x) - f(-x)}{x}\right) + q(x)f(x) \tag{1.1}$$

where $\alpha > \frac{-1}{2}$, $q$ is a $C^\infty$ real valued odd function on $\mathbb{R}$ and $Q(x) = exp\left(-\int_0^x q(t)dt\right)$. $Q$ is an even function. For $q = 0$, we obtain the classical Dunkl operator:

$$\Lambda_{\alpha}f(x) = f^{'}(x) + \left(\alpha + \frac{1}{2}\right)\left(\frac{f(x) - f(-x)}{x}\right). \tag{1.2}$$

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Corresponding Author: Akhlidj Abdellatif, Faculty of Science Ain Chock, Department of Mathematics and Informatics, Laboratoire Mathématiques Fondamentales et Appliquées, Casablanca, Maroc | E-mail: akhlidj@hotmail.fr
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We denote by $\Lambda^*_{\alpha,Q}$ the dual operator of $\Lambda_{\alpha,Q}$ given by (see[4])

$$\Lambda^*_{\alpha,Q} f(x) = f^{'}(x) + \left(\alpha + \frac{1}{2}\right)\left(\frac{f(x) - f(-x)}{x}\right) - q(x)f(x). \tag{1.3}$$

$L^p_\alpha(\mathbb{R}) = L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$ the space of measurable functions $f$ on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}} < +\infty \tag{1.4}$$

$L^p_{\alpha,Q}(\mathbb{R})$ the space of measurable functions $f$ on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,Q} = \|Qf\|_{p,\alpha} < +\infty. \tag{1.5}$$

The map defined by $\mathcal{M}_Q(f) = Qf$ is an isometric isomorphism from $L^p_{\alpha,Q}(\mathbb{R})$ onto $L^p_\alpha(\mathbb{R})$ (see[4]).
For each $\lambda \in \mathbb{C}$, there is differential-difference equation:

$$\Lambda_{\alpha,Q} u = i\lambda u \tag{1.6}$$

with the initial condition $u(0) = 1$ admits an unique $C^\infty$ solution on $\mathbb{R}$, denoted by $\psi_\lambda$ given by

$$\psi_\lambda(x) = Q(x) e_\alpha(i\lambda x) \tag{1.7}$$

where

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2\alpha+2} j_{\alpha+1}(iz), z \in \mathbb{C} \tag{1.8}$$

is the kernel Dunkl function.
$j_\alpha$ being the normalized spherical Bessel function given by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}, z \in \mathbb{C} \tag{1.9}$$

The function $j_\alpha$ is infinitely differentiable and even.

**Definition 1.1.** (see [4]) The Dunkl transform for a function $f \in L^1_\alpha(\mathbb{R})$ is defined by

$$\mathcal{F}^D_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) e_\alpha(-i\lambda x) |x|^{2\alpha+1} dx \tag{1.10}$$

The generalized Dunkl transform for a function $f \in L^1_{\alpha,Q}(\mathbb{R})$ is defined by

$$\mathcal{F}^D_{\alpha,Q}(f)(\lambda) = \int_{\mathbb{R}} f(x) \psi_{-\lambda}(x) |x|^{2\alpha+1} dx \tag{1.11}$$

**Lemma 1.1.** (see [4]) *We have the formulas*

*i)* $\mathcal{M}_Q \circ \Lambda^*_{\alpha,Q} = \Lambda_\alpha \circ \mathcal{M}_Q$

*ii)* $\mathcal{F}^D_{\alpha,Q} = \mathcal{F}^D_\alpha \circ \mathcal{M}_Q$ *where* $\mathcal{F}^D_\alpha$ *be the classical Dunkl transform.*

**Definition 1.2.** (see [4]) Let $p^{'}$ such that $\frac{1}{p} + \frac{1}{p^{'}} = 1$. The generalized translation operator is defined in $L^{p^{'}}_{\alpha,\frac{1}{Q}}(\mathbb{R})$ by

$$\mathcal{T}_{h,\alpha,Q} = Q(h)\mathcal{M}^{-1}_{\frac{1}{Q}} \circ \tau_{h,\alpha} \circ \mathcal{M}_{\frac{1}{Q}}, \tag{1.12}$$

where $\tau_{h,\alpha}$ is the translation operator associated to the classical Dunkl harmonic analysis.

