FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 30, No 2 (2015), 225–233

PROJECTIVE CURVATURE TENSOR ON GENERALIZED (κ, μ)-CONTACT METRIC MANIFOLDS

Srimayee Samui

Abstract. We study some properties of projective curvature tensor in 3- dimensional generalized (κ, μ)-contact metric manifolds.
Keywords: Projective curvature tensor, (κ, μ)-contact metric manifold, 3-dimensional generalized (κ, μ)-contact metric manifolds, N(κ)-contact metric manifolds, η-Einstein manifolds, Sasakian manifolds.

1. Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [5] introduced the notion of (κ, μ)-contact metric manifolds where κ, μ are real constants. Assuming κ, μ smooth functions, Koufogiorgos and Tsichlias [13] introduced the notion of generalized (κ, μ)-contact metric manifolds and gave several examples. Again they also show that such a manifold does not exist in a dimension greater than three. Generalized (κ, μ)-contact metric manifolds have been studied by several authors ([12], [7], [14], [1], [2]) and many others.

Apart from the conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be an $(2n + 1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in the Euclidean space, then M is said to be locally projective flat for $n \geq 1$, if and only if the well-known projective curvature tensor P vanishes. P is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \tag{1.1}$$

for all $X, Y, Z \in TM$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively at if and only if it is of constant curvature [11]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold

Received December 13, 2014.; Accepted February 12, 2015.
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C50, 53C56

to be of constant curvature. The projective curvature tensor has been studied by U. C. De and Joydeep Sengupta [19] and many others.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). Since at each point $p \in M$ the tangent space T_pM can be decomposed into a direct sum $T_pM = \varphi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is a 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$, the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \varphi(T_P) \oplus \{\xi_p\} \;\; p \in M.$$

It may be natural to consider the following particular cases: (1) the projection of the image of C in $\varphi(T_pM)$ is zero; (2) the projection of the image of C in $\{\xi_p\}$ is zero; (3) the projection of the image of $C|_{\varphi(T_pM)\times\varphi(T_pM)\times\varphi(T_pM)}$ in $\varphi(T_pM)$ is zero. An almost contact metric manifold satisfying the case (1), (2) and (3) is said to be conformally symmetric [8], ξ-conformally flat [9] and φ-conformally flat [10], respectively. In an analogous way, we define ξ-projectively flat generalized (κ, μ)-contact metric manifolds.

In [18], U. C. De and Avik De studied the projective curvature tensor in K-contact manifolds. Again in [16], Sujit Ghosh studied projective curvature tensor in $(\kappa, \ \mu)$-contact metric manifolds.

Motivated by the above study, we consider some conditions of the projective curvature tensor on 3-dimensional generalized (κ, μ)-contact metric manifolds and find some important results.

This paper is organized as follows:
After preliminaries in section 3, we characterize ξ-projectively flat generalized $(\kappa, \ \mu)$-contact metric manifolds and prove that ξ-projectively flat generalized $(\kappa, \ \mu)$-contact metric manifolds are either $N(\kappa)$-contact metric manifolds or Sasakian manifolds. In the next section, we prove that a generalized $(\kappa, \ \mu)$-contact metric manifolds are locally φ-projectively symmetric if and only if the generalized $(\kappa, \ \mu)$-contact metric manifolds are $(\kappa, \ \mu)$-contact metric manifolds. Finally, it is shown that generalized $(\kappa, \ \mu)$-contact metric manifolds satisfying $P \cdot S = 0$ are η-Einstein manifolds.

