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FRACTIONAL OSTROWSKI INEQUALITIES FOR (s, m)-GODUNOVA-LEVIN FUNCTIONS

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Abstract. In this paper, we introduce some new classes of s-Godunova-Levin functions, which are called (s, m)-Godunova-Levin functions of first and second kinds. We show that these classes contain some previously known classes of convex functions. Finally, we establish some new Ostrowski inequalities for (s, m)-Godunova-Levin functions via fractional integrals. Some special cases are also discussed.
Keywords: Convex functions, (s, m)-Godunova-Levin functions, Ostrowski inequalities.

1. Introduction

Over the years theory of convex functions has received special attention from many researchers. Consequently, the classical concepts of convex functions have been extended and generalized in various different directions using novel and innovative ideas and techniques, see [1, 2, 3, 4, 5, 7, 9, 13, 16, 17, 18, 19]. Inspired by this, Dragomir [2, 3] has introduced and investigated a new class of Godunova-Levin functions which is called s-Godunova-Levin functions of second kind. Noor et al. [19] extended the class of Godunova-Levin functions and introduced the classes of s-Godunova-Levin functions of first kind, logarithmic s-Godunova-Levin functions of first and second kinds.

The interrelationship between theory of convex functions and theory of inequalities led many researchers to extend various classical inequalities known in the literature for these newly developed generalizations of classical convex functions. For details readers are referred to [2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25].

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I, the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the

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following inequality,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right], \tag{1.1}$$

holds. This result is known in the literature as the Ostrowski inequality [20].

Motivated by this ongoing research, we introduce new notions of (s,m)-Godunova-Levin functions of first and second kind in this paper. We show that these classes unify several other known classes of convex functions. We also derive some new Ostrowski type inequalities for (s,m)-Godunova-Levin functions of second kind. Some special cases are also discussed which can be derived from our results. The ideas and techniques of this paper may stimulate further research.

2. Preliminaries

In this section, we recall some preliminary concepts. First of all, let $I = [a,b] \subset \mathbb{R}$ be an interval and $\mathbb{R}$ be the set of real numbers.

Definition 2.1. [4] A nonnegative function $f : I \to \mathbb{R}$ is said to be P-function, if

$$f(tx + (1-t)y) \le f(x) + f(y), \ \forall x, y \in I, t \in [0,1]. \tag{2.1}$$

Definition 2.2. [7] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}, \forall x, y \in I, t \in (0,1). \tag{2.2}$$

For some further details on Godunova-Levin type of functions, see [14, 22].
Using the idea of Noor et al. [19], we define a new class of (s,m)-Godunova-Levin functions of first kind.

Definition 2.3. A function $f : I \to \mathbb{R}$ is said to be (s,m)-Godunova-Levin functions of first kind or $f \in Q^1_{(s,m)}$, if $\forall s, m \in (0,1]$, we have

$$f(tx + m(1-t)y) \le \frac{1}{t^s} f(x) + m\left(\frac{1}{1-t^s}\right) f(y), \ \forall x, y \in I, t \in (0,1). \tag{2.3}$$

We would like to mention that Definition 2.3 is also introduced and studied by LI et al. [9] independently. It is obvious that for $s = 1, m = 1$ the definition of (s,m)-Godunova-Levin functions of first kind collapses to the definition of Godunova-Levin functions. For $m = 1$ in Definition 2.3 we have the definition of s-Godunova-Levin functions of first kind, which is introduced and investigated by Noor et al. [19].
Motivated by the idea of Dragomir [2, 3], we again introduce a new class of s-Godunova-Levin function of second kind and derive some Ostrowski type inequalities. This is the main motivation of this paper.

Definition 2.4. A function $f : I \to \mathbb{R}$ is said to be (s,m)-Godunova-Levin functions of second kind or $f \in Q^2_{(s,m)}$, if $s \in [0,1]$, $m \in (0,1]$, we have

$$f(tx + m(1-t)y) \le \frac{1}{t^s} f(x) + m\left(\frac{1}{(1-t)^s}\right) f(y),\ \forall x, y \in I,\ t \in (0,1). \tag{2.4}$$

It is obvious that for $s = 0, m = 1$, (s,m)-Godunova-Levin functions of second kind reduces to P-functions. If $s = 1, m = 1$, it then reduces to Godunova-Levin functions. For $m = 1$, we have the definition of s-Godunova-Levin function of second kind introduced and studied by Dragomir [2, 3].
Now we discuss a preliminary definition from fractional calculus which will be helpful in deriving our main results.

