FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 3 (2021), 519 - 528
https://doi.org/10.22190/FUMI200930038M
Original Scientific Paper

SOME NEW IDENTITIES FOR THE SECOND COVARIANT DERIVATIVE OF THE CURVATURE TENSOR

Miroslav D. Maksimović[1] and Mića S. Stanković[2]

[1]**University of Priština in Kosovska Mitrovica,**
Faculty of Sciences and Mathematics,
Department of Mathematics, 38220 Kosovska Mitrovica, Serbia
[2]**University of Niš, Faculty of Sciences and Mathematics,**
Department of Mathematics, 18000 Niš, Serbia

Abstract. In this paper, we have studied the second covariant derivative of Riemannian curvature tensor. Some new identities for the second covariant derivative have been given. Namely, identities obtained by cyclic sum with respect to three indices have been given. In the first case, two curvature tensor indices and one covariant derivative index participate in the cyclic sum, while in the second case one curvature tensor index and two covariant derivative indices participate in the cyclic sum.
Keywords: covariant derivative, curvature tensor, Riemannian manifold, second order identity

1. Introduction

The Riemannian curvature tensor R^i_{jmn} is very important in Riemannian manifold, especially when studying the theory of general relativity and quantum gravity (see [1, 8, 23]). Knowledge of the properties of curvature tensor is of great importance when studying the manifolds mentioned. Some other geometric object can be defined using curvature tensor, for example Ricci curvature tensor, scalar curvature, Weyl tensor, etc. In the articles [2, 3, 20], the curvature tensor was studied at various mappings and transformations (see also the monographs [4] and [7]).

Received September 30, 2020. accepted Jun 15, 2021.
Communicated by Dragana Cvetković-Ilić
Corresponding Author: Miroslav D. Maksimović, University of Priština in Kosovska Mitrovica, Faculty of Sciences and Mathematics, Department of Mathematics, 38220 Kosovska Mitrovica, Serbia | E-mail: miroslav.maksimovic@pr.ac.rs
2010 *Mathematics Subject Classification.* Primary 53B20; Secondary 53B21

Initially, the idea was to use three indices in cyclic sum, and thus some of the properties of the Riemannian curvature tensor were proved (the first and the second Bianchi identities). The idea of a cyclic sum was continued in the paper [6], but in the summation four indices were used: two indices of curvature tensor and two indices of covariant derivative. In the present aricle we have given the new identities for cyclic summing of the second covariant derivatives with respect to three indices. We will see that one of these identities implies Lovelock differential identity.

2. Preliminaries

Let us consider the Riemannian manifold $(\mathcal{M}_N, g)$, where $\mathcal{M}_N$ is N-dimensional manifold and g is a symmetric metric tensor. The Christoffel symbols of the first kind $\Gamma_{i\cdot jk}$ and the Christoffel symbols of the second kind Γ^i_{jk} of Riemannian manifold are defined as

$$\Gamma_{i\cdot jk} = \frac{1}{2}\left(g_{ij,k} - g_{jk,i} + g_{ki,j}\right), \tag{2.1}$$

$$\Gamma^i_{jk} = g^{ip}\Gamma_{p\cdot jk} = \frac{1}{2}g^{ip}\left(g_{pj,k} - g_{jk,p} + g_{kp,j}\right), \tag{2.2}$$

where g_{ij} and g^{ij} is the covariant and contravariant metric tensor, respectively. Hereinafter, the coma $(,)$ denotes partial derivative.

In the general case, the partial derivative of a tensor is not always a tensor, and therefore the term covariant derivative is introduced. We will use the semicolon $(;)$ for a covariant derivative in a Riemannian manifold. The covariant derivative with respect to the Christoffel symbols Γ^i_{jk} is defined as

$$t^{i_1\ldots i_A}_{j_1\ldots j_B;k} = t^{i_1\ldots i_A}_{j_1\ldots j_B,k} + \sum_{p=1}^{A} t^{i_1\ldots i_{\alpha-1}pi_{\alpha+1}\ldots i_A}_{j_1\ldots j_B}\Gamma^{i_\alpha}_{pk} - \sum_{p=1}^{B} t^{i_1\ldots i_A}_{j_1\ldots j_{\alpha-1}pj_{\alpha+1}\ldots j_B}\Gamma^{p}_{j_\alpha k}, \tag{2.3}$$

where $t^{i_1\ldots i_A}_{j_1\ldots j_B}$ is an arbitrary tensor. The Riemannian curvature tensor R^i_{jmn} of a Riemannian manifold is obtained based on Ricci identity

