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HYPERSPHERICAL AND HYPERCYLINDRICAL GENERALIZED HELICES IN THE SENSE OF HAYDEN IN $\mathbb{E}^{2n+1}$

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Abstract. In this paper, we investigate generalized helices in the sense of Hayden in $(2n+1)$-dimensional Euclidean space $\mathbb{E}^{2n+1}$. We obtain some results for such curves in $\mathbb{E}^{2n+1}$. Thereafter, we obtain two families of generalized helices which are hyperspherical and hypercylindrical generalized helices in the sense of Hayden. In addition, we give examples of hyperspherical and hypercylindrical generalized helices in the sense of Hayden in $\mathbb{E}^5$. Finally, we give examples of hyperspherical and hypercylindrical generalized helices in the sense of Hayden in $\mathbb{E}^3$ and plot the graphics of these curves with Mathematica 10.0.
Keywords: generalized helices, global submanifolds, Euclidean space

1. Introduction

Helical structures have many applications to the various branches of science such as biology, architecture, engineering, etc. [1]. One of the important research problem for differential geometry is helices. The notion of helix is stated in 3-dimensional Euclidean space by M. A. Lancret in 1802. Helix is a curve whose tangent vector field makes a constant angle with a fixed direction called the axis of

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the helix. The necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion should be constant, which is given by B. de Saint Venant in 1845 [2, 4]. If both curvature and torsion are non-zero constants, then the curve is called circular helix [2]. Also, in the n-dimensional Euclidean space, a general helix is defined similarly i.e., whose tangent vector field makes a constant angle with a fixed direction [9].

In [6], generalized helix notion is more restrictive in the n-dimensional Euclidean space for $n > 3$; a fixed direction makes a constant angle with all Frenet vector fields of the curve. This type of curves are called the generalized helix in the sense of Hayden [4]. In [6], the generalized helix in the sense of Hayden has the property that the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{n-4}}{\kappa_{n-3}}, \frac{\kappa_{n-2}}{\kappa_{n-1}}$ are constants if n is odd, where $\kappa_i \, (1 \leqslant i \leqslant n-1)$ denote ith curvature function of the curve. In this work, we study generalized helices in the sense of Hayden. For the sake of brevity, we call them generalized helices.

Notice that, a curve β is called a W-curve, if the curve has constant curvatures. Also, W-curves in $\mathbb{E}^{2n+1}$ are generalized helices [4].

This study is organized as follows: In section 2, we review differential geometry of regular curves in $\mathbb{E}^n$. In Section 3, we give a theorem for generalized helix. After that, we obtain some results for generalized helices based on angles which are between the Frenet vector fields of the curve and a fixed direction. In Section 4, we show that the family of curves in [2] are hyperspherical generalized helices. Thereafter, we obtain hypercylindrical generalized helices in $\mathbb{E}^{2n+1}$ by using a different method from [2]. Finally we give examples for such curves in $\mathbb{E}^5$ and $\mathbb{E}^3$.

2. Preliminary

In this section, we give the basic theory of local differential geometry of curves in the n-dimensional Euclidean space. For more detail and background about this space, see [3, 5].

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve in the n-dimensional Euclidean space denoted by $\mathbb{E}^n$. Recall that $\langle , \rangle$ denotes the standard inner product of $\mathbb{R}^n$ given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

for each $x = (x_1, x_2, x_3, \ldots x_n)$, $y = (y_1, y_2, y_3, \ldots y_n) \in \mathbb{R}^n$. The norm of a vector $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Let $\{V_1, V_2, V_3, \ldots V_n\}$ be the moving Frenet frame along the arbitrary curve α, where $V_i \, (1 \leqslant i \leqslant n)$ is Frenet vector field. Then, the matrix form of Frenet formulas are given by

$$(2.2)\begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-1}' \\ V_n' \end{pmatrix} = \begin{pmatrix} 0 & \nu\kappa_1 & 0 & \cdots & 0 & 0 \\ -\nu\kappa_1 & 0 & \nu\kappa_2 & \cdots & 0 & 0 \\ 0 & -\nu\kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\nu\kappa_{n-1} \\ 0 & 0 & 0 & \cdots & -\nu\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \\ V_n \end{pmatrix}$$

where $\nu = \langle \alpha', \alpha' \rangle$ and $\kappa_i \, (1 \leqslant i \leqslant n-1)$ denote the ith curvature function of the curve α [1]. To obtain $V_1, V_2, V_3, \ldots V_n$ it is sufficient to apply the Gramm-Schmidt orthogonalization process to $\alpha'(t), \alpha''(t), \ldots, \alpha^{(n)}(t)$. More precisely, $V_i \, (1 \leqslant i \leqslant n)$ and $\kappa_i \, (1 \leqslant i \leqslant n-1)$ are determined by the following formulas [8]:

$$\begin{aligned} F_1(t) &= \alpha'(t), \\ F_i(t) &= \alpha^i(t) - \sum_{j=1}^{i-1} \frac{\langle \alpha^i(t), F_j(t) \rangle}{\langle F_j(t), F_j(t) \rangle} F_j(t) \ \text{ for } 2 \leqslant i \leqslant n, \\ \kappa_i(t) &= \frac{\|F_{i+1}(t)\|}{\|F_1(t)\| \, \|F_i(t)\|} \text{ for } 1 \leqslant i \leqslant n, \\ V_i &= \frac{F_i}{\|F_i\|} \text{ for } 1 \leqslant i \leqslant n \end{aligned}$$

where $\alpha', \alpha'', \ldots, \alpha^{(n)}$ are linearly independent. Let $\beta : I \to S^n$ be a unit speed hyperspherical curve in $\mathbb{E}^{n+1}$ where I is an open interval in $\mathbb{R}$. In [10], Izumiya and Nagai defined generalized Sabban frame $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-1}\}$ of the unit speed curve β which is determined by the following formulas:

$$\begin{aligned} \mathbf{n}_1 &= \frac{\mathbf{t}' + \beta}{\|\mathbf{t}' + \beta\|}, \\ k_1 &= \|\mathbf{t}' + \beta\|, \\ \mathbf{n}_2 &= \frac{\mathbf{n}_1' + k_1\beta'}{\|\mathbf{n}_1' + k_1\beta'\|}, \\ k_2 &= \|\mathbf{n}_1' + k_1\beta'\|, \\ k_i &= \left\|\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}\right\|, \\ \mathbf{n}_i &= \frac{\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}}{\left\|\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}\right\|}, \end{aligned}$$

for $3 \leqslant i \leqslant n-2$ and $k_i \neq 0$ for all i and

$$\begin{aligned} \mathbf{n}_{n-1} &= \frac{\beta \times \mathbf{t}' \times \mathbf{n}_1 \times \cdots \times \mathbf{n}_{n-2}}{\|\beta \times \mathbf{t}' \times \mathbf{n}_1 \times \cdots \times \mathbf{n}_{n-2}\|}, \\ k_{n-1} &= \left\langle \mathbf{n}_{n-2}', \mathbf{n}_{n-1} \right\rangle \end{aligned}$$

where $k_i\,(1 \leqslant i \leqslant n-1)$ denote ith curvature function of the curve β. Also, in the same paper, Izumiya and Nagai gave the following Frenet-Serret type formula for the generalized Sabban frame of the spherical curve β.

$$\begin{pmatrix} \beta' \\ \mathbf{t}' \\ \mathbf{n}_1' \\ \vdots \\ \mathbf{n}_{n-2}' \\ \mathbf{n}_{n-1}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & k_1 & \cdots & 0 & 0 \\ 0 & -k_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \mathbf{t} \\ \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_{n-2} \\ \mathbf{n}_{n-1} \end{pmatrix}. \tag{2.3}$$

Definition 2.1. A Frenet curve of rank r for which $\kappa_1, \kappa_2, \ldots, \kappa_r$ are constants is called W-curve [7].

A unit speed W-curve of rank $2n$ has the parameterization of the form

$$\beta(s) = a_0 + \sum_{i=1}^{n} (a_i \cos \mu_i s + b_i \sin \mu_i s) \tag{2.4}$$

and a unit speed W-curve of rank $2n+1$ has the parameterization of the form

$$\beta(s) = a_0 + b_0 s + \sum_{i=1}^{n} (a_i \cos \mu_i s + b_i \sin \mu_i s) \tag{2.5}$$

where $a_0, b_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are constant vectors in $\mathbb{R}^n$ and $\mu_1 < \mu_2 < \ldots < \mu_n$ are positive real numbers. So, a W-curve of rank 1 is a straight line, a W-curve of rank 2 is a circle, a W-curve of rank 3 is a right circular helix [8].

3. Generalized Helix in $\mathbb{E}^{2n+1}$

Hayden gave the following theorems in [6].

Theorem 3.1. *Let α be a curve in a Riemannian $(2n+1)$-space, the Frenet vector fields $V_3, V_5, \ldots, V_{2n+1}$ of the curve make constant angle with a parallel vector-field along the curve, then the curve α is generalized helix; moreover, V_1 also make a constant angle with the given vector-field, and $V_2, V_4, \ldots, V_{2n}$ are each perpendicular to the given vector-field [6].*

Theorem 3.2. *Let α be a curve in a Riemannian $(2n+1)$-space, the Frenet vector fields $V_1, V_3, \ldots, V_{2n-1}$ of the curve make constant angle with a parallel vector-field along the curve, then the curve α is generalized helix; moreover, V_{2n+1} also make a constant angle with the given vector-field, and $V_2, V_4, \ldots, V_{2n}$ are each perpendicular to the given vector-field [6].*

In the light of the theorems mentioned above, we can give the following theorem.