**Remark 1.1.** The generalized dual translation operator is defined in $L^p_{\alpha,Q}(\mathbb{R})$ and is given by

$$\mathcal{T}^*_{h,\alpha,Q} = Q(h)\mathcal{M}^{-1}_{Q} \circ \tau_{-h,\alpha} \circ \mathcal{M}_{Q} \tag{1.13}$$

From [7, Theorem 2.1] we have the analog V.A. Abilov, F. V. Abilova and M. K. Kerimov's theorem for the Dunkl transform proved by M. El Hamma, R. Daher and M. Boujddaine.
Let us first recall the first and higher order finite difference which is defined as follows:
Let $f \in L^p_\alpha(\mathbb{R})$ and $k = 0, 1, 2...$

$$\Delta^0_{h,\alpha} f = f$$
$$\Delta_{h,\alpha} f = \tau_{h,\alpha} f - f = (\tau_{h,\alpha} - \mathcal{I})f$$
$$\Delta^k_{h,\alpha} f = \Delta_{h,\alpha}\left(\Delta^{k-1}_{h,\alpha}\right) f = (\tau_{h,\alpha} - \mathcal{I})^k f$$

the $k^{th}$ order modulus of continuity of a function $f \in L^p_\alpha(\mathbb{R})$ is defined as

$$\Omega_{k,\alpha}(f,\delta) = sup_{|h|\le\delta} \|\Delta^k_{h,\alpha} f\|_{p,\alpha} \tag{1.14}$$

We denote by

$$\begin{aligned} \mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha) \quad = \quad & \big\{f \in L^p_\alpha(\mathbb{R}) / \forall j \in \{1,2,..,r\} (\Lambda_\alpha)^j f \in L^p_\alpha(\mathbb{R}) \\ & \text{and } \ \Omega_{k,\alpha}((\Lambda_\alpha)^r f, \delta) = \mathcal{O}(\Psi(\delta^k) \text{ as } \delta \to 0\big\} \end{aligned}$$

The Sobolev space constructed by the operator $\Lambda_\alpha$, where $\Psi$ is any nonnegative function given on $[0, +\infty[$.

**Theorem 1.1.** [7] *Let* $f \in L^p_\alpha(\mathbb{R})$ *in the class* $\mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha)$

$$\int_{|\lambda|\ge N} |\mathcal{F}^D_\alpha(f)(\lambda)|^{p^{'}} |\lambda|^{2\alpha+1} d\lambda = \mathcal{O}\left(N^{-rp^{'}} (\Psi)^{p^{'}} \left( (\frac{c}{N})^k \right)\right) \text{ as } N \to \ +\infty$$

*where* $r = 0, 1, ...;$ $k = 1, 2, ...;$ $c > 0$ *is a fixed constant,* $\frac{1}{p} + \frac{1}{p^{'}} = 1$ *and* $\Psi$ *is any nonnegative function defined on the interval* $[0, +\infty[$.

From [11, Theorem 84], we have E.C. Titchmarsh's theorem for the Fourier transform

**Theorem 1.2.** [11] *Let $f \in L^p(\mathbb{R})$ $(1 < p \leq 2)$, and let*

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p = \mathcal{O}(h^{\gamma p}) \text{ as } h \to 0; \quad (0 < \gamma \leq 1). \tag{1.15}$$

*Then $\mathcal{F}(f) \in L^\beta(\mathbb{R})$ for $\frac{p}{p+\gamma p-1} < \beta \leq \frac{p}{p-1}$ where $\mathcal{F}(f)$ stands for the Fourier transform of $f$.*

From [6] we have an analog of E. C. Titchmarsh's Theorem for the Dunkl transform on the real line proved by M. El Hamma and R. Daher.

