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UNIQUE ECCENTRIC CLIQUE GRAPHS

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Abstract. Let G be a connected graph and ζ the set of all cliques in G. In this paper we introduce the concepts of unique (ζ,ζ)-eccentric clique graphs and self (ζ,ζ)-centered graphs. Certain standard classes of graphs are shown to be self (ζ,ζ)-centered, and we characterize unique (ζ,ζ)-eccentric clique graphs which are self (ζ,ζ)-centered.
Keywords: clique graph, graph eccentricity, connected graph.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [3]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. It is known that the this distance function d is a metric on the vertex set V. The *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The set of all vertices for which e is minimized is called the *center* of G and is denoted by $Z(G)$. The set of all vertices for which e is maximized is called the periphery of G and is denoted by $P(G)$. The concept of the center of a graph arises in the context of selection of a site at which to locate a facility in a graph. Taking into account the situation that the nature of the facility to be constructed could necessitate selecting a structure rather than a vertex to locate a facility, Slater [9]

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proposed four classes of locational problems, namely, vertex-serves-vertex, vertex-serves-structure, structure-serves-vertex and structure-serves-structure. For subsets $S, T \subseteq V$ and any vertex v in V, let $d(v, S) = \min\{d(v, u) : u \in S\}$ and $d(S, T) = \min\{d(x, y) : x \in S, y \in T\}$, respectively. The *degree* of a vertex v in a graph G, denoted by d_v or $deg\, v$, is the number of edges incident with v. Let S be a set and $F = \{S_1, S_2, \ldots, S_p\}$ a nonempty family of distinct nonempty subsets of S whose union is S. The *intersection graph* of F is denoted $\Omega(F)$ and defined by $V(\Omega(F)) = F$, with S_i and S_j adjacent whenever $i \neq j$ and $S_j \cap S_j \neq \phi$. Then a graph G is an *intersection graph* on S if there exists a family F of subsets of S for which $G \cong \Omega(F)$.

Definition 1.1. [10] Let $G = (V, E)$ be a connected graph. Let $\zeta = \{C_i : i \in I\}$ and $S = \{S_j : j \in J\}$, where each of C_i and S_j is a subset of V. Let $e_S(C_i) = max\{d(C_i, S_j) : j \in J\}$; C_i is called a (ζ, S)-*center* if $e_S(C_i) \leq e_S(C_k)$ for all $k \in I$.

Slater [10] investigated the centrality of paths by taking S to be the collection of all paths in G and ζ to be the collection of all single vertex sets in G, leading to the concepts of the *path center*, *path centroid* and *path median* of G. Let r and d represent respectively the radius and diameter of the graph G. A *clique* in G is a set S of vertices of G such that the sub graph induced by S is a maximal complete sub graph of G. Throughout the following, let ζ denote the set of all cliques in G. Santhakumaran and Arumugam [5] introduced and studied the concepts of (V, ζ)-*center*, (ζ, V)-*center* and (ζ, ζ)-*center*. Santhakumaran [7] introduced the concept of (V, ζ)-*periphery*, (ζ, V)-*periphery* and (ζ, ζ)-*periphery* and investigated their properties.

Definition 1.2. [5, 7] Let $G = (V, E)$ be a connected graph. Let $C \in \zeta$ and $v \in V$. We define the *vertex-to-clique eccentricity* by $e_1(v) = max\{d(v, C) : C$ is clique in $G\}$. The *clique-to-vertex eccentricity* $e_2(C)$ is defined by $e_2(C) = max\{d(C, v) : v \in V\}$. The *clique-to-clique eccentricity* $e_3(C)$ is defined by $e_3(C) = max\{d(C, C') : C' \in \zeta\}$. The set of all vertices for which $e_1(v)$ is minimum is called the (V, ζ)-*center* of G and is denoted by $Z_1(G)$. The set of all vertices for which $e_1(v)$ is maximum is called the (V, ζ)-*periphery* of G and is denoted by $P_1(G)$. The set of all cliques C for which $e_2(C)$ is minimum is called the (ζ, V)-*center* of G and is denoted by $Z_2(G)$. The set of all cliques C for which $e_2(C)$ is maximum is called the (ζ, V)-*periphery* of G and is denoted by $P_2(G)$. The set of all cliques C for which $e_3(C)$ is minimum is called the (ζ, ζ)-*center* of G and is denoted by $Z_3(G)$. The set of all cliques C for which $e_3(C)$ is maximum is called the (ζ, ζ)-*periphery* of G and is denoted by $P_3(G)$.