Theorem 3.3. *Let α be a curve in $\mathbb{E}^{2n+1}$. If the Frenet vector fields $V_1, V_3, V_5, \ldots, V_{2j-1}, V_{2j+3}, \ldots, V_{2n+1}$, $(1 \leqslant j \leqslant n)$ of the curve α make constant angle with a unit vector U, then the curve α is generalized helix; moreover, the vector field V_{2j+1} makes a constant angle with the given vector U, and $V_2, V_4, \ldots, V_{2n}$ are each perpendicular to the given vector U.*

Proof. Assume that the Frenet vector fields $V_1, V_3, V_5, \ldots, V_{2j-1}, V_{2j+3}, \ldots, V_{2n+1}$, $(1 \leqslant j \leqslant n)$ of the curve α make constant angle with a unit vector U. Then, we have

$$\langle V_i, U\rangle = \cos\theta_i, \quad i = 1, 3, 5, \ldots, 2j-1, 2j+1, \ldots, 2n+1. \tag{3.1}$$

If we take the derivative of 3.1 for $i = 1$ by using Frenet formulas in 2.2, we obtain that V_2 is perpendicular to U.

If we take the derivative of 3.1 for $i = 3$ by using Frenet formulas in 2.2 and the fact that $V_2 \perp U$, we obtain that V_4 is perpendicular to U.

Similarly, we take the derivative of 3.1 for $i = 5, 7, ..., 2j-1$ we obtain $V_6, V_8, \ldots V_{2j}$ each are perpendicular to U.

If we take the derivative of 3.1 for $i = 2n+1$ by using Frenet formulas in 2.2, we get V_{2n} is perpendicular to U.

If we take the derivative of 3.1 for $i = 2n-1$ by using Frenet formulas in 2.2 and the fact that $V_{2n} \perp U$, we obtain that V_{2n-2} is perpendicular to U.

Similarly, we take the derivative of 3.1 for $i = 2n-3, 2n-5, ..., 2j+3$ we obtain $V_{2n-4}, V_{2n-6}, \ldots V_{2j+2}$ each are perpendicular to U.

Finally, for $i = 2j+1$ from 2.2 we have

$$\langle V_{2j+1}, U\rangle' = \kappa_{2j+1} \langle V_{2j+2}, U\rangle - \kappa_{2j} \langle V_{2j}, U\rangle = 0 \tag{3.2}$$

since $\langle V_{2j+2}, U\rangle = 0$ and $\langle V_{2j}, U\rangle = 0$. So, $\langle V_{2j+1}, U\rangle$ is a constant. Therefore, V_{2j} makes a constant angle with U. □

The vector U is called the axes of generalized helix. It is obvious; if we take the derivative of 3.1 for $i = 2, 4, \ldots 2n$ by using 2.2 we have

$$\frac{\kappa_2}{\kappa_1} = \frac{\cos\theta_1}{\cos\theta_3}, \qquad \frac{\kappa_4}{\kappa_3} = \frac{\cos\theta_3}{\cos\theta_5}, \qquad \ldots, \qquad \frac{\kappa_{2n}}{\kappa_{2n-1}} = \frac{\cos\theta_{2n-1}}{\cos\theta_{2n+1}}. \tag{3.3}$$

From 3.3, we give the following corollary.

Corollary 3.1. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$\frac{\kappa_2\kappa_4\ldots\kappa_{2n}}{\kappa_1\kappa_3\ldots\kappa_{2n-1}} = \frac{\cos\theta_1}{\cos\theta_{2n+1}},$$

$$\cos\theta_j = \frac{\kappa_{j+1}}{\kappa_j}\cos\theta_{j+2} \text{ for } j = 1, 3, 5, \ldots, 2n-1$$

and the axis of a generalized helix has the form

$$U = \cos\theta_1 V_1 + \cos\theta_3 V_3 + \cdots + \cos\theta_{2n+1} V_{2n+1}.$$

Theorem 3.4. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \ldots \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$U = \cos\theta_1 \left(V_1 + \sum_{i=1}^{n} \frac{\kappa_1\kappa_3 \ldots \kappa_{2i-1}}{\kappa_2\kappa_4 \ldots \kappa_{2i}} V_{2i+1} \right)$$

and

$$\tan^2\theta_1 = \sum_{i=1}^{n} \left(\frac{\kappa_1\kappa_3 \ldots \kappa_{2i-1}}{\kappa_2\kappa_4 \ldots \kappa_{2i}} \right)^2$$

where θ_1 is the angle between V_1 and U.