2. Preliminaries

An odd dimensional differentiable manifold M^n is called an almost contact manifold if there is an almost contact structure (φ, ξ, η) consisting of a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η satisfying

$$\varphi^2(X) = -X + \eta(X)\xi, \;\; \eta(\xi) = 1. \tag{2.1}$$

From (2.1) it follows that

$$\varphi\xi = 0, \;\; \eta \circ \varphi = 0.$$

Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \;\; for\ all\ X, Y \in \text{TM}. \tag{2.2}$$

An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y), \ \ for\ all\ X, Y \in \mathrm{TM}. \tag{2.3}$$

Given a contact metric manifold $M^n(\varphi, \xi, \eta, g)$ we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\mathrm{L}_\xi\varphi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi = 0, \ h\varphi + \varphi h = 0, \ \nabla\xi = -\varphi - \varphi h, \ trace(h) = trace(\varphi h) = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection.
A contact metric manifold is said to be an η-Einstein manifold if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.5}$$

where a, b are smooth functions and $X, Y \in \mathrm{TM}$, S is the Ricci tensor.

Blair, Koufogiorgos and Papantoniou [5] considered the (κ, μ)-nullity condition and gave several reasons for studying it. The (κ, μ)-nullity distribution $\mathrm{N}(\kappa, \mu)$ ([5], [3]) of a contact metric manifold M is defined by

$$N(\kappa, \mu) : p \mapsto N_p(\kappa, \mu) = [U \in T_pM \mid R(X, Y)U = (\kappa I + \mu h)(g(Y, U)X - g(X, U)Y)]$$

for all $X, Y \in \mathrm{TM}$, where $(\kappa, \mu) \in \mathbb{R}^2$.

A contact metric manifold M^n with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)- contact metric manifold. Then we have

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \ \ for\ all\ X, Y \in \mathrm{TM}. \tag{2.6}$$

If $\mu = 0$, then the (κ, μ)-nullity distribution $\mathrm{N}(\kappa, \mu)$ is reduced to κ-nullity distribution $\mathrm{N}(\kappa)$ [17]. If $\xi \in N(\kappa)$, then we call contact metric manifold M an $\mathrm{N}(\kappa)$- contact metric manifold.

In a (κ, μ)-contact metric manifold the following relations hold:

$$h^2 = (\kappa - 1)\varphi^2, \tag{2.7}$$

$$(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$

$$R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \tag{2.9}$$

$$S(X, \xi) = (n - 1)\kappa\eta(X), \tag{2.10}$$

$$\begin{aligned} S(X, Y) = {} & [(n-3) - \frac{n-1}{2}\mu]g(X, Y) + \\ & [(n-3) + \mu]g(hX, Y) + \left[(3-n) + \frac{n-1}{2}(2\kappa + \mu)\right]\eta(X)\eta(Y), \end{aligned} \tag{2.11}$$

$$r=(n-1)\left(n-3+\kappa-\frac{n-1}{2}\mu\right). \tag{2.12}$$

A $(\kappa,\ \mu)$-contact metric manifold is called a generalized $(\kappa,\ \mu)$-contact metric manifold if κ, μ are smooth functions. In [13], Koufogiorgos and Tsichlias proved its existence for the 3-dimensional case, whereas greater than 3-dimensional, such manifold does not exist. In generalized $(\kappa,\ \mu)$-contact metric manifold $M^3(\varphi,\xi,\eta,g)$ the following relations hold ([13], [3]):

$$\xi\kappa=0, \tag{2.13}$$

$$\xi r=0, \tag{2.14}$$

$$h\ grad\ \mu=grad\ \mu, \tag{2.15}$$

$$R(X,Y)\xi=\kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY], \tag{2.16}$$

$$S(X,Y)=-\mu g(X,Y)+\mu g(hX,Y)+(2\kappa+\mu)\eta(X)\eta(Y), \tag{2.17}$$

$$S(X,hY)=-\mu g(X,hY)-(\kappa-1)\mu g(X,Y)+(\kappa-1)\mu\eta(X)\eta(Y), \tag{2.18}$$

