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ON CLAUSEN SERIES ${}_3F_2[-m, \alpha, \lambda+3; \beta, \lambda; 1]$ WITH APPLICATIONS*

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Abstract. In this paper, a summation theorem for the Clausen series is derived. Further, a reduction formula is obtained for the Kampé de Fériet double hypergeometric function. Some special cases are given as applications. A generalization of the reduction and linear transformation formulas is also given in the form of the general double series identity.
Key words: Clausen series, hypergemoetric function, summation theorem

1. Introduction and preliminaries

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

(1.1)
$$ {}_pF_q\left[\begin{array}{ll}(\alpha_p); & \\ & z\\ (\beta_q); & \end{array}\right] := {}_pF_q\left[\begin{array}{ll}\alpha_1, \alpha_2, \ldots, \alpha_p; & \\ & z\\ \beta_1, \beta_2, \ldots, \beta_q; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} $$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume

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that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots \; ; \quad j = 1, 2, \ldots, q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Chu-Vandermonde theorem [5, p.69, Q.No. 4]:

$$(1.2) \qquad {}_2F_1\left[\begin{array}{ccc} -M, \; A & ; & \\ & & 1 \\ B & ; & \end{array}\right] = \frac{(B-A)_M}{(B)_M}; \quad M = 0, 1, 2, \cdots,$$

such that ratio of Pochhammer symbols in r.h.s. is well defined and $A, B \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions were unified and generalized by Kampé de Fériet [2, 1] who defined a general hypergeometric function of two variables.

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [6, p.423, Eq.(26)]:

(1.3)

$$F^{p:\, q;\, k}_{\ell:\, m;\, n}\left[\begin{array}{lll} (a_p) & : (b_q) \; ; \; (c_k) \; ; & \\ & & x, \; y \\ (\alpha_\ell) & : (\beta_m) \; ; \; (\gamma_n) \; ; & \end{array}\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{\ell} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!},$$

where, for convergence,

(1.4) (i) $p + q < \ell + m + 1, \quad p + k < \ell + n + 1, \quad |x| < \infty, \quad |y| < \infty,$ or

(1.5) (ii) $p + q = \ell + m + 1, \quad p + k = \ell + n + 1$ and

$$(1.6) \qquad \begin{cases} |x|^{1/(p-\ell)} + |y|^{1/(p-\ell)} < 1, & \text{if } p > \ell \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq \ell. \end{cases}$$

An important development has been made by various authors in generalizations of the summation and transformation theorems, see [7, 4, 3]. In this work, our main motive is to find the summation theorem for the Clausen series ${}_3F_2[-m, \alpha, \lambda + 3; \beta, \lambda; 1]$ and to find its applications.

We shall use the following definition in proving our results in Sections 2 to 5:

Definition 1.1. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $r \in \mathbb{Z}^+ \cup \{0\}$, the following identity holds true:

$$\frac{(\lambda+3)_r}{(\lambda)_r} = 1 + \frac{3r}{\lambda} + \frac{3r(r-1)}{\lambda(\lambda+1)} + \frac{r(r-1)(r-2)}{\lambda(\lambda+1)(\lambda+2)}. \tag{1.7}$$

The proof of the above identity can be obtained smoothly.

2. Summation theorem

Theorem 2.1. *If γ, δ, σ are the roots of the cubic equation $Cm^3 + Dm^2 + Em + G = 0$ and α, β, λ, $-\gamma$, $-\delta$, $-\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $m \in \mathbb{N}_0$, then the following summation theorem holds true:*

(2.1)
$$ {}_3F_2\left[\begin{array}{lr} -m,\ \alpha,\ \lambda+3 & ; \\ & 1 \\ \beta,\ \lambda & ; \end{array}\right] = \frac{(-\gamma+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\gamma)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m},$$

where the coefficients C, D, E and G are the polynomials in α, β, λ given as follows:

$$\begin{aligned}
C &= -2\alpha + 3\alpha^2 - \alpha^3 + 2\lambda - 6\alpha\lambda + 3\alpha^2\lambda + 3\lambda^2 - 3\alpha\lambda^2 + \lambda^3, && (2.2)\\
D &= 12\alpha - 9\alpha^2 - 3\alpha^3 - 6\alpha\beta + 6\alpha^2\beta - 12\lambda + 27\alpha\lambda + 3\alpha^2\lambda \\
&\quad -3\alpha^3\lambda + 6\beta\lambda - 15\alpha\beta\lambda + 3\alpha^2\beta\lambda - 18\lambda^2 + 6\alpha\lambda^2 + 6\alpha^2\lambda^2 \\
&\quad +9\beta\lambda^2 - 6\alpha\beta\lambda^2 - 6\lambda^3 - 3\alpha\lambda^3 + 3\beta\lambda^3, && (2.3)\\
E &= -22\alpha - 12\alpha^2 - 2\alpha^3 + 24\alpha\beta + 6\alpha^2\beta - 6\alpha\beta^2 + 22\lambda - 21\alpha\lambda \\
&\quad -27\alpha^2\lambda - 6\alpha^3\lambda - 24\beta\lambda + 30\alpha\beta\lambda + 15\alpha^2\beta\lambda + 6\beta^2\lambda - 9\alpha\beta^2\lambda \\
&\quad +33\lambda^2 + 18\alpha\lambda^2 - 6\alpha^2\lambda^2 - 3\alpha^3\lambda^2 - 36\beta\lambda^2 - 3\alpha\beta\lambda^2 + 6\alpha^2\beta\lambda^2 \\
&\quad +9\beta^2\lambda^2 - 3\alpha\beta^2\lambda^2 + 11\lambda^3 + 12\alpha\lambda^3 + 3\alpha^2\lambda^3 - 12\beta\lambda^3 \\
&\quad -6\alpha\beta\lambda^3 + 3\beta^2\lambda^3 && (2.4)\\
G &= -12\lambda - 22\alpha\lambda - 12\alpha^2\lambda - 2\alpha^3\lambda + 22\beta\lambda + 24\alpha\beta\lambda + 6\alpha^2\beta\lambda \\
&\quad -12\beta^2\lambda - 6\alpha\beta^2\lambda + 2\beta^3\lambda - 18\lambda^2 - 33\alpha\lambda^2 - 18\alpha^2\lambda^2 - 3\alpha^3\lambda^2 \\
&\quad +33\beta\lambda^2 + 36\alpha\beta\lambda^2 + 9\alpha^2\beta\lambda^2 - 18\beta^2\lambda^2 - 9\alpha\beta^2\lambda^2 + 3\beta^3\lambda^2 - 6\lambda^3 \\
&\quad -11\alpha\lambda^3 - 6\alpha^2\lambda^3 - \alpha^3\lambda^3 + 11\beta\lambda^3 + 12\alpha\beta\lambda^3 + 3\alpha^2\beta\lambda^3 - 6\beta^2\lambda^3 \\
&\quad -3\alpha\beta^2\lambda^3 + \beta^3\lambda^3 && (2.5)\\
&= -C\gamma\delta\sigma \\
&= \lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha-1)\ (\beta-\alpha-2)\ (\beta-\alpha-3).
\end{aligned}$$

Proof. Suppose the l.h.s. of equation (2.1) is denoted by Δ, then we have

$$\Delta = \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r\ (\lambda+3)_r}{(\beta)_r\ (\lambda)_r\ r!}$$

$$= \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r}{(\beta)_r\ r!}\left[1+\frac{3r}{\lambda}+\frac{3r(r-1)}{\lambda(\lambda+1)}+\frac{r(r-1)(r-2)}{\lambda(\lambda+1)(\lambda+2)}\right]$$

$$= {}_2F_1\left[\begin{array}{ccc} -m,\ \alpha & ; & \\ & & 1 \\ \beta & ; & \end{array}\right] + \frac{3}{\lambda}\sum_{r=0}^{m-1}\frac{(-m)_{r+1}\ (\alpha)_{r+1}}{(\beta)_{r+1}\ r!}$$

$$+\frac{3}{\lambda(\lambda+1)}\sum_{r=0}^{m-2}\frac{(-m)_{r+2}\ (\alpha)_{r+2}}{(\beta)_{r+2}\ r!}+\frac{1}{\lambda(\lambda+1)(\lambda+2)}\sum_{r=0}^{m-3}\frac{(-m)_{r+3}\ (\alpha)_{r+3}}{(\beta)_{r+3}\ r!}$$

$$= {}_2F_1\left[\begin{array}{ccc} -m,\ \alpha & ; & \\ & & 1 \\ \beta & ; & \end{array}\right] + \frac{3}{\lambda}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}{}_2F_1\left[\begin{array}{ccc} -(m-1),\ \alpha+1 & ; & \\ & & 1 \\ \beta+1 & ; & \end{array}\right]+$$

$$+\frac{3}{\lambda(\lambda+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}{}_2F_1\left[\begin{array}{ccc} -(m-2),\ \alpha+2 & ; & \\ & & 1 \\ \beta+2 & ; & \end{array}\right]+$$

$$+\frac{1}{\lambda(\lambda+1)(\lambda+2)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}{}_2F_1\left[\begin{array}{ccc} -(m-3),\ \alpha+3 & ; & \\ & & 1 \\ \beta+3 & ; & \end{array}\right]. \tag{2.6}$$