(2.4)

$$t^{i_1\ldots i_A}_{j_1\ldots j_B;mn} - t^{i_1\ldots i_A}_{j_1\ldots j_B;nm} = \sum_{p=1}^{A} t^{i_1\ldots i_{\alpha-1}pi_{\alpha+1}\ldots i_A}_{j_1\ldots j_B} R^{i_\alpha}_{pmn} - \sum_{p=1}^{B} t^{i_1\ldots i_A}_{j_1\ldots j_{\alpha-1}pj_{\alpha+1}\ldots j_B} R^{p}_{j_\alpha mn},$$

where

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}. \tag{2.5}$$

Also, the Riemannian curvature tensor can be expressed in the form

$$R^i_{jmn} = \Gamma^i_{j[m,n]} + \Gamma^p_{j[m}\Gamma^i_{n]p}, \tag{2.6}$$

where $[ij]$ denotes alternation without division with respect to the indices i and j (for example, $a_{[ij]} = a_{ij} - a_{ji}$). For Ricci identity, we will use the notation below

$$t^{i_1\ldots i_A}_{j_1\ldots j_B;mn} - t^{i_1\ldots i_A}_{j_1\ldots j_B;nm} = t^{i_1\ldots i_A}_{j_1\ldots j_B;[mn]}. \tag{2.7}$$

The Riemannian curvature tensor has the following properties

1. $R^i_{jmn} = -R^i_{jnm}$, (anti-symmetry)
2. $\underset{jmn}{Cycl}\, R^i_{jmn} = 0$, (the first Bianchi identity)
3. $\underset{mnu}{Cycl}\, R^i_{jmn;u} = 0$, (the second Bianchi identity)

where $\underset{jmn}{Cycl}$ is the cyclic sum by indices j, m, n.

The covariant curvature tensor of a Riemannian manifold is defined as

$$R_{ijmn} = g_{ip} R^p_{jmn}, \tag{2.8}$$

and has the following properties:

1. $R_{ijmn} = -R_{jimn} = -R_{ijnm}$,
2. $R_{ijmn} = R_{mnij}$,
3. $\underset{\alpha\beta\gamma}{Cycl}\, R_{ijmn} = 0, \quad \{\alpha, \beta, \gamma\} \subset \{i, j, m, n\}$,
4. $\underset{mnu}{Cycl}\, R_{ijmn;u} = 0$.

Oswald Veblen showed that the following identity

$$R^i_{jmn;u} - R^i_{mju;n} + R^i_{unm;j} - R^i_{nuj;m} = 0, \tag{2.9}$$

is correct [21].

Theorem 2.1. *[6] For the curvature tensor R^i_{jmn} the identity*

$$\underset{mnuv}{Cycl}\, R^i_{jmn;uv} = \underset{mnuv}{Cycl}\, R^i_{jpm} R^p_{nuv} - R^i_{pmu} R^p_{jnv} + R^i_{pnv} R^p_{jmu}. \tag{2.10}$$

is valid.

By contracting by indices i and v in equation (2.10), one obtains the Lovelock differential identity (see [6])

$$\underset{mnu}{Cycl}\, R^p_{jmn;pu} = -\underset{mnu}{Cycl}\, R^p_{jmn} R_{pu}, \tag{2.11}$$

where R_{jm} is the Ricci curvature tensor, i.e. $R_{jm} = R^p_{jmp}$.

Theorem 2.2. *[22] The covariant curvature tensor of a Riemannian manifold satisfies the identity*

$$R_{ijmn;[uv]} + R_{mnuv;[ij]} + R_{uvij;[mn]} = 0. \tag{2.12}$$

Definition 2.1. The Riemannian manifold $(\mathcal{M}_N, g)$ is symmetric Riemannian manifold if a curvature tensor satisfies

$$R^i_{jmn;u} = 0. \tag{2.13}$$

The Riemannian manifold $(\mathcal{M}_N, g)$ is semi-symmetric if a curvature tensor satisfies

$$R^i_{jmn;[uv]} = 0. \tag{2.14}$$

3. Results

In this section, we will present new results for the cyclic sum of the second covariant derivatives of Riemannian curvature tensor.

