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ON SOME EQUIVALENCE RELATION ON NON-ABELIAN CA-GROUPS

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Abstract. A non-abelian group G is called a CA-group (CC-group) if $C_G(x)$ is abelian (cyclic) for all $x \in G \setminus Z(G)$. We say $x \sim y$ if and only if $C_G(x) = C_G(y)$. We denote the equivalence class including x by$[x]_\sim$. In this paper, we prove that if G is a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$, then $2^{r-1} \le |G'| \le 2^{\binom{r}{2}}$. where $\frac{|G|}{|Z(G)|} = 2^r, 2 \le r$ and characterize all groups whose $[x]_\sim = xZ(G)$ for all $x \in G$ and $|G| \le 100$. Also, we will show that if G is a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$, then $G \cong C_m \times Q_8$ where C_m is a cyclic group of odd order m and if G is a CC-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$, then $G \cong Q_8$.
Keywords: CA-group, CC-group, centralizer of a group, derived subgroup.

1. Introduction

Throughout this paper all groups are assumed to be finite. We denote by $Z(G)$, $C_G(x), \mathrm{Cent}(G), |\mathrm{Cent}(G)|, x^G, G'$ and $k(G)$ the center of the group G, the centralizer of $x \in G$, the set of centralizers of the group G, the number of centralizers of the group G, the conjugacy class of $x \in G$, the derived subgroup of the group G, the number of conjugacy classes of the group G, respectively. The authors in [8], denoted by $[m, n]$ the GAP ID of a group which is a label that uniquely identifies a group in GAP. The first number in $[m, n]$ is the order of the group, and the second number simply enumerates different groups of the same order. We will use usual notation, for example C_n, D_{2n} and Q_{2^n} denote the cyclic group of order n, the

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dihedral group of order $2n$ and the generalized quaternion group of order 2^n respectively. The *non-commuting graph* $\Gamma(G)$ with respect to G is a graph with vertex set $G \setminus Z(G)$ and two distinct vertices x and y, are adjacent whenever $[x, y] \neq 1$. A non-abelian group G is called a CA-group (CC-group) if $C_G(x)$ is abelian (cyclic) for all $x \in G \setminus Z(G)$. We say $x \sim y$ if and only if $C_G(x) = C_G(y)$, and $x \sim_1 y$ if and only if $xZ(G) = yZ(G)$. We denote the equivalence class including x under $\sim$ by $[x]_\sim$. The number of equivalence classes of $\sim$ and $\sim_1$ on the group G are equal with $|\mathrm{Cent}(G)|$ and $\frac{|G|}{|Z(G)|}$ respectively. The influence of $|\mathrm{Cent}(G)|$ on the group G has been investigated in [3, 2, 4]. In [5], CA-groups whose $[x]_\sim = xZ(G)$ for all $x \in G$ has been investigated. In this paper we have investigated the equivalency of above relations. We will use the following lemmas to prove the main theorems.

Lemma 1.1. *[1, Lemma 3.6] Let G be a non-abelian group. Then the following are equivalent:*

1) *G is a* CA-*group.*
2) *If $[x, y] = 1$ then $C_G(x) = C_G(y)$, where $x, y \in G \setminus Z(G)$.*
3) *If $[x, y] = [x, z] = 1$ then $[y, z] = 1$, where $x \in G \setminus Z(G)$.*
4) *If $A, B \leq G, Z(G) \lneqq C_G(A) \leqslant C_G(B) \lneqq G$, then $C_G(A) = C_G(B)$.*

Lemma 1.2. *[1, Proposition 2.6] Let G be a finite non-abelian group and $\Gamma(G)$ be a regular graph. Then G is nilpotent of class at most 3 and $G = A \times P$, where A is an abelian group and P is a p-group (p is a prime) and furthermore $\Gamma(P)$ is a regular graph.*

Lemma 1.3. *[5, Lemma 11] Let G be a non-abelian group. Then $xZ(G) \subseteq [x]_\sim$, for all $x \in G$. Also the equality happens if and only if $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$.*

Lemma 1.4. *[5, Lemma 12] Let G be a finite non-abelian group. Then the following are equivalent:*

1) *If $[x, y] = 1$, then $xZ(G) = yZ(G)$, where $x, y \in G \setminus Z(G)$.*
2) *G is a* CA-*group and $[x]_\sim = xZ(G)$, for all $x \in G$.*
3) *$[x, y] = 1$ and $[x, w] = 1$ imply that $yZ(G) = wZ(G)$, where $x, y, w \in G \setminus Z(G)$.*