Proof. It is clear from equation 3.3 and Corollary 3.1. □

Similarly, we have the following theorem.

Theorem 3.5. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \ldots \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$U = \cos\theta_{2n+1} \left(V_{2n+1} + \sum_{i=1}^{n} \frac{\kappa_2\kappa_4 \ldots \kappa_{2i}}{\kappa_1\kappa_3 \ldots \kappa_{2i-1}} V_{2i-1} \right) \tag{3.4}$$

and

$$\tan^2\theta_{2n+1} = \sum_{i=1}^{n} \left(\frac{\kappa_2\kappa_4 \ldots \kappa_{2i}}{\kappa_1\kappa_3 \ldots \kappa_{2i-1}} \right)^2 \tag{3.5}$$

where θ_{2n+1} is the angle between V_{2n+1} and U.

Proof. It is clear from equation 3.3 and Corollary 3.1. □

4. Families of Generalized Hypercylindrical and Hyperspherical Generalized Helices in $\mathbb{E}^{2n+1}$

In this section, we show that the curve in [2] is a hyperspherical generalized helix. Also, we used a W-curve to obtain a hypercylindrical generalized helix.

Lemma 4.1. $\beta : I \subset \mathbb{R} \to S^{2n}$,

$$\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_{2n+1}(t))$$

is given by

$$\beta_{2i-1}(t) = \frac{\left(1 - {c_i}^2\right)\sin\left(c_i\lambda t\right)}{\left(\sum\limits_{k=1}^{n} {c_k}^4 - {c_k}^2\right)^{1/2}},$$

$$\beta_{2i}(t) = \frac{\left(1 - {c_i}^2\right)\cos\left(c_i\lambda t\right)}{\left(\sum\limits_{k=1}^{n} {c_k}^4 - {c_k}^2\right)^{1/2}},$$

for $i = 1, 2, \ldots n$ *and*

$$\beta_{2n+1}(t) = \left(\frac{\sum\limits_{k=1}^{n} {c_k}^2 - n}{\sum\limits_{k=1}^{n} {c_k}^4 - {c_k}^2}\right)^{\frac{1}{2}}$$

where $\lambda = \left(\frac{\sum\limits_{k=1}^{n} {c_k}^4 - {c_k}^2}{\sum\limits_{k=1}^{n} {c_k}^2 - 2{c_k}^4 + {c_k}^6}\right)^{\frac{1}{2}}$ *is a constant. Then,* β *is a* W*-curve of rank* $2n$*.*

Proof. It is clear from equation 2.4. □

Theorem 4.1. *Let* $\alpha : I \subset R \to E^{2n+1}$

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_{2n+1}(t))$$

be a regular curve given by

$$\alpha_{2i-1}(t) = \frac{1}{\left(\sum_{j=1}^{n} {c_j}^2\right)^{1/2}}\left(c_i \cos(t)\cos(c_i t) + \sin(t)\sin(c_i t)\right),$$

$$\alpha_{2i}(t) = \frac{1}{\left(\sum_{j=1}^{n} {c_j}^2\right)^{1/2}}\left(\cos(c_i t)\sin(t) - c_i \cos(t)\sin(c_i t)\right),$$

for $i = 1, 2, \ldots n$ *and*

$$\alpha_{2n+1}(t) = \left(1 - \frac{n}{\sum\limits_{j=1}^{n} {c_j}^2}\right)^{1/2} \sin(t)$$

where $c_1, c_2, \ldots, c_n > 1$ *with* $c_i \neq c_j$, $1 \leqslant i < j \leqslant n$*. Then,* α *is a general helix which lies on* S^{2n} *[2].*

By means of the Teorem 4.1, we can give the following theorem.

Theorem 4.2. *Let $\alpha : I \subset R \to E^{2n+1}$*

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_{2n+1}(t))$$

be a regular curve given by

$$\begin{aligned}
\alpha_{2i-1}(t) &= \frac{1}{\left(\sum\limits_{j=1}^{n} {c_j}^2\right)^{1/2}} \left(c_i \cos(\lambda t) \cos(c_i \lambda t) + \sin(\lambda t) \sin(c_i \lambda t)\right), \\
\alpha_{2i}(t) &= \frac{1}{\left(\sum\limits_{j=1}^{n} {c_j}^2\right)^{1/2}} \left(\cos(c_i \lambda t) \sin(\lambda t) - c_i \cos(\lambda t) \sin(c_i \lambda t)\right),
\end{aligned}$$

for $i = 1, 2, \ldots n$ and

$$\alpha_{2n+1}(t) = \left(1 - \frac{n}{\sum\limits_{j=1}^{n} {c_j}^2}\right)^{1/2} \sin(\lambda t)$$

where $c_1, c_2, \ldots, c_n > 1$ with $c_i \neq c_j$, $1 \leqslant i < j \leqslant n$ and $\lambda = \left(\frac{\sum_{k=1}^{n} {c_k}^4 - {c_k}^2}{\sum_{k=1}^{n} {c_k}^2 - 2{c_k}^4 + {c_k}^6}\right)^{\frac{1}{2}}$. Then, the curve $\alpha : I \subset R \to E^{2n+1}$ is a hyperspherical generalized helix on S^{2n}.

