FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 3 (2021), 547 - 555
https://doi.org/10.22190/FUMI201114040M
Original Scientific Paper

D_a-HOMOTHETIC DEFORMATION AND RICCI SOLITIONS IN THREE DIMENSIONAL QUASI-SASAKIAN MANIFOLDS

Tarak Mandal

Department of Mathematics, Jangipur College, Murshidabad, Pin-742213, West Bengal,India

Abstract. In the present paper, we have studied curvature tensors of a quasi-Sasakian manifold with respect to the D_a-homothetic deformation. We have deduced the Ricci soliton in quasi-Sasakian manifold with respect to the D_a-homothetic deformation. We have also proved that the quasi-Sasakian manifold is not $\bar{\xi}$-projectively flat under D_a-homothetic deformation. Also, we give an example to prove the existance of quasi-Sasakian manifold.
Key words: Quasi-Sasakian manifold, D_a-homothetic deformation, Ricci soliton, Weyl projective curvature tensor.

1. Introduction

In 1967, D. E. Blair [1] introduced the notion of quasi-Sasakian structure to unify Sasakian and cosymplectic structures. The Riemannian curvature tensor of three dimensional quasi-Sasakian manifold is given by [10]

$$\begin{aligned}
R(X,Y)Z &= g(Y,Z)[(\frac{r}{2}-\beta^2)X+(3\beta^2-\frac{r}{2})\eta(X)\xi+\eta(X)(\phi grad\beta)\\
&- d\beta(\phi X)\xi]-g(X,Z)[(\frac{r}{2}-\beta^2)Y+(3\beta^2-\frac{r}{2})\eta(Y)\xi\\
&+ \eta(Y)(\phi grad\beta)-d\beta(\phi Y)\xi]+[(\frac{r}{2}-\beta^2)g(Y,Z)\\
&+ (3\beta^2-\frac{r}{2})\eta(Y)\eta(Z)-\eta(Y)d\beta(\phi Z)-\eta(Z)d\beta(\phi Y)]X
\end{aligned}$$

Received November 14, 2020. accepted February 05, 2021.
Communicated by Mića Stanković
Corresponding Author: Tarak Mandal, Department of Mathematics, Jangipur College, Murshidabad, Pin-742213, West Bengal,India | E-mail: mathtarak@gmail.com
2010 *Mathematics Subject Classification.* Primary 53C10; Secondary 53C25

$$
\begin{aligned}
&- \quad [(\frac{r}{2}-\beta^2)g(X,Z)+(3\beta^2-\frac{r}{2})\eta(X)\eta(Z)-\eta(X)d\beta(\phi Z)\\
&- \quad \eta(Z)d\beta(\phi X)]Y-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y],
\end{aligned}
$$

where β is a function on the manifold. In the paper [2], U. C. De and A. K. Mondal have proved that $\xi\beta = 0$. In a quasi-Sasakian manifold, if we consider β is a non-zero constant, then the manifold becomes β-Sasakian and if $\beta = 1$, the manifold becomes a Sasakian manifold.

The notion of D_a-homothetic deformation was introduced by Tanno [11] in 1968. In paper [8], H. G. Nagaraja, D. L. Kiran Kumar and D. G. Prakasha have studied D_a-homothetic deformation of (κ,μ)-contact metric manifolds. Nagaraja and Premalatha have studied D_a-homothetic deformation of K-contact manifolds in the paper [9].

Ricci soliton was introduced by Hamilton [4] which is the generalization of the Einstein metrics and is defined by

$$(L_X g)(Y,Z)+2S(Y,Z)+2\lambda g(Y,Z)=0,$$

where, L_X denotes the Lie-derivatives of Riemannian metric g along the vector field X, λ is a constant, S the Ricci tensor of type $(0,2)$ and Y, Z are arbitrary vector fields on the manifold. A Ricci soliton is called shrinking or steady or expanding according as λ is negative or zero or positive. Ricci solitons on three dimensional almost contact manifolds have been studied by several authors. For instance, U. C. De and A. K. Mondal studied Ricci solitons on three dimensional quasi-Sasakian manifolds [2]. S. K. Hui and colaborators have investigated Ricci solitons and their generalizations on some classes of almost contact manifolds. For details see [5], [6], [7].