Lemma 1.5. *[5, Theorem 3] Let G be a non-abelian group. The following are equivalent:*

1) *G is a* CA-*group and $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$.*
2) *$G = A \times P$, where A is an abelian group, P is a 2-group, P is a* CA-*group and $|\mathrm{Cent}(P)| = \frac{|P|}{|Z(P)|}$.*

3) *$G = A \times P$, where A is an abelian group and $C_P(x) = Z(P) \cup xZ(P)$, for all $x \in P \setminus Z(P)$.*

Lemma 1.6. *[5, Lemma 13] Let G be a non-abelian group. Let $[x]_\sim$ and $[y]_\sim$ be two different classes of $\sim$. If $[x_0, y_0] \neq 1$ for some $x_0 \in [x]_\sim$ and $y_0 \in [y]_\sim$, then $[u, v] \neq 1$ for all $u \in [x]_\sim$ and $v \in [y]_\sim$.*

Lemma 1.7. *[5, Lemma 20] Let G_1 and G_2 be two groups. Let $[g_1]_\sim = g_1Z(G_1)$, for all $g_1 \in G_1$ and $[g_2]_\sim = g_2Z(G_2)$, for all $g_2 \in G_2$. Then $[X]_\sim = XZ(G_1 \times G_2)$, for all $X \in G_1 \times G_2$.*

Lemma 1.8. *[6, Theorem 2.1] Let G be a non-abelian group and $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$. Then $\frac{G}{Z(G)}$ is an elementary abelian 2-group.*

Lemma 1.9. *[7, Corollary 2.3] Let G be a non-abelian nilpotent group. Then G is a CC-group if and only if $G \cong C_m \times Q_{2^n}$, where m and n are positive integers and m is odd.*

In Section 2 we will provide some results about the equivalency of relations.

2. Proof of the main theorems

In this section we prove the main theorems. For doing this we first prove some lemmas.

Lemma 2.1. *Let G be a CA-group. Then $C_G(x) = Z(G) \cup [x]_\sim$, for all $x \in G \setminus Z(G)$.*

Proof. Since $Z(G) \subseteq C_G(x)$ and $[x]_\sim \subseteq C_G(x)$ we have $Z(G) \cup [x]_\sim \subseteq C_G(x)$. Suppose $g \in C_G(x) \setminus Z(G)$. Then $[g, x] = 1$. By Lemma 1.1, $C_G(x) = C_G(g)$ which implies that $[x]_\sim = [g]_\sim$. Hence $g \in [x]_\sim$ and we have $C_G(x) \subseteq Z(G) \cup [x]_\sim$. Therefore $C_G(x) = Z(G) \cup [x]_\sim$, for all $x \in G \setminus Z(G)$. □

Lemma 2.2. *Let G be a non-abelian group. Then G is a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$ if and only if $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k(G))}$.*

Proof. Let G be a CA-group and $[x]_\sim = [x]_{\sim_1}$, for all $x \in G$. Let $xZ(G) \neq yZ(G)$ for some $x, y \in G \setminus Z(G)$. Since $XY \neq YX$ for all $X \in xZ(G)$ and $Y \in yZ(G)$, therefore there exists an edge between X and Y. Hence there are $|Z(G)|^2$ edges between elements of $xZ(G)$ and $yZ(G)$. Also there are $\frac{|G|}{|Z(G)|} - 1$ different classes of $xZ(G)$ for $x \in G \setminus Z(G)$. Thus $|E(\Gamma(G))| = \binom{\frac{|G|}{|Z(G)|}-1}{2}|Z(G)|^2$. Note that by [1, Lemma 3.27], $|E(\Gamma(G))| = \frac{|G|^2-k(G)|G|}{2}$. Hence $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$.

Conversely, suppose $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$. So $|G| = |Z(G)| + (k(G) - |Z(G)|)\frac{|G|}{2|Z(G)|}$. Since for all $x \in G \setminus Z(G)$, $|x^G| \leq \frac{|G|}{2|Z(G)}$ we have $|x^G| = \frac{|G|}{2|Z(G)|}$, for all $x \in G \setminus Z(G)$.

So $|C_G(x)| = 2|Z(G)|$, for all $x \in G \setminus Z(G)$. Now by [5, Lemma 15] G is a CA-group and $[x]_\sim = [x]_{\sim_1}$. $\square$

Example 2.1. Let G be a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$ and $|G| \leq 100$. Then G is one of the group with GAP ID in Table 2.1.

