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THE FEKETE-SZEGÖ PROBLEMS FOR SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SIGMOID FUNCTION

Halit Orhan[1], Gangadharan Murugusundaramoorthy[2] and Murat Çağlar[3]

[1] **Faculty of Science, Department of Mathematics**
Ataturk University, 25240 Erzurum, Turkiye
[2] **School of Advanced Sciences, Department of Mathematics**
Vellore Institue of Technology, 632014 Vellore, India
[3] **Faculty of Science, Department of Mathematics**
Erzurum Technical University, 25100 Erzurum, Turkiye

Abstract. The purpose of this article is to introduce a new subclass of analytic and bi-univalent functions, in associated with sigmoid function and to investigate the upper bounds for $|a_2|$ and $|a_3|$, where a_2, a_3 are the initial Taylor-Maclaurin coefficients. Further, we obtain the Fekete-Szegö inequalities for this subclass of the bi-univalent function class Σ. We also give several illustrative examples of the bi-univalent function class which we introduce here.
Keywords: Analytic functions; univalent functions; bi-univalent functions; Sigmoid function; upper bounds; Fekete-Szegö problem; subordination between analytic functions.

1. Introduction

Here, in this paper, we denote by $\mathcal{A}$ the class of functions of the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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Corresponding Author: Murat Çağlar, Faculty of Science, Department of Mathematics, Erzurum Technical University, 25100 Erzurum, Turkiye | E-mail: murat.caglar@erzurum.edu.tr
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which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $\mathbb{C}$ being, as usual, the set of complex numbers. Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$ (for details, see [3, 11]). Let the functions f and g be analytic in $\mathbb{D}$. We say that the function f is subordinate to g, written as $f \prec g$ in $\mathbb{D}$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function ω, analytic in $\mathbb{D}$ with $\omega(0) = 0, \quad |\omega(z)| < 1$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{D})$. It follows that

$$f(z) \prec g(z) \qquad (z \in \mathbb{D}) \ \Rightarrow \ f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

In particular, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \qquad (z \in \mathbb{D}) \ \Leftrightarrow \ f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \qquad (z \in \mathbb{D}),$$

which are analytic with $\operatorname{Re} p(z) > 0$. Here $p(z)$ is called as Caratheodory function [3]. It is well known that the following correspondence between the class $\mathcal{P}$ and the class of Schwarz functions ω exists: $p \in \mathcal{P}$ if and only if $p(z) = 1 + \omega(z)/1 - \omega(z)$. Let $\mathcal{P}(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\operatorname{Re}\{p(z)\} > \beta$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w, \qquad |w| < r_0(f); \quad r_0(f) \geqq \frac{1}{4},$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). For a brief historical account and for several interesting examples of functions in the class Σ; see the pioneering work on this subject by Srivastava *et al.* [14], which actually revived the study of bi-univalent functions in recent years (see [1, 2, 7, 10, 13, 15, 16, 17, 18]). In recent years, various subclasses of bi-univalent functions related to shell-like curves were studied in [6, 12].

Special functions plays a significant role in Geometric Function Theory and also in actual mathematical intentions for scientist and engineers. There are three types of functions namely piecewise linear function, threshold function and sigmoid function. In the hardware implementation of neural network the most important and popular activation function is the sigmoid function. Sigmoid function is often used with gradient descendent type learning algorithm. To be precise, modified sigmoid function $\psi(z) = \frac{2}{1+e^{-z}}$ has applications in problems of physics, engineering and computer science. These functions acts as a squashing function which is the output of a neuron in a neural network between certain values (usually 0 and 1 and -1 and 1). Due to differentiability of the sigmoid function it is useful in weight-learning algorithm. The sigmoid function increases the size of the hypothesis space that the network can represent. In following are some of its advantages

1. It gives real numbers between 0 and 1.
2. It maps a very large output domain to a small range of outputs.
3. It never loses information because it is one-to-one function.
4. It increases monotonically.

For more details see [4, 8]. We note that sigmoid functions holds the conditions of univalent functions. Lately, based on the techniques of Ma and Minda [9], Goel and Kumar in [5] defined the class $\mathcal{S}^*_{SG}$, based on subordination principle, as

$$\frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-w(z)}} \quad (z \in \mathbb{D}). \tag{1.3}$$

We begin by recalling each of the following lemmas which will be required in our investigation.