Let us consider the second Bianchi identity

$$\underset{mnu}{Cycl}\, R^i_{jmn;u} = 0. \tag{3.1}$$

By covariant derivative of this equation by index v we get the equation

$$\underset{mnu}{Cycl}\, R^i_{jmn;uv} = 0. \tag{3.2}$$

In the same way, we have the following identities

$$\underset{muv}{Cycl}\, R^i_{jmu;vn} = 0, \quad \underset{mvn}{Cycl}\, R^i_{jmv;nu} = 0. \tag{3.3}$$

Summing the obtained expressions (3.2) and (3.3), we have equation

$$\begin{aligned} 0 &= \underset{mnu}{Cycl}\, R^i_{jmn;uv} + \underset{muv}{Cycl}\, R^i_{jmu;vn} + \underset{mvn}{Cycl}\, R^i_{jmv;nu} \\ &= R^i_{jmn;uv} + R^i_{jnu;mv} + R^i_{jum;nv} + R^i_{jmu;vn} + R^i_{juv;mn} + R^i_{jvm;un} \\ &+ R^i_{jmv;nu} + R^i_{jvn;mu} + R^i_{jnm;vu}. \end{aligned} \tag{3.4}$$

From here, using every third addend from the previous equation, we get the identity

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} + \underset{nuv}{Cycl}\, R^i_{jnu;mv} - \underset{nuv}{Cycl}\, R^i_{jmn;vu} = 0, \tag{3.5}$$

i.e.

$$\underset{nuv}{Cycl} \left(R^i_{jmn;[uv]} + R^i_{jnu;mv} \right) = 0. \tag{3.6}$$

If we consider the Ricci identity (2.4) for $R^i_{jmn;[uv]}$, from equation (3.6) we obtain

$$\underset{nuv}{Cycl}\left(R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv} - R^i_{jmp}R^p_{nuv} + R^i_{jnu;mv}\right) = 0. \tag{3.7}$$

Since that $\underset{nuv}{Cycl}\, R^p_{nuv} = 0$ (the first Bianchi identity), it follows

$$\underset{nuv}{Cycl}\, R^i_{jnu;mv} = -\underset{nuv}{Cycl}\left(R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv}\right), \tag{3.8}$$

i.e.

$$\underset{nuv}{Cycl}\, R^i_{jnu;mv} = \underset{nuv}{Cycl}\left(R^i_{pmn}R^p_{juv} + R^i_{jpn}R^p_{muv} - R^p_{jmn}R^i_{puv}\right). \tag{3.9}$$

After changing the indices $n \to m$, $u \to n$, $m \to u$, we obtain

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right) \tag{3.10}$$

and with this we have proved the following theorem.

Theorem 3.1. *Let $(\mathcal{M}_N, g)$ be a Riemannian manifold. The Riemannian curvature tensor satisfies the identity*

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right), \tag{3.11}$$

where $\underset{mnv}{Cycl}$ is the cyclic sum with respect to the indices m, n, v.

Corollary 3.1. *Contraction by indices i and u in equation (3.11) gives the Lovelock differential identity (2.11).*

Proof.

$$\begin{aligned}\underset{mnv}{Cycl}\, R^p_{jmn;pv} &= \underset{mnv}{Cycl}\left(R^p_{spm}R^s_{jnv} + R^p_{jsm}R^s_{pnv} - R^s_{jpm}R^p_{snv}\right)\\ &= \underset{mnv}{Cycl}\left(-R^p_{smp}R^s_{jnv}\right) + \underset{mnv}{Cycl}\left(R^p_{jsm}R^s_{pnv} - R^s_{jpm}R^p_{snv}\right)\\ &= -\underset{mnv}{Cycl}\, R_{sm}R^s_{jnv} + \underset{mnv}{Cycl}\left(R^p_{jsm}R^s_{pnv} - R^p_{jsm}R^s_{pnv}\right)\\ &= -\underset{mnv}{Cycl}\, R_{sm}R^s_{jnv},\end{aligned} \tag{3.12}$$

i.e.

$$\underset{mnv}{Cycl}\, R^p_{jmn;pv} = -\underset{mnv}{Cycl}\, R^p_{jmn}R_{pv}. \tag{3.13}$$

□

If we add an expression $-\underset{nuv}{Cycl}\, R^i_{jnu;vm} = 0$ to the equation (3.6), then we have the following consequence.