In this paper we would like to study some properties of quasi-Sasakian manifold with D_a-homothetic deformation.

The paper is organized as follows: In Section 2, we have discussed some preliminaries. In Section 3, we give an example of quasi-Sasakian manifold to prove the existance of the said manifold. In Section 4, we deduced some curvature properties of quasi-Sasakian manifold with respect to the D_a-homothetic deformation. In Session 5, we study the Ricci soliton in quasi-Sasakian manifold with respect to the D_a-homothetic deformation. In the last Section, we have derived the $\bar{\xi}$-projective curvature tensor under D_a-homothetic deformation.

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [10]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1.$$

As a consequence, we get the following:

$$\phi\xi = 0, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

for all vector fields X, Y on (M)

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X, \phi Y),$$

for all X, Y on M. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed i.e., $d\Phi = 0$ [1]. The normality condition gives that the induced almost complex structure of $M \times \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M, where $[X, Y]$ is the Lie bracket. The rank of a quasi-Sasakian structure is always an odd integer [1] which is equal to 1 if the structure is cosympletic and it is equal to $(2n+1)$ if the structure is Sasakian.

For a three-dimensional quasi-Sasakian manifold, we have [2]

$$\nabla_X \xi = -\beta \phi X, \tag{2.1}$$

$$(\nabla_X \phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \tag{2.2}$$

$$(\nabla_X \eta)(Y) = -\beta g(\phi X, Y), \tag{2.3}$$

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.4}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \\ &- \eta(X)d\beta(\phi Y) - \eta(Y)d\beta(\phi X), \end{aligned}$$

$$\begin{aligned} QX &= (\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi \\ &- \eta(X)(\phi grad\beta) - d\beta(\phi X)\xi, \end{aligned}$$

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X). \tag{2.5}$$

The Weyl projective curvature tensor P of type $(1,3)$ on a Riemannian manifold (M,g) of dimension $(2n+1)$ is defined by [3]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],$$

for all X, Y, $Z \in \chi(M)$.

3. Example of quasi-Sasakian manifold of dimension three

This example is constructed by following U. C. De and A. K. Mondal in the paper [2].

Let us consider the manifold $M = \{x_1, x_2, x_3 \in \mathbb{R}^3 : x_3 \neq 0\}$ of dimension 3, where $\{x_1, x_2, x_3\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, \quad e_3 = \frac{\partial}{\partial x_3},$$

which are linearly independent at each point of M, we get

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta ij$, for all $i, j = 1, 2, 3$. Let ∇ be the Riemannian connection and R the curvature tensor of g. The 1-form η is defined by $\eta(X) = g(X, e_3)$, for any X on M, which is a contact form because $\eta \wedge d\eta \neq 0$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then we find

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X) e_3,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Hence (ϕ, e_3, η, g) defines an almost contact metric structure on M.

Using Koszul's formula, we obtain

$$\nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2} e_1,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_3} e_2 = \frac{1}{2} e_1$$

and the remaining $\nabla_{e_i} e_j = 0$, for all $i, j = 1, 2, 3$. Thus we see that the structure (ϕ, e_3, η, g) satisfies the formula $\nabla_X e_3 = -\beta \phi X$ for $\beta = -\frac{1}{2}$.

Hence the manifold is a three dimensional quasi-Sasakian manifold with the constant structure function β.

Also, from the definition of curvature tensor, the expressions curvature tensor are given by

$$R(e_1,e_2)e_1=\frac{3}{4}e_2,\quad R(e_1,e_2)e_2=-\frac{3}{4}e_1,\quad R(e_1,e_3)e_1=-\frac{1}{4}e_3,$$

$$R(e_2,e_3)e_3=\frac{1}{4}e_2,\quad R(e_1,e_3)e_3=\frac{1}{4}e_1,\quad R(e_2,e_3)e_2=-\frac{1}{4}e_3$$

and the remaining $R(e_i,e_j)e_k=0$, for all $i,j,k=1,2,3$.