**Theorem 1.3.** [6] *Let $f \in L^p_\alpha(\mathbb{R})$, and let*

$$\|\Delta_{h,\alpha} f\|_{p,\alpha} = \mathcal{O}(h^\gamma) \quad \text{as} \quad h \to 0 \quad (0 < \gamma \leq 1) \tag{1.16}$$

*Then $\mathcal{F}^D_\alpha(f) \in L^\beta_\alpha(\mathbb{R})$ for*

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \gamma p - 2} < \beta \leq \frac{p}{p-1} \tag{1.17}$$

From [5] we have the generalized E. C. Titchmarsh's theorem.

**Theorem 1.4.** [5] *Let $f \in L^p_\alpha(\mathbb{R})$ such that for all $j \in \{1, 2, ..., r\}$ we have $\Lambda^j_\alpha f \in L^p_\alpha(\mathbb{R})$ and $\|\Delta^k_h \Lambda^r_\alpha f\|_{p,\alpha} = \mathcal{O}(h^\gamma)$ as $h \to 0$ $(0 < \gamma \leq k)$. Then $\mathcal{F}^D_\alpha(f) \in L^\beta_\alpha(\mathbb{R})$ for*

$$\frac{2\alpha + 2p}{2p + 2\alpha(p-1) - 2 + \gamma p + rp} < \beta \leq \frac{p}{p-1} \tag{1.18}$$

From [11, Theorem 85], E.C. Titchmarsh's characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of their Fourier transform's norm, we have

**Theorem 1.5.** [11] *Let $0 < \gamma \leq 1$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:*

1) $\|\tau_h f - f\|_2 = \mathcal{O}(h^\gamma)$ *as* $h \to 0$
2) $\int_{|\lambda| \geq s} \mathcal{F}(f)(\lambda) d\lambda = \mathcal{O}(s^{-2\gamma})$ *as* $s \to +\infty$

From [10] M. El Hamma and R. Daher generalized the E. C. Titchmarsh's theorem see ([11], theorem 85) for the Dunkl transform on the real line.
We first need to define the $k$-Dunkl Lipschitz class.
Let $\mathcal{W}^r_{2,\alpha}$ $(r \in \{1, 2, ...\})$ be the Soblev space constructed by the Dunkl operator $\Lambda_\alpha$.

$$\mathcal{W}^r_{2,\alpha} = \{f \in L^2_\alpha(\mathbb{R}) / \Lambda^j_\alpha f \in L^2_\alpha(\mathbb{R}) \text{ for all } j \in \{1, ..., r\}\}$$

**Definition 1.3.** [10] Let $0 < \gamma \leq 1$. A function $f \in \mathcal{W}^r_{2,\alpha}$ is said to be in the $k$-Dunkl Lipschitz class, denoted by $DLip(\gamma, 2, k)$, if

$$\|\Delta^k_{h,\alpha}\Lambda^r_\alpha f\|_{2,\alpha} = \mathcal{O}(h^\gamma) \text{ as } h \to 0 \tag{1.19}$$

**Theorem 1.6.** [10] *Let $f \in \mathcal{W}^r_{2,\alpha}$. The following are equivalents*

1) $f \in DLip(\gamma, 2, k)$
2) $\int_{|\lambda|\geq s} |\lambda|^{2r}\mathcal{F}^D_\alpha(f)|\lambda|^{2\alpha+1}d\lambda = \mathcal{O}(s^{-2\gamma})$ *as* $s \to +\infty$

## 2. Main results

Our main results are inspired by the work of V.A. Abilov, F. V. Abilova, M.K. Kerimov and E. C. Titchmarsh (see [1], [2], [3],[5], [6], [7]). Briefly, we show the Hausdorff-Young inequality in the space $L^p_{\alpha,Q}(\mathbb{R})$ where $1 < p \leq 2$ and we give new estimates for the generalized Dunkl transform of a class of function $f$ in a Sobolev space that will be defined later.