Santhakumaran and Arumugam [8] also introduced and studied the concepts of (V, ζ)-*radius*, (V, ζ)-*diameter*, (ζ, V)-*radius*, (ζ, V)-*diameter*, (ζ, ζ)-*radius*, and (ζ, ζ)-*diameter* of a graph G.

Definition 1.3. [8] Let $G = (V, E)$ be a connected graph. The (V, ζ)-*radius* r_1 of G and the (V, ζ)-*diameter* d_1 of G are defined by $r_1 = min\{e_1(v) : v \in V\}$ and

$d_1 = max\{e_1(v) : v \in V\}$, respectively. The (ζ, V)-*radius* r_2 of G and the (ζ, V)-*diameter* d_2 of G are defined by $r_2 = min\{e_2(C) : C \in \zeta\}$ and $d_2 = max\{e_2(C) : C \in \zeta\}$, respectively. The (ζ, ζ)-*radius* r_3 of G and the (ζ, ζ)-*diameter* d_3 of G are defined by $r_3 = min\{e_3(C) : C \in \zeta\}$ and $d_3 = max\{e_3(C) : C \in \zeta\}$, respectively.

We observe that for any graph G, $d_1 = d_2$. However r_1 and r_2 need not be equal.

Parthasarathy and Nandakumar [4] introduced and studied unique eccentric vertex graphs.

Definition 1.4. [4] A vertex v in a connected graph G is called an *eccentric vertex* of u if $d(u, v) = e(u)$. A vertex v is called an eccentric vertex if it is an eccentric vertex of some vertex u, and is called a *non-eccentric vertex*, otherwise. A graph G is called a *unique eccentric vertex graph* (a u.e.v. graph for short) if $|E(u)| = 1$ for every $u \in V(G)$, where $E(u)$ denotes the set of all eccentric vertices of u. The unique eccentric vertex of u is denoted by u^*.

Santhakumaran [6] introduced and studied the concept of unique vertex-to-clique eccentric clique graphs.

Definition 1.5. [6] Let G be a connected graph. Any clique C in G for which $e_1(v) = d(v, C)$ is called a (V, ζ)-*eccentric clique* of the vertex v in G. We call a clique C a (V, ζ)-*eccentric clique* if it is a (V, ζ)-eccentric clique of some vertex v in G. Let $E_1(v)$ denote the set of all (V, ζ)-eccentric cliques of v. A graph G is said to be a *unique*(V, ζ)-*eccentric clique graph* if $|E_1(v)| = 1$ for every vertex v in G.

Santhakumaran [8] introduced the concept of unique clique-to-vertex eccentric vertex graphs and investigated their properties.

Definition 1.6. [8] Let G be a connected graph. Any vertex v in G for which $e_2(C) = d(C, v)$ is called a (ζ, V)-*eccentric vertex* of the clique C in G. We call a vertex v a (ζ, V)-*eccentric vertex* if it is a (ζ, V)-eccentric vertex of some clique C in G. Let $E_2(C)$ denote the set of all (ζ, C)-eccentric vertices of C. A graph G is said to be a *unique*(ζ, V)-*eccentric vertex graph* if $|E_2(C)| = 1$ for every clique C in G.

A graph G is a self-centered graph if every vertex of G is in the center $Z(G)$ of G.

The following theorem is used in the sequel.

Theorem 1.1. [4] A *u.e.v* graph G is self-centered if and only if each vertex of G is an eccentric vertex.

Centrality concepts have interesting applications in social networks [1, 2]. In a social network a clique represents a group of individuals having "a common interest" and hence centrality with respect to cliques, unique (ζ,ζ)-eccentric clique graphs and self (ζ,ζ)-centered graphs will have interesting applications in social networks. A (ζ,ζ)-eccentric clique is simply called an eccentric clique and a unique (ζ,ζ)-eccentric clique graph simply a *unique eccentric clique* (u.e.c.) *graph*.