Corollary 3.2. *The Riemannian curvature tensor satisfy the identity*

$$\underset{nuv}{Cycl}\left(R^i_{jmn;[uv]} + R^i_{jnu;[mv]}\right) = 0, \tag{3.14}$$

where $[ij]$ *denotes alternation without division with respect to the indices* i *and* j*.*

After applying Ricci identity, the previous equation takes the form

$$\begin{aligned}\underset{mnv}{Cycl}(&R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv} + R^p_{jnu}R^i_{pmv}\\ &- R^i_{pnu}R^p_{jmv} - R^i_{jpu}R^p_{nmv} - R^i_{jnp}R^p_{umv}) = 0.\end{aligned} \tag{3.15}$$

Based on Theorem (3.1) we have the consequence.

Corollary 3.3. *In a semi-symmetric Riemannian manifold the following identity*

$$\underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right) = 0. \tag{3.16}$$

holds.

Proof. Given the fact that in semi-symmetric Riemannian manifold the following is valid

$$R^i_{jmn;uv} = R^i_{jmn;vu}, \tag{3.17}$$

i.e.

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\, R^i_{jmn;vu}, \tag{3.18}$$

and since $\underset{mnv}{Cycl}\, R^i_{jmn;vu} = 0$ (the second Bianchi identity), it follows that the left hand side of equation (3.11) is equal to zero, thus completing the proof. □

Corollary 3.4. *The equation (3.16) is valid in symmetric Riemannian manifold.*

Below we present the result obtained by cyclic sum of the second covariant derivatives of curvature tensor, when one curvature tensor index and two covariant derivative indices participate in the cyclic sum.

Theorem 3.2. *Let $(\mathcal{M}_N, g)$ be a Riemannian manifold. The Riemannian curvature tensor satisfy the following identity*

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl} \Bigg(C^i_{jmnuv} - R^i_{jmn,uv} + R^i_{jmn,p}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p} - R^p_{jmn}R^i_{uvp} + R^i_{psn}B^{sp}_{muvj} + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm} + \sum_{\beta=1}^{3}\left(R^i_{j_1 p j_3}A^p_{j_\beta uv} - R^p_{j_1 s j_3}B^{si}_{j_\beta uvp}\right)\Bigg), \tag{3.19}$$

where

$$A^i_{jmn} = -\Gamma^i_{jm,n} + \Gamma^p_{jn}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{pj},$$
$$B^{pi}_{jmnu} = \Gamma^p_{jm}\Gamma^i_{nu} + \Gamma^p_{jn}\Gamma^i_{mu},$$
$$C^i_{jmnuv} = C^i_{jmnu,v} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv} - C^i_{jmnp}\Gamma^p_{uv},$$
$$C^i_{jmnu} = R^i_{jmn,u} + R^i_{jmu,n},\ j_1 = j,\ j_2 = m,\ j_3 = n,$$

and $\underset{nuv}{Cycl}$ is the cyclic sum with respect to the indices n, u, v.

Proof. First, we have identity

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} = R^i_{jmn;uv} + R^i_{jmu;vn} + R^i_{jmv;nu} = \left(R^i_{jmn;u}\right)_{;v} + \left(R^i_{jmu;v}\right)_{;n} + \left(R^i_{jmv;n}\right)_{;u}.$$

Further, we get the following equation

$$\begin{aligned}
&\left(R^i_{jmn;u}\right)_{;v} + \left(R^i_{jmu;v}\right)_{;n} + \left(R^i_{jmv;n}\right)_{;u} = \\
&= \left(R^i_{jmn;u}\right)_{,v} + R^p_{jmn;u}\Gamma^i_{pv} - R^i_{pmn;u}\Gamma^p_{jv} - R^i_{jpn;u}\Gamma^p_{mv} - R^i_{jmp;u}\Gamma^p_{nv} - R^i_{jmn;p}\Gamma^p_{uv} \\
&+ \left(R^i_{jmu;v}\right)_{,n} + R^p_{jmu;v}\Gamma^i_{pn} - R^i_{pmu;v}\Gamma^p_{jn} - R^i_{jpu;v}\Gamma^p_{mn} - R^i_{jmp;v}\Gamma^p_{un} - R^i_{jmu;p}\Gamma^p_{vn} \\
&+ \left(R^i_{jmv;n}\right)_{,u} + R^p_{jmv;n}\Gamma^i_{pu} - R^i_{pmv;n}\Gamma^p_{ju} - R^i_{jpv;n}\Gamma^p_{mu} - R^i_{jmp;n}\Gamma^p_{vu} - R^i_{jmv;p}\Gamma^p_{nu}.
\end{aligned}$$