Lemma 1.1. [4] *Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \tag{1.4}$$

then $\Phi(z) \in \mathcal{P}$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.2. [4] *Let*

$$\begin{aligned} \Phi_{m,n}(z) = 2h(z) &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \\ &= 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + ... \end{aligned} \tag{1.5}$$

then $|\Phi_{m,n}(z)| < 2$.

Lemma 1.3. [4] *If $\Phi(z) \in \mathcal{P}$ is starlike, then f is a normalized univalent function of the form* (1.1).

Setting $m = 1$, Fadipe-Joseph *et al.* [4] remarked that

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$. As such, $|c_n| \leq 2$, $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and the result is sharp for each n. Lately there has been considerably large number of sequels to the aforementioned work of Srivastava *et al.* [14], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [1, 2, 7, 10, 11, 13, 18] and references therein. Stimulated by the earlier works on Σ; in the present work, we introduce the following new subclass of bi-univalent function, as given in Definition 1.1 and obtain the initial estimates of the coefficients a_2 and a_3. Simultaneously, we also obtain the corresponding Fekete-Szegö functional inequalities. We noted that these results are new and not studied sofar in association with sigmoid function.

Definition 1.1. A function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$, $\kappa \geq 0$, $\vartheta \geq 1$, $\delta \geq 0$, if the following conditions are satisfied:

$$(1-\vartheta)\left(\frac{f(z)}{z}\right)^{\kappa} + \vartheta f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} + \delta z f''(z) \prec \Phi(z)$$

and

$$(1-\vartheta)\left(\frac{g(w)}{w}\right)^{\kappa} + \vartheta g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} + \delta w g''(w) \prec \Phi(w),$$

where $g(w) = f^{-1}(w)$ assumed as in (1.2).

By suitably specializing the parameters κ, ϑ and δ, the class $\mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$ reduces to various new subclasses, as illustrated the following remark:

Remark 1.1.

1. For $\delta = 0$, we let $\mathcal{G}_{\Sigma}^{\kappa,0,\vartheta}(\Phi) \equiv \mathcal{N}_{\Sigma}^{\kappa,\vartheta}(\Phi)$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{N}_{\Sigma}^{\kappa,\vartheta}(\Phi(z))$, if

$$(1-\vartheta)\left(\frac{f(z)}{z}\right)^{\kappa} + \vartheta f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} \prec \Phi(z)$$

   and

$$(1-\vartheta)\left(\frac{g(w)}{w}\right)^{\kappa} + \vartheta g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} \prec \Phi(w),$$

   where $g(w) = f^{-1}(w)$ assumed as in (1.2).

2. For $\vartheta = 1$ and $\delta = 0$, we let $\mathcal{G}_{\Sigma}^{\kappa,0,1}(\Phi) \equiv \mathcal{B}_{\Sigma}^{\kappa}(\Phi)$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{B}_{\Sigma}^{\kappa}(\Phi)$, if

$$f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} \prec \Phi(z) \quad \textit{and} \quad g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} \prec \Phi(w),$$

   where $g(w) = f^{-1}(w)$ assumed as in (1.2).

3. For $\vartheta = 1$ and $\delta = 0 = \kappa$, we let $\mathcal{G}_{\Sigma}^{0,0,1}(\Phi) \equiv \mathcal{S}_{\Sigma}(\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{S}_{\Sigma}(\Phi)$, if

$$\frac{zf'(z)}{f(z)} \prec \Phi(z) \quad \textit{and} \quad \frac{wg'(w)}{g(w)} \prec \Phi(w),$$

   where $g(w) = f^{-1}(w)$ assumed as in (1.2).