Table 2.1: The GAP ID of group G where $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$ and $|G| \leq 100$.

$[8,3]$	$[8,4]$						
$[16,3]$	$[16,4]$	$[16,6]$	$[16,11]$	$[16,12]$	$[16,13]$		
$[24,10]$	$[24,11]$						
$[32,2]$	$[32,4]$	$[32,5]$	$[32,12]$	$[32,17]$	$[32,22]$	$[32,23]$	$[32,24]$
$[32,25]$	$[32,26]$	$[32,37]$	$[32,38]$	$[32,46]$	$[32,47]$	$[32,48]$	
$[40,11]$	$[40,12]$						
$[48,21]$	$[48,22]$	$[48,24]$	$[48,45]$	$[48,46]$	$[48,47]$		
$[56,9]$	$[56,10]$						
$[64,3]$	$[64,17]$	$[64,27]$	$[64,29]$	$[64,44]$	$[64,51]$	$[64,56]$	$[64,57]$
$[64,58]$	$[64,59]$	$[64,73]$	$[64,74]$	$[64,75]$	$[64,76]$	$[64,77]$	$[64,78]$
$[64,79]$	$[64,80]$	$[64,81]$	$[64,82]$	$[64,84]$	$[64,85]$	$[64,86]$	$[64,87]$
$[64,103]$	$[64,112]$	$[64,115]$	$[64,126]$	$[64,184]$	$[64,185]$	$[64,193]$	$[64,194]$
$[64,195]$	$[64,196]$	$[64,197]$	$[64,198]$	$[64,247]$	$[64,248]$	$[64,261]$	$[64,262]$
$[64,263]$							
$[72,10]$	$[72,11]$	$[72,37]$	$[72,38]$				
$[80,21]$	$[80,22]$	$[80,24]$	$[80,46]$	$[80,47]$	$[80,48]$		
$[88,9]$	$[88,10]$						
$[96,45]$	$[96,47]$	$[96,48]$	$[96,52]$	$[96,54]$	$[96,55]$	$[96,60]$	$[96,162]$
$[96,163]$	$[96,165]$	$[96,166]$	$[96,167]$	$[96,221]$	$[96,222]$	$[96,223]$	

Theorem 2.1. *Let G be a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $2^{r-1} \leq |G'| \leq 2^{\binom{r}{2}}$, where $\frac{|G|}{|Z(G)|} = 2^r, 2 \leq r$.*

Proof. Let G be a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$. First we show that $|G'| \leq 2^{\binom{r}{2}}$. Since $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemmas 1.8 and 1.3, we find that $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G), g^2 \in Z(G)$, for all $g \in G$ and G' is an elementary abelian 2-group. Since G is a non-abelian group, there exist $x, y \in G$ such that $[x, y] = z \neq 1$ and $[x, xy] \neq 1$ and $[y, xy] \neq 1$. By Lemma 1.4, $xZ(G) \neq yZ(G)$, $xZ(G) \neq xyZ(G)$ and $yZ(G) \neq xyZ(G)$. Let $H_1 = Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G)$. Since $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $H_1 \leq G$. By Lemma 1.6, none of the elements of $xZ(G)$ are commute with elements of $yZ(G)$ and $xyZ(G)$. Also none of the elements of $yZ(G)$ are commute with elements of $xyZ(G)$. Therefore $Z(H_1) = Z(G)$. Since $G' \leq Z(G)$ and $t^2 = 1$, for all $t \in G^{'}$, we have the following:

$$[x, y]^{-1} = [y, x] = [x, y] = [x, xy] = [y, yx] = z, [eu, fw] = [e, f],$$

for all $e, f \in \{x, y, xy\}$ and for all $u, w \in Z(G)$. Hence

$$\begin{aligned} H_1' &= \langle [g_1, h_1] | g_1, h_1 \in H_1 \rangle = \langle [eu, fw] | e, f \in \{x, y, xy\}, u, w \in Z(G) \rangle \\ &= \langle [e, f] | e, f \in \{x, y, xy\} \rangle = \langle [x, y] \rangle = \langle z \rangle = \{1, z\}. \end{aligned}$$