After developing the remaining covariant derivatives on the right hand side of equality and grouping expressions using basic operations for the Ricci calculus, we get

$$\begin{aligned}
\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl} \Bigg(& R^i_{jmn,uv} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv} \\
& - R^i_{jmp,n}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p} - R^p_{jmn}R^i_{uvp} + R^i_{pmn}A^p_{juv} + R^i_{jpn}A^p_{muv} \\
& + R^i_{jmp}A^p_{nuv} - R^p_{smn}B^{si}_{juvp} - R^p_{jsn}B^{si}_{muvp} - R^p_{jms}B^{si}_{nuvp} + R^i_{psn}B^{sp}_{muvj} \\
& + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm}\Bigg),
\end{aligned}$$

where

$$A^i_{jmn} = -\Gamma^i_{jm,n} + \Gamma^p_{jn}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{pj}, \quad B^{pi}_{jmnu} = \Gamma^p_{jm}\Gamma^i_{nu} + \Gamma^p_{jn}\Gamma^i_{mu},$$
$$C^i_{jmnu} = R^i_{jmn,u} + R^i_{jmu,n}.$$

If we introduce notation

$$C^i_{jmnuv} = C^i_{jmnu,v} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv} - C^i_{jmnp}\Gamma^p_{uv},$$

the previous equation takes the form

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl}\Big(C^i_{jmnuv} - R^i_{jmu,nv} + C^i_{jmnp}\Gamma^p_{uv} - R^i_{jmp,n}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p}$$
$$- R^p_{jmn}R^i_{uvp} + R^i_{pmn}A^p_{juv} + R^i_{jpn}A^p_{muv} + R^i_{jmp}A^p_{nuv} - R^p_{smn}B^{si}_{juvp}$$
$$- R^p_{jsn}B^{si}_{muvp} - R^p_{jms}B^{si}_{nuvp} + R^i_{psn}B^{sp}_{muvj} + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm}\Big)$$

and, from here, after rearranging, we obtain identity (3.19). This ends the proof. □

4. Conclusion

The first part of the Results section was devoted to the result we obtained by cyclic sum with respect to two indices of curvature tensor and one index of covariant derivative, i.e. $\underset{mnv}{Cycl}\, R^i_{jmn;uv}$. Due to anti-symmetry property of Riemannian curvature tensor R^i_{jmn}, the result we got has a simple form. Following the identity (3.11) obtained, we also listed three consequences implied by Theorem 3.1. In the second part of Results section, we present the cyclic sum $\underset{nuv}{Cycl}\, R^i_{jmn;uv}$ over known quantities, i.e. Riemannian curvature tensor and Christoffel symbols of the second kind.

For further research, one can observe cyclic sum of the second covariant derivatives in other manifolds, as the curvature tensor is an interesting geometric object in other manifolds [25], as well as in studying various mappings and transformations in other manifolds (see [5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 24, 26, 27]).

Acknowledgement

The authors were supported by the research project 174025 and 174012 of the Serbian Ministry of Science (451-03-68/2020-14/200123 and 451-03-68/2020-14/200124)

REFERENCES

1. X. Calmet: *Vanishing of quantum gravitational corrections to vacuum solutions of general relativity at second order in curvature.* Physics Letters B. **787** (2018), 36–38.