2. Unique Eccentric Clique (u.e.c.) Graphs

Definition 2.1. Let G be a connected graph and let C be a clique in G. Any clique C' in G for which $e_3(C) = d(C, C')$ is called a (ζ,ζ)-*eccentric clique* of the clique C in G. We call a clique C' a (ζ,ζ)-*eccentric clique* if it is a (ζ,ζ)-eccentric clique of some clique C in G. A graph G is said to be a *unique*(ζ,ζ)-*eccentric clique graph* if $|E_3(C)| = 1$ for every C in ζ, where $E_3(C)$ denotes the set of all (ζ,ζ)-eccentric cliques of C. The unique (ζ,ζ)-eccentric clique of G is denoted by C^*. A (ζ,ζ)-eccentric clique is simply called an eccentric clique and a unique (ζ,ζ)-eccentric clique graph simply a *unique eccentric clique* (u.e.c.) *graph*.

Definition 2.2. A graph G is called a $self(\zeta,\zeta)$-*centered graph* if every clique of G is in the (ζ,ζ)-center $Z_3(G)$ of G.

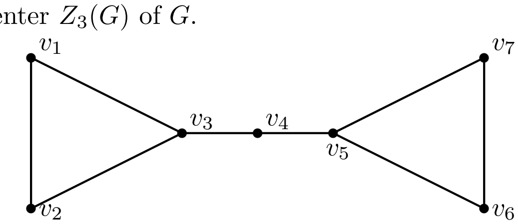

Figure 2.1: G_1

Example 2.1. For the graph G_1 given in Figure 2.1, the cliques are $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5\}$ and $C_4 = \{v_5, v_6, v_7\}$. It is easily seen that $e_3(C_1) = 2$, $e_3(C_2) = 1$, $e_3(C_3) = 1$ and $e_3(C_4) = 2$. The eccentric cliques of C_1, C_2, C_3 and C_4 are C_4, C_4, C_1 and C_1, respectively and G_1 is a u.e.c graph. Also, $Z_3(G_1) = \{C_2, C_3\}$ and $P_3(G_1) = \{C_1, C_4\}$.

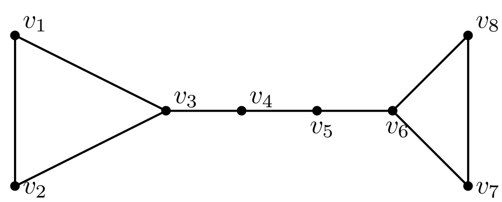

Figure 2.2: G_2

Example 2.2. For the graph G_2 given in Figure 2.2, the cliques are $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5\}$, $C_4 = \{v_5, v_6\}$ and $C_5 = \{v_5, v_6, v_7\}$. It is easy to see that $e_3(C_1) = 3$, $e_3(C_2) = 2$, $e_3(C_3) = 1$, $e_3(C_4) = 2$ and $e_3(C_5) = 3$. Thus $E_3(C_3) = \{C_1, C_5\}$ and so G_2 is not a u.e.c graph. Also, $Z_3(G_2) = \{C_3\}$ and $P_3(G_2) = \{C_1, C_5\}$.

Remark 2.1. If C is a (ζ,ζ)-peripheral clique in G, then it is a (ζ,ζ)-eccentric clique in G. However, a (ζ,ζ)-eccentric clique need not be a (ζ,ζ)-peripheral clique. For the graph G_3 in Figure 2.3, the (ζ,ζ)-eccentricities are written alongside of the edges, $C_1 = \{v_1, u_1\}$ and $C_2 = \{u_2, v_2\}$ are the (ζ,ζ)-peripheral cliques, $C_3 = \{x_1, x_3\}$ and $C_4 = \{y_1, y_2\}$ are (ζ,ζ)-eccentric cliques which are not (ζ,ζ)-peripheral cliques.