Using Chu-Vandermonde theorem (1.2) in r.h.s. of equation (2.6), we obtain

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}+\frac{3}{\lambda}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}\frac{(\beta-\alpha)_{m-1}}{(\beta+1)_{m-1}}+\frac{3}{\lambda(\lambda+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}\frac{(\beta-\alpha)_{m-2}}{(\beta+2)_{m-2}}+$$

$$+\frac{1}{\lambda(\lambda+1)(\lambda+2)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}\frac{(\beta-\alpha)_{m-3}}{(\beta+3)_{m-3}}$$

$$= \frac{(\beta-\alpha)_m}{(\beta)_m}+\frac{3(-m)_1\ (\alpha)_1}{\lambda}\frac{(\beta-\alpha)_{m-1}}{(\beta)_m}+\frac{3(-m)_2\ (\alpha)_2}{\lambda(\lambda+1)}\frac{(\beta-\alpha)_{m-2}}{(\beta)_m}+$$

$$+\frac{(-m)_3\ (\alpha)_3}{\lambda(\lambda+1)(\lambda+2)}\frac{(\beta-\alpha)_{m-3}}{(\beta)_m}$$

$$= \frac{(\beta-\alpha)_m}{(\beta)_m}\left[1-\frac{3m\ \alpha}{\lambda\ (\beta-\alpha+m-1)}+\frac{3(-m)_2\ (\alpha)_2}{\lambda(\lambda+1)\ (\beta-\alpha+m-2)_2}+\right.$$

$$\left.+\frac{(-m)_3\ (\alpha)_3}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-3)_3}\right]$$

$$= \frac{(\beta-\alpha)_m}{(\beta)_m}\cdot$$

$$\cdot\left[\frac{\Omega(\alpha,\beta,\lambda,m)}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right], \tag{2.7}$$

where

$$\begin{aligned}\Omega(\alpha,\beta,\lambda,m) &= \lambda(\lambda+1)(\lambda+2)(\beta-\alpha+m-1)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\ &-3m\alpha(\lambda+1)(\lambda+2)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\ &+3(-m)(-m+1)(\alpha)(\alpha+1)(\lambda+2)(\beta-\alpha+m-3)\\ &+(-m)(-m+1)(-m+2)(\alpha)(\alpha+1)(\alpha+2).\end{aligned}$$

Equation (2.7) can be written as

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{Cm^3+Dm^2+Em+G}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right], \tag{2.8}$$

Since γ, δ, σ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$, therefore equation (2.8) can be written as:

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{C(m-\gamma)(m-\delta)(m-\sigma)}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right]. \tag{2.9}$$

On simplification, we get assertion (2.1). □

3. Application in reducibility of the Kampé de Fériet function

The application of summation Theorem 2.1 is given by proving the following reduction formula:

Theorem 3.1. *For $b_1,\cdots,b_B$, α, β, λ, $-\gamma$, $-\delta$, $-\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$, the following reduction formula holds true:*

$$F^{A:0;2}_{B:0;2}\left[\begin{matrix}(a_A) & : & - & ; & \alpha,\ \lambda+3 & ; \\ & & & & & & z,\ -z\\ (b_B) & : & - & ; & \beta,\ \lambda & ;\end{matrix}\right]=$$

$$ {}_{A+4}F_{B+4}\left[\begin{matrix}a_1,\cdots,a_A,\ -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & & z\\ b_1,\cdots,b_B,\ -\gamma,\ -\delta,\ -\sigma,\ \beta & ;\end{matrix}\right], \tag{3.1}$$

subject to the convergence conditions:

$$\begin{cases}|z|<\frac{1}{2}, & \text{if } A=B+1\\ |z|<\infty, & \text{if } A\le B,\end{cases}$$

where γ, δ, σ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$ and C,D,E,G are given by equations (2.2)-(2.5).