Proof. After straightforward calculations, we obtain

$$||\alpha(t)|| = 1, \qquad \alpha'(t) = \omega \cos t \, \beta(t),$$

where $\omega = \left(\frac{\sum_{k=1}^{n} {c_k}^4 - {c_k}^2}{\sum_{k=1}^{n} {c_k}^2}\right)^{\frac{1}{2}}$ and β is the W-curve in Lemma 4.1. Since $||\alpha(t)|| = 1$ the curve α lies on S^{2n}. If we apply the Gramm-Schmidt orthogonalization process to the curve α

$$\begin{aligned}
F_1(t) &= \omega \cos t \, \beta(t), \\
F_2(t) &= \omega \cos t \, \mathbf{t}(t), \\
F_i(t) &= \omega \cos t \, k_1(t) k_2(t) \ldots k_{i-2}(t) \mathbf{n}_{i-2}(t) \text{ for } 3 \leqslant i \leqslant n
\end{aligned}$$

where $k_i \, (1 \leqslant i \leqslant n-1)$ is the curvature functions of the curve β. Now, we can calculate the curvature functions $\kappa_i, (1 \leqslant i \leqslant n-1)$ of the curve α.

$$\begin{aligned}
\kappa_1(t) &= \frac{\|F_2(t)\|}{\|F_1(t)\|^2} = \omega^{-1} \sec t, \\
\kappa_i(t) &= \frac{\|F_{i+1}(t)\|}{\|F_1(t)\| \, \|F_i(t)\|} = \omega^{-1} k_{i-1}(t) \sec t
\end{aligned}$$

for $2 \leqslant i \leqslant 2n$. Since the curvature functions k_i are constants for $1 \leqslant i \leqslant 2n-1$, the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{2n-1}}{\kappa_{2n}}$ are constants. Therefore, α is a hyperspherical generalized helix on S^{2n}. $\square$

Corollary 4.1. *From Theorem 4.2, the Frenet vector fields of the curve α are*

$$V_1 = \beta, \qquad V_2 = \mathbf{t}, \qquad V_3 = \mathbf{n}_1, \quad \ldots, \quad V_{2n+1} = \mathbf{n}_{2n-1} \tag{4.1}$$

where $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{2n-1}\}$ is the generalized Sabban frame of the unit speed curve β.

Example 4.1. If we choose $c_1 = 2$ and $c_2 = 4$ in Theorem 4.2, then

$$\alpha(t) = \begin{pmatrix} \frac{\cos(\lambda t)\cos(2\lambda t)}{\sqrt{5}} + \frac{\sin(2\lambda t)\sin(\lambda t)}{2\sqrt{5}}, \frac{\cos(2\lambda t)\sin(\lambda t)}{2\sqrt{5}} - \frac{\cos(\lambda t)\sin(2\lambda t)}{\sqrt{5}}, \\ \frac{2\cos(\lambda t)\cos(4\lambda t)}{\sqrt{5}} + \frac{\sin(4\lambda t)\sin(\lambda t)}{2\sqrt{5}}, \frac{\cos(4\lambda t)\sin(\lambda t)}{2\sqrt{5}} - \frac{2\cos(\lambda t)\sin(4\lambda t)}{\sqrt{5}}, \frac{3\sin(\lambda t)}{\sqrt{10}} \end{pmatrix}$$

where $\lambda = \sqrt{\frac{7}{101}}$.

After straightforward calculations, we obtain the Frenet vector fields of the curve α