4. D_a-homothetic deformation

Let (M,ϕ,ξ,η,g) be a 3-dimensional quasi-Sasakian manifold. A D_a-homothetic deformation is defind by

$$\bar{\phi}=\phi,\quad \bar{\xi}=\frac{1}{a}\xi,\quad \bar{\eta}=a\eta,\quad \bar{g}=ag+a(a-1)\eta\otimes\eta, \tag{4.1}$$

with a being a positive constant [8].

If $M(\phi,\xi,\eta,g)$ is a quasi-Sasakian manifold with Riemannian connection ∇ and $\bar{\nabla}$ be the connection of the D_a-homothetic deformed quasi-Sasakian manifold $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ which is calculated from ∇ and g. Then the relation between the connections ∇ and $\bar{\nabla}$ is given by

$$\bar{\nabla}_XY=\nabla_XY+(1-a)[\eta(Y)\phi X+\eta(X)\phi Y], \tag{4.2}$$

for any vector fields X, Y, on M.

The Riemannian curvature tensor $\bar{R}$ of $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is given by

$$\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{[X,Y]}Z. \tag{4.3}$$

Using (2.1), (2.2), (4.1) and (4.2) in (4.3), we get

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z+(1-a)\beta[g(X,Z)\eta(Y)\xi\\ &- g(Y,Z)\eta(X)\xi-2g(\phi X,Y)\phi Z-g(\phi X,Z)\phi Y\\ &+ g(\phi Y,Z)\phi X-2\eta(Y)\eta(Z)X+2\eta(X)\eta(Z)Y]\\ &- (1-a)^2[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X].\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W)+(1-a)\beta[g(X,Z)\eta(Y)\eta(W)\\ &- g(Y,Z)\eta(X)\eta(W)-2g(\phi X,Y)g(\phi Z,W)\\ &- g(\phi X,Z)g(\phi Y,W)+g(\phi Y,Z)g(\phi X,W)\\ &- 2g(X,W)\eta(Y)\eta(Z)+2g(Y,W)\eta(X)\eta(Z)]\\ &- (1-a)^2[g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)],\end{aligned} \tag{4.5}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Let, $\{e_i\}, (i = 1,2,3)$ be the orthonormal basis of the tangent space of the manifold. Putting $X = W = e_i$, in (4.5) and summing over i, we get

$$\begin{aligned} \bar{S}(Y,Z) &= S(Y,Z) + 2(1-a)\beta[g(Y,Z) - 3\eta(Y)\eta(Z)] \\ &+ 2(1-a)^2\eta(Y)\eta(Z). \end{aligned} \tag{4.6}$$

From which,

$$\bar{Q}Y = QY + 2(1-a)\beta(Y - 3\eta(Y)\xi) + 2(1-a)^2\eta(Y)\xi,$$

From (2.2) and (4.2), we get

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - (1-a)\eta(Y)\phi^2X.$$

Also, from (2.1) and (4.2), we obtain

$$\bar{\nabla}_X\bar{\xi} = \frac{1-a-\beta}{a}\phi X. \tag{4.7}$$

Thus, from (2.3), (4.1), (4.2) and (4.7), we get

$$(\bar{\nabla}_X\bar{\eta})Y = a^2(1-a-\beta)g(\phi X,Y).$$

Thus, we can state the following

Theorem 4.1. For a D_a-homothetically deformed quasi-Sasakian manifold $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$, the followings hold

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + (1-a)\beta[g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi - 2g(\phi X,Y)\phi Z - g(\phi X,Z)\phi Y \\ &+ g(\phi Y,Z)\phi X - 2\eta(Y)\eta(Z)X + 2\eta(X)\eta(Z)Y] \\ &- (1-a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]. \end{aligned}$$

$$\begin{aligned} \bar{S}(Y,Z) &= S(Y,Z) + 2(1-a)\beta[g(Y,Z) - 3\eta(Y)\eta(Z)] \\ &+ 2(1-a)^2\eta(Y)\eta(Z). \end{aligned}$$