$$S(X,\xi)=2\kappa\eta(X), \tag{2.19}$$

$$QX=\mu(hX-X)+(2\kappa+\mu)\eta(X)\xi, \tag{2.20}$$

$$r=2(\kappa-\mu). \tag{2.21}$$

$$\begin{aligned}(\nabla_X h)Y &= \{(1-\kappa)g(X,\varphi Y)\\ &\quad -g(X,\varphi hY)\}\xi-\eta(Y)\{(1-\kappa)\varphi X\\ &\quad +\varphi hX\}-\mu\eta(X)\varphi hY,\end{aligned} \tag{2.22}$$

$$(\nabla_X\varphi)Y=\{g(X,Y)+g(X,hY)\}\xi-\eta(Y)(X+hX). \tag{2.23}$$

3. ξ-projectively flat generalized $(\kappa,\ \mu)$-contact metric manifolds

Assume that M^3 is a ξ-projectively flat $(\kappa,\ \mu)$-contact metric manifold. So we have

$$P(X,Y)\xi=0. \tag{3.1}$$

From (1.1) we have in 3-dimensional generalized (κ, μ)-contact metric manifold,

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}[S(Y, Z)X - S(X, Z)Y]. \tag{3.2}$$

Putting $Z = \xi$ in (3.2) we obtain

$$P(X, Y)\xi = R(X, Y)\xi - \frac{1}{2}[S(Y, \xi)X - S(X, \xi)Y]. \tag{3.3}$$

Using (2.16) and (2.19) we get

$$\mu[\eta(Y)hX - \eta(X)hY] = 0. \tag{3.4}$$

From (3.4) we can conclude either $\mu = 0$ or

$$\eta(Y)hX = \eta(X)hY. \tag{3.5}$$

Putting $Y = \xi$ in (3.5) we have

$$hX = 0. \tag{3.6}$$

If $\mu = 0$, then M^3 is an $N(\kappa)$-contact metric manifold.
If $h = 0$, then M^3 is a Sasakian manifold.

Hence we can state the following:

Theorem 3.1. *Let M be a 3-dimensional ξ-projectively flat generalized (κ, μ)-contact metric manifold. Then M is either an $N(\kappa)$-contact metric manifold or a Sasakian manifold.*

4. Locally φ-projectively symmetric generalized (κ, μ)-contact metric manifolds

Definition 1. A contact metric manifold is said to be locally φ-symmetric if the manifold satisfy the following:

$$\varphi^2((\nabla_X R)(Y, Z)W) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [15].

In this paper, we study locally φ-projectively symmetric 3-dimensional generalized (κ, μ)-contact metric manifolds. A generalized (κ, μ)-contact manifold is called φ-projectively symmetric if the condition

$$\varphi^2((\nabla_X P)(Y, Z)W) = 0, \tag{4.2}$$

holds on the manifold, where X, Y, Z, W are orthogonal to ξ.

Let us consider M be a 3-dimensional generalized (κ, μ)-contact metric manifold. Taking covariant differentiation of (3.2) we have

$$
\begin{aligned}
((\nabla_W P)(X, Y)Z) = &-\{(W\kappa) + (W\mu)\}[g(Y, Z)X - g(X, Z)Y] \\
&+ (W\mu)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - \\
&g(hX, Z)Y] - \frac{1}{2}\{(W\mu)[g(hY, Z)X - g(Y, Z)X] \\
&- (W\mu)[g(hX, Z)Y - g(X, Z)Y]\},
\end{aligned} \tag{4.3}
$$

for all vector fields X, Y, Z, W orthogonal to ξ.
Operating φ^2 to the equation (4.3) we obtain

$$
\begin{aligned}
\varphi^2((\nabla_W P)(X, Y)Z) = &(W\kappa)[g(Y, Z)X - g(X, Z)Y] \\
&+ \frac{1}{2}(W\mu)[g(Y, Z)X - g(X, Z)Y] \\
&- (W\mu)[g(Y, Z)hX - g(X, Z)hY] \\
&- \frac{1}{2}(W\mu)[g(hY, Z)X - g(hX, Z)Y],
\end{aligned} \tag{4.4}
$$

for all vector fields X, Y, Z, W orthogonal to ξ.
Thus, from (4.4) we conclude that if κ and μ are constants, then M is locally φ-projectively symmetric.