**Proposition 2.1.** (Hausdorff-Young inequality) *Let be $1 < p \leq 2$, $1 \leq p' \leq +\infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p_{\alpha,Q}(\mathbb{R})$. Then, $\mathcal{M}^{-1}_Q \circ \mathcal{F}^D_{\alpha,Q}(f) \in L^{p'}_{\alpha,Q}(\mathbb{R})$ and there exists a constant $C > 0$ satisfying*

$$\|\mathcal{M}^{-1}_Q \circ \mathcal{F}^D_{\alpha,Q}(g)\|_{p',\alpha,Q} \leq C\|g\|_{p,\alpha,Q} \quad \textit{for all } g \in L^p_{\alpha,Q}(\mathbb{R}) \tag{2.1}$$

*Proof.* Let $f \in L^p_{\alpha,Q}(\mathbb{R})$ then, $\mathcal{M}_Q(f) \in L^p_\alpha(\mathbb{R})$ and by the Hausdorff-Young inequality tied to the classical Dunkl harmonic analysis (see ([6], [7])), we have $\mathcal{F}^D_\alpha \circ \mathcal{M}_Q(f) \in L^{p'}_\alpha(\mathbb{R})$ and, there exists a constant positive $C > 0$ such that

$$\|\mathcal{F}^D_\alpha \mathcal{M}_Q(g)\|_{p',\alpha} \leq C\|\mathcal{M}_Q(g)\|_{p,\alpha} \quad \text{for all } g \in L^p_{\alpha,Q}(\mathbb{R})$$

therefore

$$\|\mathcal{M}^{-1}_Q \circ \mathcal{F}^D_{\alpha,Q}(g)\|_{p',\alpha,Q} \leq C\|g\|_{p,\alpha,Q} \quad \text{for all } g \in L^p_{\alpha,Q}(\mathbb{R})$$

□

Let $f \in L^p_{\alpha,Q}(\mathbb{R})$ and $k = 0, 1, 2, \ldots$ put

$$\Delta^0_{h,\alpha,Q}f = f$$

$$\Delta_{h,\alpha,Q}f = \mathcal{T}^*_{h,\alpha,Q}f - Q(h)f = (\mathcal{T}^*_{h,\alpha,Q} - Q(h)\mathcal{I})f$$

$$\Delta^k_{h,\alpha,Q}f = \Delta_{h,\alpha,Q}\left(\Delta^{k-1}_{h,\alpha,Q}\right)f = (\mathcal{T}^*_{h,\alpha,Q} - Q(h)\mathcal{I})^k f \tag{2.2}$$

the $k^{th}$ order generalized modulus of continuity of a function $f \in L^p_{\alpha,Q}(\mathbb{R})$ is defined as

$$\Omega_{k,\alpha,Q}(f,\delta) = sup_{|h|\leq\delta}\|\Delta^k_{h,\alpha,Q}f\|_{p,\alpha,Q} \tag{2.3}$$

We denote by

$$\begin{aligned}\mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q}) &= \Big\{f \in L^p_{\alpha,Q}(\mathbb{R})/\forall j \in \{1,2,..,r\}(\Lambda^*_{\alpha,Q})^j f \in L^p_{\alpha,Q}(\mathbb{R}) \\ &\quad \text{and } \Omega_{k,\alpha,Q}((\Lambda^*_{\alpha,Q})^r f,\delta) = \mathcal{O}(\Psi(\delta^k) \text{ as } \delta \to 0\Big\}\end{aligned}$$

that the Sobolev space constructed by the dual operator $\Lambda^*_{\alpha,Q}$, where $\Psi$ is any nonnegative function defined on $[0,+\infty[$