A natural question that arises is whether $E_3(C) \bigcap P_3(G) \neq \phi$ for every C in ζ. However, there are graphs which contain C such that $E_3(C) \bigcap P_3(G) = \phi$. For the graph G_3 given in Figure 2.3, $P_3(G_3) = \{C_1, C_2\}$ and $E_3(C_4) = \{C_3\}$. We observe that $|P_3(G)| \geq 2$ for any non-complete graph G.

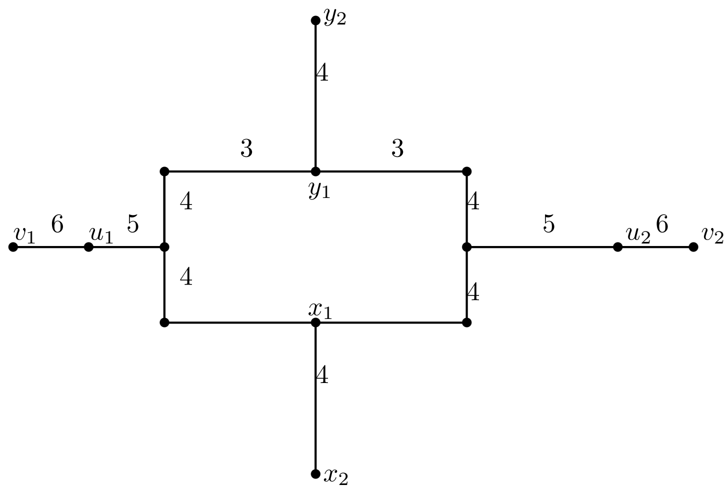

Figure 2.3: G_3

For any connected graph G, the *clique graph* H of G is the intersection graph of the family of all cliques in G. Thus, the vertices of H are the cliques of G. Two vertices C and D in H are adjacent in H if and only if C and D have a vertex common in G. Two cliques in G are called *adjacent* if they have a vertex in common. The distance in H is denoted by d_H.

The following theorem on the clique graph H of a graph G has several applications in facility location problems in real life situations.

Theorem 2.1. Let G be any connected graph and H its clique graph. Then $d_H(C,D) = d(C,D) + 1$ for any two cliques C and D in G.

Proof. Let C and D be two cliques in G. Suppose that C and D are adjacent in G. Then $d(C,D) = 0$. Now, since C and D are adjacent vertices in H, $d_H(C,D) = 1$ so that $d_H(C,D) = d(C,D) + 1$. Now, suppose that C and D are not adjacent in G. Let $d(C,D) = p \geq 1$. Hence there exist vertices $u_0 \in C$ and $u_p \in D$ such that $d(u_0, u_p) = p$. Let $P : u_0, u_1, u_2, \ldots, u_{p-1}, u_p$ be a shortest $u_0 - u_p$ path in G such that none of the $u_i (1 \leq i \leq p-1)$ belongs to C or D. Let C_i be a clique containing the edge $u_{i-1}u_i (1 \le i \le p)$. Since P is a shortest path in G, the cliques $C, C_1, C_2, \ldots, C_p, D$ are all distinct and $Q : C, C_1, C_2, \ldots, C_p, D$ is a $C-D$ shortest path in H so that $d_H(C, D) = p + 1 = d(C, D) + 1$. □

Theorem 2.2. Let G be any connected graph and H its clique graph. For any clique C in G, let $e_H(C)$ denotes the eccentricity of the vertex C in H. Then

(i) $e_3(C) = e_H(C) - 1$

(ii) $Z_3(G) = Z(H)$

(iii) $P_3(G) = P(H)$

(iv) $d_3 = d_H - 1$

(v) $r_3 = r_H - 1$

Proof. (i) By definition
$$\begin{aligned} e_3(C) &= max\{d(C, C') : C' \text{ is a clique in } G\} \\ &= max\{d_H(C, C') - 1 : C' \text{ is a vertex in } H\} \quad \text{(by Theorem 2.1)} \\ &= max\{d_H(C, C') : C' \text{ is a vertex in } H\} - 1 \\ &= e_H(C) - 1. \end{aligned}$$

Thus (i) is proved and now (ii) and (iii) follow from the definitions of $Z_3(G)$, $Z(H)$, $P_3(G)$ and $P(H)$. Also (iv) and (v) follow from (i). □

Corollary 2.1. A connected graph G is self (ζ, ζ)- centered if and only if its clique graph H is self-centered.