Proof. Suppose l.h.s. of equation (3.1) is denoted by Φ, then we have

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_{m+n}\ (\alpha)_n\ (\lambda+3)_n (-1)^n z^{m+n}}{\prod_{i=1}^{B}(b_i)_{m+n}\ (\beta)_n\ (\lambda)_n\ m!\ n!}$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m} \frac{z^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n\ (\alpha)_n\ (\lambda+3)_n}{(\beta)_n\ (\lambda)_n\ n!}$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m} \frac{z^m}{m!}\ {}_3F_2\left[\begin{array}{lr} -m,\ \alpha,\ \lambda+3 & ; \\ & 1 \\ \beta,\ \lambda & ; \end{array}\right]. \tag{3.2}$$

Using Theorem 2.1 in r.h.s. of above equation, it follows that

$$\Phi = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m} \frac{(-\gamma+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\gamma)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m} \frac{z^m}{m!}. \tag{3.3}$$

In view of equation (3.3), reduction formula (3.1) follows. □

4. Applications in linear transformations

If γ, δ, σ are the roots of the cubic equation $Cm^3 + Dm^2 + Em + G = 0$ and C, D, E, G are given by equations (2.2)-(2.5), we prove the following consequences of Theorem 3.1:

I. Taking $A = B = 0$ in equation (3.1), we get the following transformation formula:

$$ {}_4F_4\left[\begin{array}{lr} -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & z \\ -\gamma,\ -\delta,\ -\sigma,\ \beta & ; \end{array}\right] =$$

$$\exp(z)\ {}_2F_2\left[\begin{array}{lr} \alpha,\ \lambda+3 & ; \\ & -z \\ \beta,\ \lambda & ; \end{array}\right], \tag{4.1}$$

where $|z| < \infty$ and α, β, λ, $-\gamma$, $-\delta$, $-\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

II. Taking $A = 1$, $a_1 = a$, $B = 0$ in equation (3.1) and using binomial theorem, we get the following transformation formula:

$$ {}_5F_4\left[\begin{array}{lr} a,\ -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & z \\ -\gamma,\ -\delta,\ -\sigma,\ \beta & ; \end{array}\right]$$

$$(4.2) \qquad = (1-z)^{-a}\ {}_3F_2\left[\begin{array}{lll} a,\ \alpha,\ \lambda+3 & ; & \\ & & \frac{-z}{1-z} \\ \beta,\ \lambda & ; & \end{array}\right],$$

where $|z|<1$, $\left|\frac{-z}{1-z}\right|<1$ and $\alpha,\ \beta,\ \lambda,\ -\gamma,\ -\delta,\ -\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

5. General double series identity

Theorem 5.1. *Let $\{\Theta(\ell)\}_{\ell=1}^{\infty}$ is bounded sequence of arbitrary complex numbers, $\Theta(0)\neq 0$ and $\alpha,\ \beta,\ \lambda,\ -\gamma,\ -\delta,\ -\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\sum_{m,n=0}^{\infty} \frac{\Theta(m+n)\ (\alpha)_n\ (\lambda+3)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n} \frac{z^{m+n}}{m!\ n!}$$

$$(5.1) \qquad = \sum_{m=0}^{\infty} \frac{\Theta(m)\ (-\gamma+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\gamma)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m} \frac{z^m}{m!},$$

where $\gamma,\ \delta,\ \sigma$ are the roots of cubic equation $Cm^3+Dm^2+Em+G=0$ and C, D, E, G are given by equations (2.2)-(2.5) *with each of the multiple series involved is absolutely convergent.*

Remark 5.1. *For $\Theta(\ell) = \frac{\prod_{i=1}^{A}(a_i)_\ell}{\prod_{i=1}^{B}(b_i)_\ell}$, the above series identity reduces to the reduction formula* (3.1).

Appendix

The roots γ, δ, σ of the cubic equation $Cm^3+Dm^2+Em+G=0$ are calculated by using *Wolfram Mathematica 9.0* Software. The values of γ, δ and σ are given as follows:

$$\gamma = -\frac{D}{3C} - \frac{2^{1/3}\left(-D^2+3CE\right)}{3C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}} + \frac{\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{3\times 2^{1/3}C}$$

$$\delta = -\frac{D}{3C} + \frac{\left(1+i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}$$

$$-\frac{\left(1-i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}$$

$$\sigma=-\frac{D}{3C}$$

$$+\frac{\left(1-i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}$$

$$-\frac{\left(1+i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}$$

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