Conversely, let us consider that M is locally φ-projectively symmetric.
From (4.2) and (4.4) we have

$$
\begin{aligned}
&(W\kappa)[g(Y, Z)X - g(X, Z)Y] \\
&+ \frac{1}{2}(W\mu)[g(Y, Z)X - g(X, Z)Y] \\
&- (W\mu)[g(Y, Z)hX - g(X, Z)hY] \\
&- \frac{1}{2}(W\mu)[g(hY, Z)X - g(hX, Z)Y] = 0.
\end{aligned} \tag{4.5}
$$

Taking inner product with U of (4.5) we get

$$
\begin{aligned}
&(W\kappa)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\frac{1}{2}(W\mu)[g(Y, Z)g(X, U) \\
&- g(X, Z)g(Y, U)] - (W\mu)[g(Y, Z)g(hX, U) - g(X, Z)g(hY, U)] \\
&- \frac{1}{2}(W\mu)[g(hY, Z)g(X, U) - g(hX, Z)g(Y, U)] = 0.
\end{aligned} \tag{4.6}
$$

Contracting X and Z we obtain

$$
(-2(W\kappa) - (W\mu))g(Y, U) + \frac{1}{2}(W\mu)g(hY, U) = 0. \tag{4.7}
$$

From (4.7) we have

$$(-2(W\kappa) - (W\mu))Y + \frac{1}{2}(W\mu)hY = 0. \tag{4.8}$$

Applying h on both sides of (4.8) we get

$$(-2(W\kappa) - (W\mu))hY + \frac{1}{2}(W\mu)h^2Y = 0. \tag{4.9}$$

Taking trace on both sides of (4.9) and using $trace(h) = 0$ we obtain μ is constant. Thus, from (4.9) we can conclude that κ is also constant.

Therefore, we can state the following:

Theorem 4.1. *Let M be a 3-dimensional generalized (κ, μ)-contact metric manifold. M is locally φ-projectively symmetric if and only if M is a (κ, μ)-contact metric manifold.*

5. Generalized (κ, μ)-contact metric manifolds satisfying $P \cdot S = 0$

Let M^3 be a generalized (κ, μ)-contact metric manifold satisfying $P \cdot S = 0$, which implies that

$$S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0. \tag{5.1}$$

Putting $X = U = \xi$ in (5.1) and using (2.19) we have

$$S(P(\xi, Y)\xi, V) = 2\kappa\eta(P(\xi, Y)V). \tag{5.2}$$

Again putting $X = \xi$ in (3.2) we obtain

$$\begin{aligned} P(\xi, Y)Z = {} & \kappa[g(Y, Z)\xi - \eta(Z)Y] \\ & + \mu[-\eta(Z)hY + g(hY, Z)\xi] \\ & - \frac{1}{2}[-\mu g(Y, Z)\xi + \mu g(hY, Z)\xi + (2\kappa + \mu)\eta(Y)\eta(Z)\xi] - \kappa\eta(Z)Y. \end{aligned} \tag{5.3}$$

Putting $Z = \xi$ in (5.3) we have

$$P(\xi, Y)\xi = 2\kappa Y - \mu hY. \tag{5.4}$$

Using (5.3) and (5.4) in (5.2 we get

$$\begin{aligned} \mu g(hY, V) = {} & \left[\frac{\kappa\mu + \mu^2(\kappa - 1) - 2\kappa^2}{3\kappa - \mu}\right] g(Y, V) \\ & \left[\frac{-\mu^2(\kappa - 1) - \kappa(2\kappa + \mu) + 4\kappa^2}{3\kappa - \mu}\right] \eta(Y)\eta(V). \end{aligned} \tag{5.5}$$