**Lemma 2.1.** *Let $f \in L^p_{\alpha,Q}(\mathbb{R})$ we have*

*i)* $\Omega_{k,\alpha,Q}(f,\delta) = \left(sup_{|h|\leq\delta}Q(h)^k\right)\Omega_{k,\alpha}(\mathcal{M}_Q f,\delta)$

*ii)* $\mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q}) = \mathcal{M}_Q^{-1}\left(\mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha)\right)$

*Proof.* i) Let $f \in L^p_{\alpha,Q}(\mathbb{R})$

$$\begin{aligned}\Omega_{k,\alpha,Q}(f,\delta) &= sup_{|h|\leq\delta}\left\|\Delta^k_{h,\alpha,Q}f\right\|_{p,\alpha,Q} \\ &= sup_{|h|\leq\delta}\left\|\left(\mathcal{T}^*_{h,\alpha,Q} - Q(h)\mathcal{I}\right)^k f\right\|_{p,\alpha,Q} \\ &= sup_{|h|\leq\delta}\left\|\Big(Q(h)\mathcal{M}_Q^{-1}\circ\tau_{-h,\alpha}\circ\mathcal{M}_Q - Q(h)\mathcal{I}\Big)^k f\right\|_{p,\alpha,Q} \\ &= \Big(sup_{|h|\leq\delta}\left(Q(h)\right)^k\Big)\, sup_{|h|\leq\delta}\left\|\mathcal{M}_Q^{-1}\circ\Delta^k_{-h,\alpha}\circ\mathcal{M}_Q f\right\|_{p,\alpha,Q} \\ &= \Big(sup_{|h|\leq\delta}\left(Q(h)\right)^k\Big)\, sup_{|h|\leq\delta}\left\|\Delta^k_{-h,\alpha}\circ\mathcal{M}_Q f\right\|_{p,\alpha} \\ &= \Big(sup_{|h|\leq\delta}\left(Q(h)\right)^k\Big)\,\Omega_{k,\alpha}\left(\mathcal{M}_Q f,\delta\right).\end{aligned}$$

ii) It is equivalent to prove that

$$f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q}) \Leftrightarrow \mathcal{M}_Q f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha)$$

Consider $f \in L^p_{\alpha,Q}(\mathbb{R})$

$$\begin{aligned}f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q}) &\Leftrightarrow \text{for all } j \in \{0,1,...,r\}(\Lambda^*_{\alpha,Q})^j f \in L^p_{\alpha,Q}(\mathbb{R}) \\ &\qquad \text{and } \Omega_{k,\alpha,Q}((\Lambda^*_{\alpha,Q})^r f,\delta) = \mathcal{O}\left(\Psi(\delta^k)\right) \text{ as } \delta \to 0 \\ &\Leftrightarrow \text{for all } j \in \{0,1,...,r\}\mathcal{M}_Q^{-1}\circ(\Lambda_\alpha)^j\circ\mathcal{M}_Q f \in L^p_{\alpha,Q}(\mathbb{R}) \\ &\qquad \text{and } \Omega_{k,\alpha,Q}(\mathcal{M}_Q^{-1}\circ(\Lambda_\alpha)^r\circ\mathcal{M}_Q f,\delta) = \mathcal{O}\left(\Psi(\delta^k)\right) \text{ as } \delta \to 0 \\ &\Leftrightarrow \text{for all } j \in \{0,1,...,r\}(\Lambda_\alpha)^j\circ\mathcal{M}_Q f \in L^p_\alpha(\mathbb{R}) \\ &\qquad \text{and } \Omega_{k,\alpha}(\Lambda^r_\alpha\mathcal{M}_Q f,\delta) = \mathcal{O}\left(\Psi(\delta^k)\right) \text{ as } \delta \to 0 \\ &\Leftrightarrow \mathcal{M}_Q f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha)\end{aligned}$$

□

**Theorem 2.1.** *For all $f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q})$ where $1 < p \leq 2$ we have,*

$$\int_{|\lambda|\geq N} |\mathcal{F}^D_{\alpha,Q}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d\lambda = \mathcal{O}\left(N^{-rp'}(\Psi)^{p'}\left((\frac{c}{N})^k\right)\right) \quad as\ N \to \ +\infty$$

*where $c$ is a positive constant, $r = 0, 1, 2...; k = 1, 2, ...;$ $p'$ is the conjugate exponent of $p$ and $\Psi$ is a nonnegative function given on* $[0, +\infty[$.