2. L. Chongshan: *On concircular transformations in Riemannian spaces.* J. Austral. Math. Soc. **40** (1986), 218–225.
3. J. Cui, J. C. Yong, H. T. Yun and P. Zhao: *On a projective conformal semi-symmetric connection.* Filomat. **33**, No.12 (2019), 3901–3912.
4. L.P. Eisenhart: *Riemannian spaces*, Princeton Univ. Press, 1967.
5. I. Hinterleitner and J. Mikeš: *Geodesic mappings onto Weyl manifolds*, J. Appl. Math. **2**, (2009), 125–133.
6. C. A. Mantica and L. G. Molinari: *A second order identity for the Riemann tensor and applications.* Colloq. Math. **122** (2011), 69–82.
7. J. Mikeš: *Differential geometry of special mappings*, Palacky Univ. Press, Olomouc, 2015.
8. P. van Nieuwenhuizen and C. C. Wu: *On integral relations for invariants constructed from three Riemann tensors and their applications in quantum gravity.* J. Math. Phys. **18**, 182 (1977).
9. M.Z. Petrović: *Generalized para-Kahler spaces in Eisenhart's sense admitting a holomorphically projective mapping.* Filomat **33**, No.13 (2019), 4001–4012.
10. R. Prasad and A. Haseeb: *Conformal curvature tensor on K-contact manifolds with respect to the quarter-symmetric metric connection.* Facta Universitatis, Ser. Math. Inform. **32**, No.4 (2017), 503–514.
11. M. S. Stankovic: *First type almost geodesic mappings of general affine connection spaces.* Novi Sad J. Math. **29**, No. 3 (1999), 313–323.
12. M. S. Stanković: *On a canonic almost geodesic mappings of the second type of affine spaces.* Filomat **13**, (1999), 105–114.
13. M. S. Stanković: *On a special almost geodesic mappings of third type of affine spaces.* Novi Sad J. Math. Vol. **31**, No. 2, 2001, 125–135.
14. M. S. Stanković: *Special equitorsion almost geodesic mappings of the third type of non-symmetric affine connection spaces.* Applied Mathematics and Computation, **244**, (2014), 695–701.
15. M. S. Stanković and S. M. Minčić: *New special geodesic mappings of generalized Riemannian space.* Publ. Inst. Math. (Beograd) (N. S) **67**(81) (2000), 92–102.
16. M. S. Stanković, S. M. Minčić and Lj. S. Velimirović: *On holomorphically projective mappings of generalized Kahlerian spaces.* Matematicki vesnik **54** (2002), 195–202.
17. M. S. Stanković, S. M. Minčić and Lj. S. Velimirović: *On equitorsion holomorphically projective mappings of generalised Kahlerian spaces.* Czechosl. Math. J., **54** (129) (2004), No. 3, 701-715.
18. M. S. Stanković, M. Lj. Zlatanović and Lj. S. Velimirović: *Equitorsion holomorphically projective mappings of generalized Kahlerian space of the second kind*, International Electronic Journal of Geometry, Vol. **3**, No. 2 (2010), 26–39.
19. M. S. Stanković, M. Lj. Zlatanović and N. O. Vesić: *Basic equations of G-almost geodesic mappings of the second type, which have the property of reciprocity*, Czechosl. Math. J., (2015) Vol. **65**, No. 3, pp. 787–799.
20. M. Tani: *On a conformally flat Riemannian space with positive Ricci curvature.* Tohoku Math. Journ. **19**, 2 (1967).

21. O. Veblen: *Normal coordinates for the geometry of paths.* Proceedings of Nat. Acad. of Sciences **8** (1922), 192–197.

22. A. G. Walker: *On Ruse's spaces of recurrent curvature.* In: Proceedings of the London Mathematical Society. **52** (1950), 36–64.

23. E. Zakhary and J. Carminati: *On the problem of algebraic completeness for the invariants of the Riemann tensor I.* J. Math. Phys. **42** (2001).

24. M.Lj. Zlatanović, I. Hinterleitner and M.S. Najdanović: *On equitorsion concircular tensors of generalized Riemannian spaces.* Filomat. **28**, No.3 (2014), 463–471.

25. M.Lj. Zlatanović, S.M. Minčić and M.Z. Petrović: *Curvature tensors and pseudotensors in a generalized Finsler space.* Facta Universitatis, Ser. Math. Inform. **30**, No.5 (2015), 741–752.

26. M.Lj. Zlatanović and V.M. Stanković: *Some invariants of holomorphically projective mappings of generalized Kahlerian spaces.* J. Math. Anal. Appl., Vol. **450**, (2017), 480–489.

27. M.Lj. Zlatanović and V.M. Stanković: *Geodesic mapping onto Kählerian space of the third kind.* J. Math. Anal. Appl., Vol. **458**, (2018), 601–610.