Definition 2.5. [8] Let $a \ge 0$ and $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^\alpha_{a^+} f$ and $J^\alpha_{b^-} f$ of order $\alpha > 0$ are defined by

$$J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$J^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where $\Gamma(.)$ is the Gamma function.

Lemma 2.1. *[25] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f' \in L_1[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$, we have*

$$\left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)\right]$$
$$= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)b)dt.$$

3. Main Results

In this section, we derive our main results.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ and $f' \in L_1[a,b]$. If $|f'|$ is (s,m)-Godunova-Levin function of second kind and $|f'(x)| \le M$, then, for $\alpha > 0$, we have*

$$\left|\left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(b)\right]\right|$$
$$\le \min\{\vartheta_1(a,b;m,\alpha;x), \vartheta_2(a,b;m,\alpha;x)\},$$

where

$$\vartheta_1(a,b;m,\alpha;x) = \frac{M}{1+\alpha-s}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right] + \frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\left[\frac{(x-a)^{\alpha+1}|f'\left(\frac{b}{m}\right)|+(b-x)^{\alpha+1}|f'\left(\frac{a}{m}\right)|}{b-a}\right],$$

and

$$\vartheta_2(a,b;m,\alpha;x) = \left[\frac{m}{1+\alpha-s}|f'(x/m)| + M\frac{\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\right]\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].$$

Proof. Using Lemma 2.1 and the fact that $|f'|$ is (s,m)-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)\right]\right| \\
&= \left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)dt\right| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}|f'(tx+(1-t)a)|dt + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}|f'(tx+(1-t)b)|dt \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left[\frac{1}{t^s}|f'(x)| + \frac{m}{(1-t)^s}\left|f'\left(\frac{a}{m}\right)\right|\right]dt \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left[\frac{1}{t^s}|f'(x)| + \frac{m}{(1-t)^s}\left|f'\left(\frac{b}{m}\right)\right|\right]dt \\
&\le \frac{M}{1+\alpha-s}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right] \\
&\quad + \frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\left[\frac{(x-a)^{\alpha+1}|f'\left(\frac{b}{m}\right)|+(b-x)^{\alpha+1}|f'\left(\frac{a}{m}\right)|}{b-a}\right].
\end{aligned} \tag{3.1}$$

Similarly

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(b)\right]\right| \\
&\le \left[\frac{m}{1+\alpha-s}\left|f'\left(\frac{x}{m}\right)\right| + M\frac{\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\right]\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].
\end{aligned} \tag{3.2}$$

This completes the proof. □

Note that for $\alpha = 1$, Theorem 3.1 collapses to following result for (s, m)-Godunova-Levin function of second kind.

Corollary 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$ for all $x \in [a, b]$. If $|f'|$ is (s, m)-Godunova-Levin function of second kind and $|f'(x)| \le M$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \le \min\{\vartheta_1(a, b; m, 1; x), \vartheta_2(a, b; m, 1; x)\}, \tag{3.3}$$

where

$$\begin{aligned} &\vartheta_1(a, b; m, 1; x) \\ &= \frac{M}{2-s}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right] \\ &\quad + \frac{m}{(1-s)(2-s)}\left[\frac{(x-a)^2|f'\left(\frac{b}{m}\right)| + (b-x)^2|f'\left(\frac{a}{m}\right)|}{b-a}\right], \end{aligned}$$

and

$$\vartheta_2(a, b; m, 1; x) = \left[\frac{m}{2-s}|f'(x/m)| + \frac{M}{(1-s)(2-s)}\right]\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right].$$

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ and $f' \in L_1[a, b]$. If $|f'|$ is (s, m)-Godunova-Levin function of second kind and $|f'(x)| \le M$, then, for $\alpha > 0$, we have*

$$\begin{aligned} &\left|\left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x^-}^\alpha f(a) + J_{x^+}^\alpha f(b)\right]\right| \\ &\le \min\{\varphi_1(a, b; m, \alpha; x), \varphi_2(a, b; m, \alpha; x)\}, \end{aligned}$$

where

$$\begin{aligned} &\varphi_1(a, b; m, \alpha; x) \\ &= \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\left(\frac{M^q}{1-s} + \frac{m}{1-s}|f'\left(\frac{a}{m}\right)|^q\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}}{b-a}\right. \\ &\quad \left. + \left(\frac{M^q}{1-s} + \frac{m}{1-s}|f'\left(\frac{b}{m}\right)|^q\right)^{\frac{1}{q}}\frac{(b-x)^{\alpha+1}}{b-a}\right], \end{aligned}$$

and

$$\varphi_2(a, b; m, \alpha; x) = \left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\frac{m}{1-s}|f'(x/m)|^q + \frac{M^q}{1-s}\right]^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a}\right],$$

respectively.