*Proof.* By ii) of Lemma 2.3 we have $f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda^*_{\alpha,Q})$ is equivalent to $\mathcal{M}_Q f \in \mathcal{W}^{r,k}_{p,\Psi}(\Lambda_\alpha)$. Then, by Theorem 1.5 (see [7]) we have

$$\int_{|\lambda|\geq N} |\mathcal{F}^D_{\alpha}\mathcal{M}_Q(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d\lambda = \mathcal{O}\left(N^{-rp'}(\Psi)^{p'}\left((\frac{c}{N})^k\right)\right) \quad \text{as } N \to \ +\infty$$

By ii) of lemma 1.2 we have $\mathcal{F}^D_\alpha \circ \mathcal{M}_Q = \mathcal{F}^D_{\alpha,Q}$. Therefore, we obtain the result. □

**Corollary 2.1.** *Let $\Psi(t) = t^\beta$ and $f \in \mathcal{W}^{r,k}_{p,t^\beta}(\Lambda^*_{\alpha,Q})$ where $\beta > 0$ and $1 < p \leq 2$. Then,*

$$\int_{|\lambda|\geq N} |\mathcal{F}^D_{\alpha,Q}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d\lambda = \mathcal{O}\left(N^{-rp'-p'k\beta}\right) \quad as\ N \to \ +\infty$$

The next theorem is devoted to establish the E.C. Titchmarsh's theorem ([5], [6]) in the generalized Dunkl operator setting by means of the differences of higher orders.

**Theorem 2.2.** *Let $0 < \gamma \leq k$, $f \in L^p_{\alpha,Q}(\mathbb{R})$ such that for all $j \in \{1, 2, ..., r\}$, $(\Lambda^*_{\alpha,Q})^j f \in L^p_{\alpha,Q}(\mathbb{R})$ and*

$$\|\Delta^k_{-h,\alpha,Q}(\Lambda^*_{\alpha,Q})^r f\|_{p,\alpha,Q} = \mathcal{O}(Q(h)^k h^\gamma) \quad as\ h \to 0$$

*Then*

$$\mathcal{M}^{-1}_Q \circ \mathcal{F}^D_{\alpha,Q}(f) \in L^\beta_{\alpha,Q}(\mathbb{R})$$

*for*

$$\frac{2\alpha p + 2p}{2p + 2\alpha(p-1) - 2 + \gamma p + rp} < \beta \leq \frac{p}{p-1}$$

*Proof.* Let $f \in L^p_{\alpha,Q}(\mathbb{R})$ such that for all $j \in \{1, 2, ..., r\}$, $(\Lambda^*_{\alpha,Q})^j f \in L^p_{\alpha,Q}(\mathbb{R})$ by i) lemma 1.2 and by the remark 1.4 we have for all $j \in \{1, 2, ..., r\}$, $(\Lambda_\alpha)^j \circ \mathcal{M}_Q f \in$