$$\begin{aligned}
V_1(t) &= \left(-\frac{\sin(2\lambda t)}{2\sqrt{7}}, -\frac{\cos(2\lambda t)}{2\sqrt{7}}, -\frac{5\sin(4\lambda t)}{2\sqrt{7}}, -\frac{5\cos(4\lambda t)}{2\sqrt{7}}, \frac{1}{\sqrt{14}}\right), \\
V_2(t) &= \left(-\frac{\cos(2\lambda t)}{\sqrt{101}}, \frac{\sin(2\lambda t)}{\sqrt{101}}, -\frac{10\cos(4\lambda t)}{\sqrt{101}}, \frac{10\sin(4\lambda t)}{\sqrt{101}}, 0\right), \\
V_3(t) &= \left(-\frac{73\sin(2\lambda t)}{2\sqrt{7189}}, -\frac{73\cos(2\lambda t)}{2\sqrt{7189}}, \frac{55\sin(4\lambda t)}{2\sqrt{7189}}, \frac{55\cos(4\lambda t)}{2\sqrt{7189}}, \frac{101}{\sqrt{14378}}\right), \\
V_4(t) &= \left(-\frac{10\cos(2\lambda t)}{\sqrt{101}}, \frac{10\sin(2\lambda t)}{\sqrt{101}}, \frac{\cos(4\lambda t)}{\sqrt{101}}, -\frac{\sin(4\lambda t)}{\sqrt{101}}, 0\right), \\
V_5(t) &= \sqrt{\frac{2}{1027}}\left(20\sin(2\lambda t), 20\cos(2\lambda t), -\sin(4\lambda t), -\cos(4\lambda t), \frac{15\sqrt{2}}{2}\right).
\end{aligned}$$

It is clear that the Frenet vector fields V_1, V_3 and V_5 of the curve α make constant angles $\theta_1 = \arccos\frac{1}{\sqrt{14}}$, $\theta_3 = \arccos\frac{101}{\sqrt{14378}}$ and $\theta_5 = \arccos\frac{15}{\sqrt{1027}}$ with vector $U = (0,0,0,0,1)$, respectively.

Also, after straightforward calculations, we have the curvatures of the curve α

$$\kappa_1(t) = \frac{1}{21}\sqrt{505}\sec(\lambda t), \quad \kappa_2(t) = \frac{1}{21}\sqrt{\frac{5135}{101}}\sec(\lambda t), \quad \kappa_3(t) = 40\sqrt{\frac{5}{103727}}\sec(\lambda t)$$

and

$$\kappa_4(t) = \frac{4}{3}\sqrt{\frac{1010}{7189}}\sec(\lambda t).$$

Since, α lies on hypersphere $S^4 = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}^5 \middle| \sum_{i=1}^{5} x_i^2 = 1\right\}$, then α is a hyperspherical generalized helix in $\mathbb{E}^5$.

Now, we have the following theorem for a curve γ which is integration of the curve β in Lemma 4.1.

Theorem 4.3. *Let* $\gamma : I \subset R \to E^{2n+1}$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_{2n+1}(t))$$

be a regular curve given by

$$\gamma_{2i-1}(t) = \frac{\left(c_i{}^2 - 1\right)\left(\sum\limits_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6\right)^{\frac{1}{2}}}{c_i\left(\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2\right)} \cos\left(c_i \lambda t\right),$$

$$\gamma_{2i}(t) = \frac{\left(1 - c_i{}^2\right)\left(\sum\limits_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6\right)^{\frac{1}{2}}}{c_i\left(\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2\right)} \sin\left(c_i \lambda t\right),$$

for $i = 1, 2, \ldots n$ *and*

$$\gamma_{2n+1}(t) = \left(\frac{\sum\limits_{k=1}^{n} c_k{}^2 - n}{\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2}\right)^{\frac{1}{2}} t$$

where $\lambda = \left(\frac{\sum_{k=1}^{n} c_k{}^4 - c_k{}^2}{\sum_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6}\right)^{\frac{1}{2}}$ *and* $c_1, c_2, \ldots, c_n > 1$ *with* $c_i \neq c_j$, $1 \leqslant i < j \leqslant n$. *Then,* γ *is a generalized helix which lies on hypercylinder*

$$\frac{1}{n\lambda^2 \sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2}\left(\frac{x_1^2 + x_2^2}{\left(\frac{c_1^2-1}{c_1}\right)^2} + \frac{x_3^2 + x_4^2}{\left(\frac{c_2^2-1}{c_2}\right)^2} + \cdots + \frac{x_{2n-1}^2 + x_{2n}^2}{\left(\frac{c_n^2-1}{c_n}\right)^2}\right) = 1.$$

Proof. After straightforward calculations, we have $\gamma'(t) = \beta(t)$ where β is a W-curve in Lemma 4.1. If we apply the Gramm-Schmidt orthogonalization process to the curve γ, we have

$$\begin{aligned} F_1(t) &= \beta(t), \\ F_2(t) &= \mathbf{t}(t), \\ F_i(t) &= k_1(t)k_2(t)\ldots k_{i-2}(t)\mathbf{n}_{i-2}(t) \text{ for } 3 \leqslant i \leqslant 2n-1, \end{aligned}$$

where $k_i\,(1 \leqslant i \leqslant n-1)$ is the curvature functions of the curve β. Now, we can calculate the curvature functions $\kappa_i, (1 \leqslant i \leqslant n-1)$ of the curve γ.