4. For $\kappa = 1$, we let $\mathcal{G}_{\Sigma}^{1,\delta,\vartheta}(\Phi) \equiv \mathcal{M}_{\Sigma}^{\delta,\vartheta}(\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{M}_{\Sigma}^{\delta,\vartheta}(\Phi)$, if
$$(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) + \delta z f''(z) \prec \Phi(z)$$
and
$$(1-\vartheta)\frac{g(w)}{w} + \vartheta g'(w) + \delta w g''(w) \prec \Phi(w),$$
where $g(w) = f^{-1}(w)$ assumed as in (1.2).
5. For $\vartheta = \kappa = 1$, we let $\mathcal{G}_{\Sigma}^{1,\delta,1}(\Phi) \equiv \mathcal{Q}_{\Sigma}(\delta, \Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{Q}_{\Sigma}(\delta, \Phi)$, if
$$f'(z) + \delta z f''(z) \prec \Phi(z)$$
and
$$g'(w) + \delta w g''(w) \prec \Phi(w),$$
where $g(w) = f^{-1}(w)$ assumed as in (1.2).
6. For $\kappa = 1$ and $\delta = 0$, we let $\mathcal{G}_{\Sigma}^{1,0,\vartheta}(\Phi), \equiv \mathcal{F}_{\Sigma}(\vartheta, \Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{F}_{\Sigma}(\vartheta, \Phi(z))$, if
$$(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) \prec \Phi(z)$$
and
$$(1-\vartheta)\frac{g(w)}{w} + \vartheta g'(w) \prec \Phi(w),$$
where $g(w) = f^{-1}(w)$ assumed as in (1.1).
7. For $\vartheta = 1$, $\kappa = 1$ and $\delta = 0$, we have the class $\mathcal{G}_{\Sigma}^{1,0,1}(\Phi) \equiv \mathcal{H}_{\Sigma}(\Phi)$. A function $f \in \Sigma$ as assumed in (1.1) is said to be in $\mathcal{H}_{\Sigma}(\Phi)$, if
$$f'(z) \prec \Phi(z) \quad \textit{and} \quad g'(w) \prec \Phi(w),$$
where $g(w) = f^{-1}(w)$ assumed as in (1.2).

In order to prove our results for the functions in the class $\mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$, we recall the following lemma.

Lemma 1.4. [11] *If $p \in \mathcal{P}$, then $|p_i| \leqq 2$ for each i, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{D}$, for which*
$$\operatorname{Re}\{p(z)\} > 0 \qquad (z \in \mathbb{D}),$$
where
$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{D}).$$
In particular, the equality holds for all n for the next function
$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n.$$

In the following section , we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$ and its special cases. Also, Fekete-Szegö inequality for functions in this subclass.

2. Coefficient estimates and Fekete-Szegö inequality

In order to discuss coefficient estimates and Fekete-Szegö inequality for $f \in \mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$ we let the functions s and t in $\mathcal{P}$ given by

$$s(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \textit{and} \quad t(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \;\; (z, w \in \mathbb{D}). \tag{2.1}$$

Letting

$$s(z) = 1 + c_1 z + c_2 z^2 + \cdots = \frac{1 + u(z)}{1 - u(z)}, \tag{2.2}$$

we imply that

$$\begin{aligned} u(z) = \frac{s(z)-1}{s(z)+1} &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots} \\ &= \frac{c_1}{2} z + \left(c_2 - \frac{c_1^2}{2}\right) \frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) \frac{z^3}{2} + \cdots, \end{aligned}$$

so that

$$\begin{aligned} \Phi(u(z)) &= \frac{2}{1 + e^{-u(z)}} \\ &= 1 + \frac{1}{4} c_1 z + \left(\frac{1}{4} c_2 - \frac{1}{8} c_1^2\right) z^2 + \left(\frac{11}{192} c_1^3 - \frac{1}{4} c_2 c_1 + \frac{1}{4} c_3\right) z^3 \\ &\quad + \left(\frac{1}{4} c_1^2 c_2 - \frac{1}{2} c_3 c_1 - \frac{1}{4} c_2^2 + \frac{1}{2} c_4\right) z^4 + \ldots. \end{aligned} \tag{2.3}$$

Similarly, from (2.1) we have

$$\Phi(v(w)) = 1 + \frac{1}{4} d_1 w + \left(\frac{1}{4} d_2 - \frac{1}{8} d_1^2\right) w^2 + \left(\frac{11}{192} d_1^3 - \frac{1}{4} d_2 d_1 + \frac{1}{4} d_3\right) w^3 \ldots. \tag{2.4}$$

We now prove our first result asserted by Theorem 2.1 below.