$L^p_\alpha(\mathbb{R})$ and

$$\begin{aligned}
&\left\|\Delta^k_{-h,\alpha,Q}\left((\Lambda^*_{\alpha,Q})^r f\right)\right\|_{p,\alpha,Q}\\
=\ & \left\|\left(\mathcal{T}^*_{-h,\alpha,Q}-Q(-h)\mathcal{I}\right)^k\circ(\Lambda^*_{\alpha,Q})^r(f)\right\|_{p,\alpha,Q}\\
=\ & \|(Q(h))^k\left(\mathcal{M}_Q^{-1}\circ\tau_{h,\alpha}\circ\mathcal{M}_Q-\mathcal{I}\right)^k\circ\mathcal{M}_Q^{-1}\circ\Lambda^r_\alpha\circ\mathcal{M}_Q(f)\|_{p,\alpha,Q}\\
=\ & Q(h)^k\|\mathcal{M}_Q^{-1}\circ\Delta^k_{h,\alpha}\circ\Lambda^r_\alpha\circ\mathcal{M}_Q(f)\|_{p,\alpha,Q}\\
=\ & Q(h)^k\|\mathcal{M}_Q^{-1}\circ\Delta^k_{h,\alpha}\Lambda^r_\alpha\circ\mathcal{M}_Q(f)\|_{p,\alpha,Q}\\
=\ & Q(h)^k\|\Delta^k_{h,\alpha}\circ\Lambda^r_\alpha\circ\mathcal{M}_Q(f)\|_{p,\alpha}.
\end{aligned}$$

Since we have by hypothesis

$$\|\Delta^k_{-h,\alpha,Q}\left((\Lambda^*_{\alpha,Q})^r f\right)\|_{p,\alpha,Q}=\mathcal{O}\left(Q(h)^k h^\gamma\right)\quad\text{as } h\to 0$$

we obtain

$$\|\Delta^k_{h,\alpha}\left((\Lambda_\alpha)^r(\mathcal{M}_Q f)\right)\|_{p,\alpha}=\mathcal{O}\left(h^\gamma\right)\quad\text{as } h\to 0$$

Then by Theorem proved in [5] we deduce $\mathcal{F}^D_\alpha\circ\mathcal{M}_Q(f)\in L^\beta_\alpha(\mathbb{R})$ where

$$\frac{2\alpha p+2p}{2p+2\alpha(p-1)-2+\gamma p+rp}<\beta\le\frac{p}{p-1}$$

Therefore,

$$\mathcal{M}_Q^{-1}\circ\mathcal{F}^D_{\alpha,Q}(f)\in L^\beta_{\alpha,Q}(\mathbb{R}).$$

□

Let $\mathcal{W}^r_{2,\alpha,Q}$ ($r\in\{1,2,...\}$) be the Soblev space constructed by the dual Dunkl operator $\Lambda^*_{\alpha,Q}$.

$$\mathcal{W}^r_{2,\alpha,Q}=\{f\in L^2_{\alpha,Q}(\mathbb{R})/(\Lambda^*_{\alpha,Q})^j f\in L^2_{\alpha,Q}(\mathbb{R})\ \text{ for all } j\in\{1,...,r\}\}$$

**Definition 2.1.** Let $0<\gamma\le 1$. A function $f\in\mathcal{W}^r_{2,\alpha,Q}$ is said to be in the $k$-$Q$-Dunkl Lipschitz class, denoted by $DLip(\gamma,2,k,Q)$, if

$$\|\Delta^k_{-h,\alpha,Q}(\Lambda^*_{\alpha,Q})^r f\|_{2,\alpha,Q}=\mathcal{O}((Q(h))^k h^\gamma)\ \text{ as } h\to 0 \tag{2.4}$$

**Lemma 2.2.** *Let $0<\gamma\le 1$ and $k\in\{0,1,2,...\}$ we have*

$$DLip(\gamma,2,k,Q)=\mathcal{M}_Q^{-1}\left(DLip(\gamma,2,k)\right) \tag{2.5}$$

**Theorem 2.3.** *Let $f\in\mathcal{W}^r_{2,\alpha,Q}$. The following are equivalents*

*1) $f\in DLip(\gamma,2,k,Q)$*

*2) $\int_{|\lambda|\ge s}|\lambda|^{2r}\mathcal{F}^D_{\alpha,Q}(f)|\lambda|^{2\alpha+1}d\lambda=\mathcal{O}(s^{-2\gamma})$ as $s\to+\infty$*