$$\bar{Q}Y = QY + 2(1-a)\beta(Y - 3\eta(Y)\xi) + 2(1-a)^2\eta(Y)\xi,$$

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - (1-a)\eta(Y)\phi^2X.$$

$$\bar{\nabla}_X\bar{\xi} = \frac{1-a-\beta}{a}\phi X.$$

$$(\bar{\nabla}_X\bar{\eta})Y = a^2(1-a-\beta)g(\phi X,Y).$$

5. Ricci soliton in three dimensional quasi-Sasakian manifold with respect to the D_a-homothetic deformation

Let $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a D_a-homothetically deformed quasi-Sasakian manifold of dimension 3. A Ricci soliton $(\bar{g}, V, \lambda)$ is defined on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ as

$$(\bar{L}_V \bar{g})(X, Y) + 2\bar{S}(X, Y) + 2\lambda\bar{g}(X, Y) = 0, \tag{5.1}$$

where $\bar{L}_V \bar{g}$ denotes the Lie derivative of Riemannian metric $\bar{g}$ along a vector field V, $\bar{S}$ is the Ricci tensor of type $(0, 2)$ on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$.

Let us suppose that the vector field V is the Reeb vector field $\bar{\xi}$ on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. Then from (5.1), we have

$$(\bar{L}_{\bar{\xi}} \bar{g})(X, Y) + 2\bar{S}(X, Y) + 2\lambda\bar{g}(X, Y) = 0. \tag{5.2}$$

Now, from (2.1) and (4.2), we have

$$\begin{aligned} (\bar{L}_{\bar{\xi}} \bar{g})(X, Y) &= \bar{g}(\bar{\nabla}_X \bar{\xi}, Y) + \bar{g}(X, \bar{\nabla}_Y \bar{\xi}) \\ &= 0. \end{aligned} \tag{5.3}$$

Therefore, from (4.1), (5.2) and (5.3), we get

$$\bar{S}(X, Y) = -\lambda\bar{g}(X, Y). \tag{5.4}$$

Putting $Y = Z = \xi$ in (4.6), we get

$$\bar{S}(\xi, \xi) = 2(\beta + a - 1)^2. \tag{5.5}$$

Putting $X = Y = \bar{\xi}$ in (5.4) and using (5.5), we get

$$\lambda = -\frac{2(\beta + a - 1)^2}{a^2}.$$

Thus we can state the following

Theorem 5.1. If a D_a-homothetically deformed quasi-Sasakian manifold of dimension three admits Ricci soliton, then the Ricci soliton is shrinking.

6. $\bar{\xi}$-projective curvature tensor on quasi-Sasakian manifold with respect to D_a-homothetic deformation

Definition 6.1. The $\bar{\xi}$-projective curvature tensor of type $(1, 3)$ on a quasi-Sasakian manifold of dimension $(2n+1)$ with respect to D_a-homothetic deformation is given by [8]

$$\bar{P}(X, Y)\bar{\xi} = \bar{R}(X, Y)\bar{\xi} - \frac{1}{2n}[S(Y, \bar{\xi})X - S(X, \bar{\xi})Y],$$

for any X, Y on M.
A quasi-Sasakian manifold of dimension n is said to be $\bar{\xi}$-projectively flat with respect to the D_a-homothetic deformation if $\bar{P}(X,Y)\bar{\xi}=0$.

The Weyl projective curvature tensor $\bar{P}$ of a three dimensional quasi-Sasakian manifold under D_a-homothetic deformation is defined by [8]

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{2}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y]. \tag{6.1}$$

Interchanging X and Y,we get

$$\bar{P}(Y,X)Z=\bar{R}(Y,X)Z-\frac{1}{2}[\bar{S}(X,Z)Y-\bar{S}(Y,Z)X]. \tag{6.2}$$

Adding (6.1) and (6.2), we get by using the property $\bar{R}(X,Y)Z=-\bar{R}(Y,X)Z$

$$\bar{P}(X,Y)Z+\bar{P}(Y,X)Z=0.$$

Also, from (6.1) by using first Bianchi identity $\bar{R}(X,Y)Z+\bar{R}(Y,Z)X+\bar{R}(Z,X)Y=0$, we get

$$\bar{P}(X,Y)Z+\bar{P}(Y,Z)X+\bar{P}(Z,X)Y=0.$$

Thus the Weyl projective curvature tensor under D_a-homothetic deformation in a quasi-Sasakian manifold is skew-symmetric and cyclic.