Theorem 2.1. *Let f be assumed as in* (1.1) *and $f \in \mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$. Then*

$$|a_2| \leq min\{F_1, F_2, F_3\},$$

where

$$F_1 = \frac{1}{2(\vartheta + \kappa + 2\delta)},$$

$$F_2 = \sqrt{\frac{2}{(2\vartheta + \kappa)(\kappa + 1) + 12\delta}}$$

and

$$F_3 = \frac{1}{\sqrt{(2\vartheta + \kappa)(\kappa + 1) + 12\delta + 4(\vartheta + \kappa + 2\delta)^2}}.$$

Also

$$|a_3| \leq min\{G_1, G_2, G_3\},$$

where

$$G_1 = \frac{2(\vartheta+\kappa+2\delta)^2 + (2\vartheta+\kappa+6\delta)}{4\,(2\vartheta+\kappa+6\delta)\,(\vartheta+\kappa+2\delta)^2},$$

$$G_2 = \frac{8\,(2\vartheta+\kappa+6\delta) + [(2\vartheta+\kappa)(\kappa+1)+12\delta]}{2\,(2\vartheta+\kappa+6\delta)\,[(2\vartheta+\kappa)(\kappa+1)+12\delta]},$$

$$G_3 = \frac{(2\vartheta+\kappa+6\delta) + 2\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+(\vartheta+\kappa+2\delta)^2\right]}{2\,(2\vartheta+\kappa+6\delta)\,[(2\vartheta+\kappa)(\kappa+1)+12\delta+(\vartheta+\kappa+2\delta)^2]}.$$

Proof. Since $f \in \mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$, from the Definition 1.1, we have

$$(1-\vartheta)\left(\frac{f(z)}{z}\right)^{\kappa} + \vartheta f'(z)\left(\frac{f(z)}{z}\right)^{\kappa-1} + \delta z f''(z) = \Phi(s(z)) \tag{2.5}$$

and for $g = f^{-1}$

$$(1-\vartheta)\left(\frac{g(w)}{w}\right)^{\kappa} + \vartheta g'(w)\left(\frac{g(w)}{w}\right)^{\kappa-1} + \delta w g''(w) = \Phi(t(w)), \tag{2.6}$$

where $z,\ w \in \mathbb{D}$. By virtue of (2.3), (2.4), (2.5) and (2.6), we have

$$(\vartheta+\kappa+2\delta)\,a_2 = \frac{c_1}{4}, \tag{2.7}$$

$$(2\vartheta+\kappa)\left[\left(\frac{\kappa-1}{2}\right)a_2^2 + \left(1+\frac{6\delta}{2\vartheta+\kappa}\right)a_3\right] = \frac{1}{4}c_2 - \frac{1}{8}c_1^2, \tag{2.8}$$

$$-(\vartheta+\kappa+2\delta)\,a_2 = \frac{d_1}{4}, \tag{2.9}$$

and

$$(2\vartheta+\kappa)\left[\left(\frac{\kappa+3}{2}+\frac{12\delta}{2\vartheta+\kappa}\right)a_2^2 - \left(1+\frac{6\delta}{2\vartheta+\kappa}\right)a_3\right] = \frac{1}{4}d_2 - \frac{1}{8}d_1^2. \tag{2.10}$$

From (2.7) and (2.9), we obtain

$$c_1 = -d_1,$$

and

$$2\left(\vartheta+\kappa+2\delta\right)^2 a_2^2 = \frac{c_1^2+d_1^2}{16}, \tag{2.11}$$

$$a_2^2 = \frac{c_1^2+d_1^2}{32\left(\vartheta+\kappa+2\delta\right)^2}\ . \tag{2.12}$$

Now, applying Lemma 1.4, we obtain

$$|a_2| \leq \frac{1}{2(\vartheta+\kappa+2\delta)}. \tag{2.13}$$

By adding (2.8) and (2.10), we have

$$[(2\vartheta+\kappa)(\kappa+1)+12\delta]a_2^2 = \frac{1}{4}(c_2+d_2) - \frac{1}{8}\left(c_1^2+d_1^2\right). \tag{2.14}$$

Again by applying Lemma 1.4, we obtain,

$$|a_2| \leq \sqrt{\frac{2}{(2\vartheta+\kappa)(\kappa+1)+12\delta}}. \tag{2.15}$$

Now, by substituting (2.11) in (2.14), we reduce that

$$a_2^2 = \frac{c_2+d_2}{4\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2\right]}. \tag{2.16}$$

Now, applying Lemma 1.4, we obtain

$$|a_2| \leq \frac{1}{\sqrt{(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2}}. \tag{2.17}$$