**Corollary 2.2.** *Let $f\in DLip(\gamma,2,k,Q)$. Then*

$$\int_{|\lambda|\ge s}\mathcal{F}^D_{\alpha,Q}(f)|\lambda|^{2\alpha+1}d\lambda=\mathcal{O}(s^{-2\gamma-2r})\quad\text{as } s\to+\infty \tag{2.6}$$

## 3. Conclusion

In this work, via the isometric isomorphism operator $\mathcal{M}_Q$ from $L^p_{\alpha,Q}(\mathbb{R})$ onto $L^p_{\alpha}(\mathbb{R})$, and from the formulas of the generalized Dunkl transform and the generalized dual Dunkl translation given as follows:

$$\mathcal{F}^D_{\alpha,Q} = \mathcal{F}^D_{\alpha} \circ \mathcal{M}_Q$$

where $\mathcal{F}^D_{\alpha}$ be the classical Dunkl transform and

$$\mathcal{T}^*_{h,\alpha,Q} = Q(h)\mathcal{M}^{-1}_Q \circ \tau_{-h,\alpha} \circ \mathcal{M}_Q$$

where $\tau_{-h,\alpha}$ is the translation operator associated to the classical Dunkl harmonic analysis, we were able to establish, without using calculations, the generalization Theorem 2.1 in [7], Theorem 2.2 in [5] and Theorem 2.3 in [10].

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## REFERENCES

1. V. A. Abilov and F. V. Abilova, *Approximation of Functions by Fourier-Bessel Sums*, Izv. Vyssh. Uchebn. Zaved., Mat., No. **8**, 3-9 (2001).
2. V.A.Abilov, F.V.Abilova and M.K.Kerimov, *Some Remarks Concerning the Fourier Transform in the Space* $\mathrm{L}_2(\mathbb{R})$ Zh. Vychisl. Mat. Mat. Fiz. **48**, 939-945 (2008) [Comput. Math. Math. Phys. 48, 885-891].
3. V.A.Abilov, F.V.Abilova and M.K.Kerimov, *Some Remarks Concerning the Fourier Transform in the Space* $\mathrm{L}_2(\mathbb{R}^n)$, Zhurnal Vychislitel'noi Matematiki iMatematicheskoi Fiziki, 2008, Vol. **48** No. 12, pp. 2113-2120.
4. E. A. Al Zahrani and M. A. Mourou, *The continuous wavelet transform associated with a Dunkl type operator on the real line*, Advances in Pure Mathematics, 2013, Vol. **3**, pp. 443-450.
5. M. Boujeddaine, R. Daher and M. El Hamma, *Sobolev Spaces for the Dunkl operator on the real line*, Asia Pacific Journal of Mathematics, Vol. **1**, No. 3 (2014) pp. 176-181.
6. M. El Hamma and R. Daher, *Growth properties of the Dunkl Transform on the real line*, Int. Journal of Math. Analysis, Vol. **6**, No. 44 (2012) pp. 2173-2177.
7. M. EL Hamma, R. Daher and M. Boujeddaine, *On estimates for the Dunkl transform in the space* $L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, Nonlinear Studies, 2014, Vol. **21**, No. 3 pp.435-441.
8. M. El Hamma and R. Daher, *On Estimates for the Dunkl transform in the space* $\mathrm{L}_{2,\alpha}(\mathbb{R})$, Int. J. Analysis and Applications. Vol. **6**, No. 1 (2014), pp. 53-62.
9. M. El Hamma and R. Daher, *On Estimates for the Dunkl transform in the space* $\mathrm{L}_{2,\alpha}(\mathbb{R})$, Int. J. Analysis and Applications. Vol. **6**, No. 1 (2014), pp. 53-62.

10. M. El Hamma and R. Daher, *Genaralization of Titchmarsh 's theorem for the Dunkl transform on the real line On Estimates for the Dunkl transform*, Analele Universitatii de Vest, Timisoara, Seria Matematica- Informatica, LI, **2** (2013), 47-55.

11. E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals*, Oxford University Press, Amen House, London, E. C. 4, 1948.