$$\begin{aligned}\kappa_1 &= \frac{\|F_2\|}{\|F_1\|^2} = 1,\\ \kappa_i &= \frac{\|F_{i+1}\|}{\|F_1\|\,\|F_i\|} = k_{i-1},\end{aligned}$$

for $2 \leqslant i \leqslant 2n$. Since the curvature functions k_i are constants for $1 \leqslant i \leqslant 2n-1$, the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{2n-1}}{\kappa_{2n}}$ are constants. Therefore, γ is a hypercylindrical generalized helix. $\square$

Corollary 4.2. *From Theorem 4.3, the Frenet vector fields of the curve γ are*

$$V_1 = \beta, \qquad V_2 = \mathbf{t}, \qquad V_3 = \mathbf{n}_1, \quad \ldots, \quad V_{2n+1} = \mathbf{n}_{2n-1} \tag{4.2}$$

where $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{2n-1}\}$ is the generalized Sabban frame of the unit speed curve β.

Example 4.2. If we choose $c_1 = 3$ and $c_2 = 4$ in Theorem 4.3, then

$$\gamma(t) = \begin{pmatrix} \frac{4\sqrt{29}}{39}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), -\frac{4\sqrt{29}}{39}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{15\sqrt{29}}{104}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), -\frac{15\sqrt{29}}{104}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{\sqrt{23}}{2\sqrt{78}}t \end{pmatrix}$$

After straightforward calculations, we obtain the Frenet vector fields of the curve γ

$$\begin{aligned}
V_1(t) &= \begin{pmatrix} \frac{-2\sqrt{2}}{\sqrt{39}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{-2\sqrt{2}}{\sqrt{39}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-5\sqrt{3}}{2\sqrt{26}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{-5\sqrt{3}}{2\sqrt{26}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{\sqrt{23}}{2\sqrt{78}} \end{pmatrix},\\
V_2(t) &= \begin{pmatrix} \frac{-2}{\sqrt{29}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{2}{\sqrt{29}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-5}{\sqrt{29}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{5}{\sqrt{29}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), 0 \end{pmatrix},\\
V_3(t) &= \begin{pmatrix} \frac{-19\sqrt{2}}{\sqrt{4043}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{-19\sqrt{2}}{\sqrt{4043}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{85}{2\sqrt{8086}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{85}{2\sqrt{8086}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{29\sqrt{23}}{2\sqrt{8086}} \end{pmatrix},\\
V_4(t) &= \begin{pmatrix} -\frac{5}{\sqrt{29}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{5}{\sqrt{29}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{2}{\sqrt{29}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), -\frac{2}{\sqrt{29}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), 0 \end{pmatrix},\\
V_5(t) &= \begin{pmatrix} \frac{5\sqrt{23}}{\sqrt{933}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{5\sqrt{23}}{\sqrt{933}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-\sqrt{69}}{2\sqrt{311}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{-\sqrt{69}}{2\sqrt{311}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{35}{2\sqrt{933}} \end{pmatrix}.
\end{aligned}$$

It is clear that the Frenet vector fields V_1, V_3 and V_5 of the curve γ make constant angles $\theta_1 = \frac{\sqrt{23}}{2\sqrt{78}}$, $\theta_3 = \frac{29\sqrt{23}}{2\sqrt{8086}}$ and ,$\theta_5 = \frac{35}{2\sqrt{933}}$ with vector $U = (0, 0, 0, 0, 1)$, respectively.
Also, after straightforward calculations, we have the curvatures of the curve γ

$$\kappa_1 = 1, \quad \kappa_2 = \frac{\sqrt{311}}{29\sqrt{3}}, \quad \kappa_3 = \frac{455}{29\sqrt{933}}, \quad \kappa_4 = \sqrt{\frac{299}{622}}.$$

Since, γ lies on the hypercylinder $\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}^5 \middle| \frac{x_1^2+x_2^2}{\frac{16}{351}} + \frac{x_3^2+x_4^2}{\frac{150}{1664}} = 1\right\}$, then γ is a hypercylindrical generalized helix in $\mathbb{E}^5$.

Remark 4.1. Even if the curve α and γ have different curvatures, they have same Frenet vectors.