By subtracting (2.10) from (2.8), we obtain

$$a_3 = \frac{c_2-d_2}{8(2\vartheta+\kappa+6\delta)} + a_2^2. \tag{2.18}$$

Hence by Lemma 1.4, we have

$$|a_3| \leq \frac{|c_2|+|d_2|}{8(2\vartheta+\kappa+6\delta)} + |a_2|^2 = \frac{1}{2(2\vartheta+\kappa+6\delta)} + |a_2|^2.$$

In view of (2.11), we get

$$|a_3| \leq \frac{2(\vartheta+\kappa+2\delta)^2+(2\vartheta+\kappa+6\delta)}{4(2\vartheta+\kappa+6\delta)(\vartheta+\kappa+2\delta)^2}.$$

By using (2.15)

$$|a_3| \leq \frac{8(2\vartheta+\kappa+6\delta)+[(2\vartheta+\kappa)(\kappa+1)+12\delta]}{2(2\vartheta+\kappa+6\delta)[(2\vartheta+\kappa)(\kappa+1)+12\delta]}.$$

Then in view of (2.16), we obtain

$$|a_3| \leq \frac{2(2\vartheta+\kappa+6\delta)+\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+(\vartheta+\kappa+2\delta)^2\right]}{2(2\vartheta+\kappa+6\delta)\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+(\vartheta+\kappa+2\delta)^2\right]}.$$

Similarly, we can prove the following theorem. □

Theorem 2.2. *Let f be given by* (1.1) *and $f \in \mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$. Then for $\nu \in \mathbb{R}$,*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \frac{1}{2(2\vartheta+\kappa+6\delta)} & ; 0 \leq |h(\nu)| \leq \frac{1}{4(2\vartheta+\kappa+6\delta)} \\ 4\,|h(\nu)| & ; |h(\nu)| \geq \frac{1}{4(2\vartheta+\kappa+6\delta)} \end{cases}$$

where

$$h(\nu) = \frac{1-\nu}{2\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2\right]}.$$

Proof. From (2.18), for $\nu \in \mathbb{R}$, we have

$$a_3 - \nu a_2^2 = \frac{c_2 - d_2}{8\,(2\vartheta+\kappa+6\delta)} + (1-\nu)\,a_2^2. \tag{2.19}$$

By substituting (2.16) in (2.19), we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{(c_2 - d_2)}{8\,(2\vartheta+\kappa+6\delta)} + \frac{(1-\nu)(c_2+d_2)}{4\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2\right]} \\ &= \left(h(\nu) + \frac{1}{8\,(2\vartheta+\kappa+6\delta)}\right)c_2 + \left(h(\nu) - \frac{1}{8\,(2\vartheta+\kappa+6\delta)}\right)d_2, \end{aligned} \tag{2.20}$$

where

$$h(\nu) = \frac{1-\nu}{4\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2\right]}.$$

Thus by taking modulus of (2.20),

$$|a_3 - \nu a_2^2| \leq \left|h(\nu) + \frac{1}{4\,(2\vartheta+\kappa+6\delta)}\right| + \left|h(\nu) - \frac{1}{4\,(2\vartheta+\kappa+6\delta)}\right|$$

and

$$h(\nu) = \frac{1-\nu}{2\left[(2\vartheta+\kappa)(\kappa+1)+12\delta+4(\vartheta+\kappa+2\delta)^2\right]}. \tag{2.21}$$

Thus, we conclude that

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \frac{1}{2(2\vartheta+\kappa+6\delta)} & ; 0 \leq |h(\nu)| \leq \frac{1}{4(2\vartheta+\kappa+6\delta)} \\ 4\,|h(\nu)| & ; |h(\nu)| \geq \frac{1}{4(2\vartheta+\kappa+6\delta)} \end{cases} \tag{2.22}$$

where $h(\nu)$ is given by (2.21). □

By taking $\nu = 1$ in above theorem one can easily state the following:

Remark 2.1. Let f be given by (1.1) and $f \in \mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$. Then

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2(2\vartheta+\kappa+6\delta)}.$$

3. Corollaries and consequences

In this section, we give coefficient estimates and Fekete-Szegö inequalities for the subclasses of $\mathcal{G}_{\Sigma}^{\kappa,\delta,\vartheta}(\Phi)$.