Using (4.1), (4.4) and (4.6) in (6.1) , we get

$$\begin{aligned}
\bar{P}(X,Y)Z &= R(X,Y)Z+(1-a)\beta[g(X,Z)\eta(Y)\xi\\
&- g(Y,Z)\eta(X)\xi-2g(\phi X,Y)\phi Z-g(\phi X,Z)\phi Y\\
&+ g(\phi Y,Z)\phi X-2\eta(Y)\eta(Z)X+2\eta(X)\eta(Z)Y]\\
&- (1-a)^2[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X]\\
&- \frac{1}{2}[S(Y,Z)X+2(1-a)\beta\{g(Y,Z)X-3\eta(Y)\eta(Z)X\}\\
&+ 2(1-a)^2\eta(Y)\eta(Z)X-S(X,Z)Y\\
&- 2(1-a)\beta\{g(X,Z)Y-3\eta(X)\eta(Z)Y\}\\
&- 2(1-a)^2\eta(X)\eta(Z)Y].
\end{aligned} \tag{6.3}$$

Replacing Z by $\bar{\xi}$ in (6.3), using (2.4), (2.5) and (4.1), we get

$$\bar{P}(X,Y)\bar{\xi}=\frac{1}{a}[(-(X\beta)\phi Y+(Y\beta)\phi X)+\frac{1}{2}(-(\phi X\beta)Y+(\phi Y\beta)X)].$$

Thus we can say
Theorem 6.1. The $\bar{\xi}$-projective curvature tensor on a D_a-homothetically deformed quasi-Sasakian manifold of dimension 3 is given by

$$\bar{P}(X,Y)\bar{\xi} = \frac{1}{a}[(-(X\beta)\phi Y + (Y\beta)\phi X) + \frac{1}{2}(-(\phi X\beta)Y + (\phi Y\beta)X)].$$

and the manifold is not $\bar{\xi}$-projectively flat with respect to the D_a-homothetic deformation.

Acknowledgement. The author is thankful to the referee for his/her valuable suggestions for the improvement of the paper.

REFERENCES

1. D. E. Blair: *The theory of quasi-Sasakian structure.* J. Diff. Geom. 1(1967), 331-345.
2. U. C. De and A. K. Mondal: 3-*dimensional quasi-Sasakian manifolds and Ricci solitons.* SUT J. Math, Vol. **48**, No 1(2012), 71-81.
3. U. C. De and A. A. Shaikh: *Differential Geometry of manifolds.* Narosa Pub. House, New Delhi, 2007.
4. R. S. Hamilton: *The Ricci flow on surfaces.* Contemporary Mathematics, **71** (1988), 237-261.
5. S. K. Hui and D. Chakraborty: *Ricci almost solitons on concircular Ricci pseudosymmetric β-Kenmotsu manifolds.* Hacettepe Journal of Mathematics and Statistics, **47(3)** (2018), 579-587.
6. S. K. Hui, S. S. Shukla and D. Chakraborty: *η-Ricci solitons on η-Einstein Kenmotsu manifolds.* Global J. Adv. Res. Clas. Mod. Geom., **6(1)** (2017), 1-6.
7. S. K. Hui, S. Uddin and D. Chakraborty: *Generalized Sasakian-space-forms whose metric is η-Ricci almost solitons.* Differental Geometry and Dynamical Systems, **19** (2017), 45-55.
8. H. G. Nagaraja, D. L. Kiran Kumar and D. G. Prakasha: *D_a-Homothetic Deformation And Ricci solitions in (κ,μ)-contact metric manifolds.* Konuralp J. of Math., **7**(1) (2019), 122-127.
9. H. G. Nagaraja and C. R. Premalatha: *D_a-Homothetic deformation of K-contact manifolds.* International Scholarly Research Notice, Vol-2013.
10. A. Sarkar, A. Sil and A. K. Paul: *On three dimensional quasi-Sasakian manifolds and magnetic curves.* Applied Mathematics E-notes, **19** (2019), 55-64.
11. S. Tanno: *The topology of contact Riemannian manifolds.* Illinois Journals of Mathematics, Vol **12**, pp. 700-717, 1968.