Theorem 2.3. If C_1 and C_2 are two adjacent cliques in a connected graph G, then $|e_3(C_1) - e_3(C_2)| \le 1$.

Proof. We first prove that if u and v are two adjacent vertices in G, then $|e(u) - e(v)| \le 1$. Suppose that $e(u) \ge e(v)$. Let u_1 be an eccentric vertex of u so that $e(u) = d(u, u_1)$. Then $e(u) = d(u, u_1) \le d(u, v) + d(v, u_1) \le 1 + e(v)$, and so $e(u) - e(v) \le 1$. It follows that $|e(u) - e(v)| \le 1$. Now, let H denote the clique graph of G. If C_1 and C_2 are two adjacent cliques in G, then C_1 and C_2 are two adjacent vertices in H and hence $|e_H(C_1) - e_H(C_2)| \le 1$. Hence by Theorem 2.2(i), $|e_3(C_1) + 1 - e_3(C_2) - 1| \le 1$ so that $|e_3(C_1) - e_3(C_2)| \le 1$. □

Theorem 2.4. If C_1 and C_2 are two adjacent cliques in a u.e.c graph G and $e_3(C_1) \ne e_3(C_2)$, then $C_1^* = C_2^*$, where C_1^* and C_2^* denote respectively the unique eccentric cliques of C_1 and C_2.

Proof. We may assume without loss of generality that $e_3(C_1) < e_3(C_2)$. Let $\zeta' = \zeta - \{e_3^*(C_1)\}$. Then $d(C_1, C_1^*) = e_3(C_1)$ and since G is a u.e.c graph, $d(C_1, C) \leq e_3(C_1) - 1$ for all C in ζ'. Since C_1 and C_2 are adjacent and $e_3(C_1) < e_3(C_2)$, it follows that $d(C_2, C) \leq 1 + d(C_1, C)$ for all cliques C in G. Hence $e_3(C_2) > e_3(C_1) \geq d(C_2, C)$ for all C in ζ'. Thus $e_3(C_1) > d(C_2, C)$ for all C in ζ' so that $C_2^* = C_1^*$. □

Corollary 2.2. In an u.e.c graph, any clique C with $e_3(C) = d_3 - 1$ is adjacent to at most one (ζ, ζ)- peripheral clique.

Proof. Suppose that C is adjacent to two distinct (ζ, ζ)- peripheral cliques C_1 and C_2. Since $e_3(C_1) = e_3(C_2) = d_3$ and $e_3(C) = d_3 - 1$, it follows from Theorem 2.4 that $C_1^* = C^* = C_2^*$. Hence $d(C^*, C_1) = d(C^*, C_2) = d_3$ so that C^* has two distinct eccentric cliques C_1 and C_2, which is a contradiction. □

In the following part, we will give certain classes of graphs which are self (ζ, ζ)-centered.

If a graph G is complete, then G is the only clique of G and $e_3(G) = 0$ so that G is self (ζ, ζ)- centered. If G is an even cycle $C_{2p} (p \geq 2)$, then $e_3(C) = p - 1$ for any clique C in G. If G is an odd cycle $C_{2p+1} (p \geq 2)$, then again $e_3(C) = p - 1$ for any clique C in G. If $G = C_3$, then $e_3(G) = 0$. Hence every cycle is self (ζ, ζ)-centered.

Theorem 2.5. Any complete bipartite graph $G = K_{p,q}$ is self (ζ, ζ)-centered.

Proof. If G is a star, then each clique C is an edge and since $e_3(C) = 0$, it follows that $Z_3(G) = \zeta$ so that G is self (ζ, ζ)-centered. It G is not a star, let the partite sets of G be $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, $p > 1$ and $q > 1$. Then any clique C in G is of the form $C = x_i y_j$ ($1 \leq i \leq p$ and $1 \leq j \leq q$) and $e_3(C) = 1$. Hence $Z_3(G) = \zeta$ so that G is self (ζ, ζ)- centered. □

Remark 2.2. For a bipartite graph G, Theorem 2.5 is not true. For the graph G_4 given in Figure 2.4, $Z_3(G_4) = \{\{v_1, v_3\}, \{v_3, v_4\}, \{v_6, v_7\}, \{v_3, v_6\}\}$ and so G_4 is not self (ζ, ζ)-centered.