Corollary 3.1. *Let* $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ *be in the class* $\mathcal{N}_{\Sigma}^{\kappa,\vartheta}(\Phi)$. *Then*

$$|a_2|\leq min\{F_1,F_2,F_3\},$$

where

$$F_1=\frac{1}{2\left(\vartheta+\kappa\right)},$$

$$F_2=\sqrt{\frac{2}{(2\vartheta+\kappa)(\kappa+1)}}$$

and

$$F_3=\frac{1}{\sqrt{(2\vartheta+\kappa)(\kappa+1)+4(\vartheta+\kappa)^2}}.$$

Also

$$|a_3|\leq min\{G_1,G_2,G_3\},$$

where

$$G_1=\frac{2(\vartheta+\kappa)^2+(2\vartheta+\kappa)}{4\left(2\vartheta+\kappa\right)\left(\vartheta+\kappa\right)^2},$$

$$G_2=\frac{\kappa+9}{2\left[(2\vartheta+\kappa)(\kappa+1)\right]},$$

$$G_3=\frac{(2\vartheta+\kappa)+2\left[(2\vartheta+\kappa)(\kappa+1)+(\vartheta+\kappa)^2\right]}{2\left(2\vartheta+\kappa\right)\left[(2\vartheta+\kappa)(\kappa+1)+(\vartheta+\kappa)^2\right]}$$

and for $\nu\in\mathbb{R}$,

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\frac{1}{2(2\vartheta+\kappa)} & ;0\leq|h(\nu)|\leq\frac{1}{4(2\vartheta+\kappa)}\\ 4\left|h(\nu)\right| & ;|h(\nu)|\geq\frac{1}{4(2\vartheta+\kappa)}\end{cases}$$

where $h(\nu)=\frac{1-\nu}{2[(2\vartheta+\kappa)(\kappa+1)+4(\vartheta+\kappa)^2]}$.

Remark 3.1. By taking $\vartheta=1$ in above corollary one can easily state the result for the function class $\mathcal{B}_{\Sigma}^{\kappa}(\Phi)$. Further by taking $\kappa=0$ one can derive the above results for $f\in\mathcal{S}_{\Sigma}(\Phi)$.

Corollary 3.2. *Let* $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ *be in the class* $\mathcal{M}_{\Sigma}^{\delta,\vartheta}(\Phi)$. *Then*

$$|a_2|\leq min\{F_1,F_2,F_3\},$$

where

$$F_1 = \frac{1}{2\left(\vartheta + 1 + 2\delta\right)},$$

$$F_2 = \sqrt{\frac{1}{2\vartheta + 1 + 6\delta}}$$

and

$$F_3 = \frac{1}{\sqrt{2(2\vartheta + 1 + 6\delta) + 4(\vartheta + 1 + 2\delta)^2}}.$$

Also

$$|a_3| \leq min\{G_1, G_2, G_3\},$$

where

$$G_1 = \frac{2(\vartheta + 1 + 2\delta)^2 + (2\vartheta + 1 + 6\delta)}{4\left(2\vartheta + 1 + 6\delta\right)\left(\vartheta + 1 + 2\delta\right)^2},$$

$$G_2 = \frac{5}{2\left[2\vartheta + 1 + 6\delta\right]},$$

$$G_3 = \frac{6\left(2\vartheta + 1 + 6\delta\right) + \left(\vartheta + 1 + 2\delta\right)^2}{2\left(2\vartheta + 1 + 6\delta\right)\left[2(2\vartheta + 1 + 6\delta) + (\vartheta + 1 + 2\delta)^2\right]}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \frac{1}{2(2\vartheta+1+6\delta)} & ; 0 \leq |h(\nu)| \leq \frac{1}{4(2\vartheta+1+6\delta)} \\ 4\left|h(\nu)\right| & ; |h(\nu)| \geq \frac{1}{4(2\vartheta+1+6\delta)} \end{cases}$$

where $h(\nu) = \frac{1-\nu}{4[(2\vartheta+1+6\delta)+2(\vartheta+1+2\delta)^2]}$.

4. Conclusions

Making use of the above said corollaries, suitably specificating the parameters as mentioned in Remark 1.1 ,one can easily obtain upper bounds for the coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $|a_3 - \nu a_2^2|$ for function classes illustrated in Remark 1.1.

5. Conflict of interest

The authors declare that they have no conflict of interest.

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