Using (5.5) in (2.17) we have

$$S(Y,V) = ag(Y,V) + b\eta(Y)\eta(V), \tag{5.6}$$

where

$$a = \frac{-2\kappa\mu + \mu^2\kappa - 2\kappa^2 - 2\mu^2}{3\kappa - \mu}$$

and

$$b = \frac{-\mu^2\kappa + 8\kappa^2}{3\kappa - \mu}.$$

From (5.6) we can state the following:

Theorem 5.1. *Let M be a 3-dimensional generalized (κ,μ)-contact metric manifold satisfying $P \cdot S = 0$. Then M is an η-Einstein manifold.*

REFERENCES

1. A. Sarkar, U. C. De and M. Sen: *Some results on generalized (κ,μ)-contact metric manifolds*. Acta Universitatis Apulensis., **32** (2012), 49-59.
2. A. Yildiz, U. C. De and A. Cetinkaya: *On some classes of 3-dimensional generalized (κ,μ)-contact metric manifolds*, submitted.
3. B. J. Papantoniou: *Contact Riemannian manifolds satisfying $R(\xi,X)\cdot R = 0$ and $\xi \in (\kappa,\mu)$-nullity distribution*, Yokohama Math. J. **40**(1993), 149-161.
4. D. E. Blair: *Two remarks on contact metric structures*, Tohoku Math. J., **29** (1977), 319-324.
5. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou: *Contact metric manifold satisfying a nullity condition*, Israel J.Math. **91** (1995), 189-214.
6. E. Boeckx: *A full classificstion of contact metric (κ,μ)-spaces*, Illinois J.Math. **44** (2000), 212-219 .
7. F. Gouli-Andreon and P. J. Xenos: *A class of contact metric 3-manifolds with $\xi \in N(\kappa,\ \mu)$ and $\kappa,\ \mu$ functions*, Algebras. Group and Geom. **17** (200), 401-407.
8. G. Zhen: *On conformal symmetric K-contact manifolds*, Chinese Quart. J. Math. **7**(1992), 5-10.
9. G. Zhen, J. L. Cabrerizo, L. M. Fernández and M. Fernández: *On ξ-conformally flat contact metric manifols*, Indian J. Pure Appl. Math. **28** (1997), 725-734.
10. J. L. Cabrerizo, L. M. Fernández., M. Fernández and G. Zhen: *The structure of a clss of K-contact manifolds*, Acta Math. Hungar. **82** (4)(1999), 331-340.
11. K. Yano and S. Bochner: *Curvature and Betti numbers*, Annals of mathematics studies, **32** (Princeton university press)(1953).
12. S. Ghosh and U. C. De: *On a class of generalized (κ,μ)-contact metric manifolds*, Proceeding of the Jangjeon Math. Society. **13** (2010), 337-347.
13. T. Koufogiorgos and C. Tsichlilias, *On the existance of new class of contact metric manifolds*, Canad. Math.Bull. XX(Y) (2000), 1-8.
14. T. Koufogiorgos and C. Tsichlilias: *Generalized (κ,μ)-contact metric manifolds with $\| grad\ \kappa \| = constant$*, J. Geom. **78** (2003), 54-65.

15. T. Takahashi: *Sasakian φ-symmetric spaces*, Tôhoku Math. J. **(2)29** (1977), 91-113.

16. S. Ghosh: *On a class of (κ, μ)- contact manifolds*, Bull. Cal. Math. Soc. **102** (2010), 219-226.

17. S. Tanno: *Ricci curvatures of contact Riemannian manifolds*, Tôhoku Math. J. **40** (1988), 441-448.

18. U. C. De and A. De: *On some curvature properties of K-contact manifolds*, Extracta Mathematicae **27** (2012), 125-134.

19. U. C. De and J. Sengupta: *On a Type of SemiSymmetric Metric Connection on an almost contact metric connection*, Facta Universitatis (Niš) Ser. Math. Inform. **16** (2001), 8796.

Srimayee Samui
Umeschandra College
13, Surya Sen Street
Kolkata-700012, West Bengal, India
srimayee.samui@gmail.com