Proof. Using Lemma 2.1, the Hölder inequality and the fact that $|f'|$ is (s,m)-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|\\
&=\left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)dt\right|\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\le\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1}{t^s}|f'(x)|^q+\frac{m}{(1-t)^s}\left|f'\left(\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1}{t^s}|f'(x)|^q+\frac{m}{(1-t)^s}\left|f'\left(\frac{b}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\left(\frac{M^q}{1-s}+\frac{m}{1-s}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}}{b-a}\right.\\
&\qquad\left.+\left(\frac{M^q}{1-s}+\frac{m}{1-s}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^{\alpha+1}}{b-a}\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|\\
&\le\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\frac{m}{1-s}|f'(x/m)|^q+\frac{M^q}{1-s}\right]^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].
\end{aligned}$$

This completes the proof. □

For $\alpha=1$, Theorem 3.2 collapses to following result for (s,m)-Godunova-Levin function of second kind.

Corollary 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|$ is (s,m)-Godunova-Levin function of second kind and $|f'(x)|\le M$,*

then, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq \min\{\varphi_1(a,b;m,1;x),\varphi_2(a,b;m,1;x)\},$$

where

$$\begin{aligned}\varphi_1(a,b;m,1;x) &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[\left(\frac{M^q}{1-s}+\frac{m}{1-s}|f'\left(\frac{a}{m}\right)|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}\\ &\quad+\left(\frac{M^q}{1-s}+\frac{m}{1-s}|f'\left(\frac{b}{m}\right)|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\Bigg],\end{aligned}$$

and

$$\varphi_2(a,b;m,1;x)=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{m}{1-s}|f'(x/m)|^q+\frac{K^q}{1-s}\right]^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],$$

respectively.

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|$ is (s,m)-Godunova-Levin function of second kind and $|f'(x)|\leq M$, then, for $\alpha>0$, we have*

$$\begin{aligned}&\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)\right]\right|\\ &\leq \min\{\rho_1(a,b;m,\alpha;x),\rho_2(a,b;m,\alpha;x)\},\end{aligned}$$

where

$$\begin{aligned}&\rho_1(a,b;m,\alpha;x)\\ &=\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg[\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{M^q}{1+\alpha-s}+\frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}|f'\left(\frac{a}{m}\right)|^q\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{M^q}{1+\alpha-s}+\frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}|f'\left(\frac{b}{m}\right)|^q\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

and

$$\begin{aligned}&\rho_2(a,b;m,\alpha;x)\\ &=\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right)\\ &\quad\times\left(\frac{m}{1+\alpha-s}|f'\left(\frac{x}{m}\right)|^q+\frac{M^q\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

respectively.

Proof. Using Lemma 2.1, the power mean inequality and the fact that $|f'|$ is (s,m)-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|\\
&=\left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)dt\right|\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left(\frac{1}{t^s}|f'(x)|^q+\frac{m}{(1-t)^s}|f'\left(\frac{a}{m}\right)|^q\right)dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left(\frac{1}{t^s}|f'(x)|^q+\frac{m}{(1-t)^s}|f'\left(\frac{b}{m}\right)|^q\right)dt\right)^{\frac{1}{q}}\\
&\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{M^q}{1+\alpha-s}+\frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}|f'\left(\frac{a}{m}\right)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{M^q}{1+\alpha-s}+\frac{m\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}|f'\left(\frac{b}{m}\right)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|\\
&\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right)\right.\\
&\quad\left.\times\left(\frac{m}{1+\alpha-s}|f'\left(\frac{x}{m}\right)|^q+\frac{M^q\Gamma(1-s)\Gamma(\alpha+1)}{\Gamma(2+\alpha-s)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

When $\alpha=1$ in Theorem 3.3, we have the following result.