Example 4.3. If we choose $c_1 = 2$ and in Theorem 4.2, then

$$\alpha(t) = \left(\frac{2\cos\frac{t}{\sqrt{3}}\cos\frac{2t}{\sqrt{3}} + \sin\frac{t}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}}{2}, \frac{\cos\frac{2t}{\sqrt{3}}\sin\frac{t}{\sqrt{3}} - 2\cos\frac{t}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}}{2}, \sqrt{\frac{3}{4}}\sin\frac{t}{\sqrt{3}} \right)$$

After straightforward calculations, we obtain the Frenet vector fields of the curve α

$$\begin{aligned} T_\alpha(t) &= \left(-\frac{\sqrt{3}}{2}\sin\frac{2t}{\sqrt{3}}, -\frac{\sqrt{3}}{2}\cos\frac{2t}{\sqrt{3}}, \frac{1}{2} \right), \\ N_\alpha(t) &= \left(-\cos\frac{2t}{\sqrt{3}}, \sin\frac{2t}{\sqrt{3}}, 0 \right), \\ B_\alpha(t) &= \left(\frac{1}{2}\sin\frac{2t}{\sqrt{3}}, \frac{1}{2}\cos\frac{2t}{\sqrt{3}}, \frac{\sqrt{3}}{2} \right). \end{aligned}$$

It is clear that the Frenet vector fields T_α and B_α of the curve α make constant angles $\theta_1 = \arccos\frac{1}{2}$ and $\theta_3 = \arccos\frac{\sqrt{3}}{2}$ with vector $U = (0,0,1)$, respectively. Also, after straightforward calculating, we have the curvatures of the curve α

$$\kappa_1 = \sec\frac{t}{\sqrt{3}}, \quad \kappa_2 = \frac{1}{\sqrt{3}}\sec\frac{t}{\sqrt{3}}.$$

Since, α lies on $S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{E}^3 \middle| \sum_{i=1}^{3} x_i^2 = 1 \right\}$, then α is a spherical generalized helix in $\mathbb{E}^3$.

Example 4.4. If we choose $c_1 = 2$ and in Theorem 4.3, then

$$\gamma(t) = \left(\frac{3}{4}\cos\frac{2t}{\sqrt{3}}, -\frac{3}{4}\sin\frac{2t}{\sqrt{3}}, \frac{t}{2} \right).$$

After straightforward calculations, we obtain the Frenet vector fields of the curve γ

$$\begin{aligned} T_\gamma(t) &= \left(-\frac{\sqrt{3}}{2}\sin\frac{2t}{\sqrt{3}}, -\frac{\sqrt{3}}{2}\cos\frac{2t}{\sqrt{3}}, \frac{1}{2} \right), \\ N_\gamma(t) &= \left(-\cos\frac{2t}{\sqrt{3}}, \sin\frac{2t}{\sqrt{3}}, 0 \right), \\ B_\gamma(t) &= \left(\frac{1}{2}\sin\frac{2t}{\sqrt{3}}, \frac{1}{2}\cos\frac{2t}{\sqrt{3}}, \frac{\sqrt{3}}{2} \right). \end{aligned}$$

It is clear that the Frenet vector fields T_γ and B_γ of the curve makes constant angles $\theta_1 = \arccos\frac{1}{2}$ and $\theta_3 = \arccos\frac{\sqrt{3}}{2}$ with vector $U = (0,0,1)$, respectively. Also, after straightforward calculating, we have the curvatures of γ

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{\sqrt{3}}.$$

Since, γ lies on $\frac{x_1^2 + x_2^2}{\left(\frac{3}{4}\right)^2} = 1$, then α is a circular helix in $\mathbb{E}^3$.

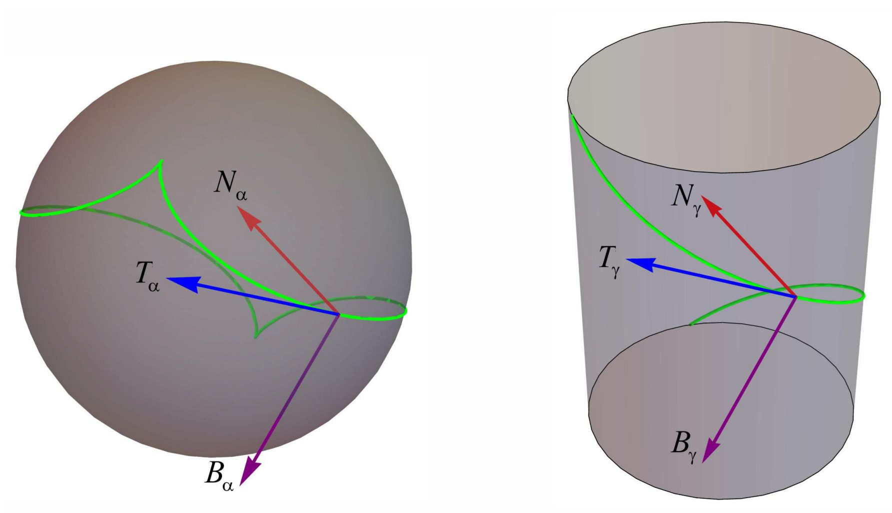

FIG. 4.1: Frenet vectors of the curves α and γ for $t = \frac{\pi}{6}$ in Example 4.3 and 4.4.

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