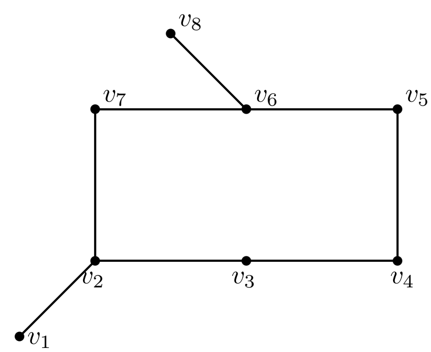

Figure 2.4: G_4

Theorem 2.6. If G is a connected graph such that every pair of cliques in G has a common vertex, then G is self (ζ, ζ)-centered.

Proof. Since $d(C, C') = 0$ for any two cliques C and C', it follows that $e_3(C) = 0$ for any clique C in G. Thus $Z_3(G) = \zeta$ so that G is self(ζ, ζ)- centered. □

Corollary 2.3. If G is a graph with n vertices and maximum degree $\Delta = n - 1$, then G is self (ζ, ζ) - centered.

Proof. Let $S = \{v \in V : deg\, v = n - 1\}$. Since $S \subseteq C$ for any clique C, the result follows. □

The following theorem gives a characterization for a u.e.c graph to be self (ζ, ζ)-centered.

Theorem 2.7. A u.e.c graph is self (ζ, ζ)-centered if and only if each clique of G is eccentric.

Proof. Let G be a self (ζ, ζ)-centered graph. For any clique C in G, let C^* be an eccentric clique of C so that $e_3(C^*) = e_3(C) = d(C^*, C)$. Hence C is an eccentric clique of C^*. Thus each clique of G is eccentric.

Let G be a u.e.c graph. Suppose that each clique of G is eccentric. First, we prove that every vertex of H is eccentric in H. Let C be any vertex of H. Then C is a clique in G. Since each clique of G is eccentric, there exists a clique C_1 in G such that $e_3(C_1) = d(C_1, C)$. By Theorem 2.2(i), $e_H(C_1) - 1 = d_H(C_1, C) - 1$ and so $e_H(C_1) = d_H(C_1, C)$. Thus every vertex in H is eccentric. Now, we prove that H is u.e.v graph. Let C be a vertex in H having two distinct eccentric vertices, say C_1 and C_2. Then $e_H(C) = d_H(C, C_1) = d_H(C, C_2)$. By Theorems 2.1 and 2.2(i), $e_3(C) + 1 = d(C, C_1) + 1 = d(C, C_2) + 1$, which gives $e_3(C) = d(C, C_1) = d(C, C_2)$ so that C_1 and C_2 are two distinct eccentric cliques of C in G, contradicting the hypothesis that G is a u.e.c graph. Thus H is a u.e.v graph such that every vertex in H is an eccentric vertex. Hence by Theorem 1.1, H is self centered. By Corollary 2.1, G is self (ζ, ζ) - centered. □

Corollary 2.4. A u.e.c graph G is self (ζ, ζ) - centered if and only if $C^{**} = C$ for every clique C in G.

Proof. Suppose that G is self (ζ, ζ) - centered. In a self (ζ, ζ) - centered graph, C^* is an eccentric clique of C if and only if C is an eccentric clique of C^*. Hence it follows that $C^{**} = C$ for every clique C in G. Conversely, suppose that $C = C^{**}$ for every clique C in G. Then C is the unique eccentric clique of C^*. Thus $e_3(C^*) = d(C^*, C)$ so that each clique C in G is eccentric. Hence by Theorem 2.7, G is self (ζ, ζ) - centered. □

Characterizing all self (ζ, ζ) - centered graphs seems to be a very difficult problem and we leave it as an open question.

Problem 2.1. Characterize self (ζ, ζ) - centered graphs.

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