Corollary 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L_1[a,b]$ for all $x\in[a,b]$. If $|f'|$ is (s,m)-Godunova-Levin function of second kind and*

$|f'(x)| \le M$, *then, we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx]\right| \le \min\{\rho_1(a,b;m,1;x), \rho_2(a,b;m,1;x)\},$$

where

$$\begin{aligned}\rho_1(a,b;m,1;x) \quad = \quad & \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg[\frac{(x-a)^2}{b-a}\left(\frac{M^q}{2-s} + \frac{m}{(1-s)(2-s)}|f'\Big(\frac{a}{m}\Big)|^q\right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2}{b-a}\left(\frac{M^q}{2-s} + \frac{m}{(1-s)(2-s)}|f'\Big(\frac{b}{m}\Big)|^q\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

and

$$\begin{aligned}&\rho_2(a,b;m,1;x) \\ &= \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{(x-a)^2+(b-x)^2}{b-a}\right)\left(\frac{m}{2-s}|f'\Big(\frac{x}{m}\Big)|^q + \frac{M^q}{(1-s)(2-s)}\right)^{\frac{1}{q}}\right],\end{aligned}$$

respectively.

Remark 3.1. Note that for $x = \frac{a+b}{2}$ in all above results, we have mid-point inequalities and for $x = a$ and $x = b$, we have further new results.

Remark 3.2. We would like to mention here that using the same analysis one can prove the similar results for (s,m)-Godunova-Levin functions of first kind. We leave the details for interested readers. For some recent investigations on (s,m)-Godunova-Levin functions of first kind see [9].

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REFERENCES

1. G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
2. S. S. Dragomir, Inequalities of Hermite-Hadamard type for *h*-convex functions on linear spaces, preprint, (2014).
3. S. S. Dragomir, *n*-points inequalities of Hermite-Hadamard type for *h*-convex functions on linear spaces, preprint, (2014).

4. S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math, 21 (1995), 335-341.
5. S. S. Dragomir and C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia 2000.
6. S. S. Dragomir and T. M. Rassias, Ostrowski Type Inequalities and Applications in Numerical Integration,
7. E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) Numerical mathematics and mathematical physics (Russian), 138-142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985.
8. U. N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput. 218(3), 860-865 (2011).
9. M. Li, J.-R. Wang and W. Wei, Some fractional Hermite-Hadamard inequalities for convex and Godunova-Levin functions, FACTA Universitatis (NIS) Ser. Math. Inform. 30(2) (2015).
10. Z. Lin, J.-R. Wang and W. Wei, Fractional Hermite-Hadamard inequalities through r-convex functions via power means, FACTA Universitatis (NIS) Ser. Math. Inform. 30(2) (2015), 129-145.
11. W. Liu and J. Park, A generalization of the companion of Ostrowski-like inequality and applications, Appl. Math. Inf. Sci. 7(1), 273-278 (2013).
12. M. V. Mihai, M. A. Noor, K. I. Noor and M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Appl. Math. Comput. 252 (2015) 257-262.
13. V.G. Mihean, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania, 1993.
14. D. S. Mitrinovic and J. Pecaric, Note on a class of functions of Godunova and Levin, C. R. Math. Rep. Acad. Sci. Can. 12, 33-36, (1990).
15. M. A. Noor and M. U. Awan, Some integral inequalities for two kinds of convexities via fractional integrals, Trans. J. Math. Mech., 5(2), (2013).
16. M. A. Noor, K. I. Noor and M. U. Awan, Geometrically relative convex functions, Appl. Math. Infor. Sci. 8(2), 607-616, (2014).
17. M. A. Noor, K. I. Noor and M. U. Awan, Integral inequalities for harmonically s-Godunova-Levin functions, FACTA Universitatis (NIS) Ser. Math. Inform. 29(4) (2014), 415-424.
18. M. A. Noor, K. I. Noor, M. U. Awan and S. Costache, Some integral inequalities for harmonically h-convex functions, U.P.B. Sci. Bull., Series A, 77(1), 2015.
19. M. A. Noor, K. I. Noor, M. U. Awan and S. Khan, Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions, Appl. Math. Inf. Sci. 8, No. 6, 2865-2872 (2014).
20. A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel, 10 (1938), 226-227.
21. M.E. Ozdemir, H. Kavurmaci and E. Set, Ostrowski's type inequalities for (α, m)-convex functions, Kyungpook Math. J. 50, 371-378, (2010).
22. M. Radulescu. S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions, Math. Inequal. Appl. 12(4), 853-862, (2009).

23. M. Z. Sarikaya and H. Budak, Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals, FACTA Universitatis (NIS) Ser. Math. Inform. 29(4) (2014), 371-384.

24. M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling 57 (2013), 2403-2407.

25. M. Tunc, Ostrowski-type inequalities via h-convex functions with applications to special means, J. Ineq. Appl. 2013,2013:326.

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