Thus $|H_1'| = 2 \leq 2^{\binom{2}{2}}$ and $\frac{|H_1|}{|Z(H_1)|} = \frac{4|Z(G)|}{|Z(G)|} = 2^2$. If $G = H_1$ then proof is complete, so assume that $G \neq H_1$. Hence there exists $a \in G \setminus H_1$. Let $H_2 = H_1\langle a \rangle$. Since $a^2 \in Z(G)$ we have

$$\begin{aligned} H_2 = \quad & H_1\langle a \rangle = H_1 \cup aH_1 = Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G) \\ & \cup aZ(G) \cup axZ(G) \cup ayZ(G) \cup axyZ(G) \end{aligned}$$

and since $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $H_2 \leq G$. By Lemma 1.6 $Z(H_2) = Z(G)$. Let $[a, x] = t_1, [a, y] = t_2$. Therefore $1 \neq [a, xy] = [a, x][a, y] = t_1t_2$. In above we had $[x, y] = [x, xy] = [y, xy] = z$. On the other hand $[e_1u, f_1w] = [e_1, f_1]$, for all $u, w \in Z(G)$ and for all $e_1, f_1 \in \{x, y, xy, a, ax, ay, axy\}$. Also $[g_2, h_2k_2] = [g_2, h_2][g_2, k_2]$, for all $g_2, h_2, k_2 \in H_2$. Hence

$$\begin{aligned} H_2' = \quad & \langle [g_2, h_2] | g_2, h_2 \in H_2 \rangle = \langle [e_1u, f_1w] | e_1, f_1 \in \{x, y, xy, a, ax, ay, axy\} \rangle \\ & = \langle [x, y], [a, x], [a, y] \rangle = \langle z, t_1, t_2 \rangle. \end{aligned}$$

Therefore $|H_2'| \leq 2^{\binom{3}{2}}$ and $\frac{|H_2|}{|Z(H_2)|} = \frac{8|Z(G)|}{|Z(G)|} = 2^3$. If $G = H_2$, then the proof is complete. Let $G \neq H_2$. Therefore there exists $b \in G \setminus H_2$. Let $H_3 = H_2\langle b \rangle$. Let $[b, x] = l_1, [b, y] = l_2, [b, a] = l_3$. By a Similar calculation we have, $Z(H_3) = Z(G)$ and $H_3' = \langle z, t_1, t_2, l_1, l_2, l_3 \rangle$. Hence $|H_3'| \leq 2^6 = 2^{\binom{4}{2}}$ and $\frac{|H_3|}{|Z(H_3)|} = \frac{16|Z(G)|}{|Z(G)|} = 2^4$. By continuing this process, we have the following subgroups: $Z(G) \leq H_1 \leq H_2 \leq ... \leq H_i \leq ... \leq G$, such that $Z(H_i) = Z(G)$, $|H_i'| \leq 2^{\binom{i+1}{2}}$, $\frac{|H_i|}{|Z(H_i)|} = 2^{i+1}$. Since G is finite, there exists $2 \leq r$, such that $G = H_{r-1}, |G'| \leq 2^{\binom{r}{2}}$ and $\frac{|G|}{|Z(G)|} = \frac{|H_{r-1}|}{|Z(H_{r-1})|} = 2^r$. Since $[w]_\sim = wZ(G)$, for all $w \in G \setminus Z(G)$, so by Lemma 2.1, $|w^G| = \frac{|G|}{|C_G(w)|} = \frac{|G|}{2|Z(G)|}$, for all $w \in G \setminus Z(G)$. Consequently, as $w^G \subseteq wG'$, we have $\frac{|G|}{2|Z(G)|} = 2^{r-1} \leq |G'|$. □

Theorem 2.2. *Let G be a non-abelian* CC*-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $G \cong C_m \times Q_8$ where C_m is a cyclic group of odd order m.*

Proof. Let G be a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Therefore G is a CA-group. By lemma 1.3, $|\text{Cent}(G)| = \frac{|G|}{|Z(G)|}$ and by lemma 1.5, $G \cong A \times P$ where A is an abelian group and P is a 2-group. Hence G is a nilpotent group. By lemma

1.9, $G \cong C_m \times Q_{2^n}$ where C_m is a cyclic group of order odd m. Since $[x]_\sim = xZ(G)$ for all $x \in G$, we have by lemma 1.3, that $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$ and by Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group which implies that $G' \leq Z(G)$. Hence $(C_m \times Q_{2^n})' \subseteq Z(C_m \times Q_{2^n})$ and $1 \times Q'_{2^n} \subseteq C_m \times Z(Q_{2^n}) \cong C_m \times C_2$. Therefore $Q'_{2^n} \cong C_2$ and $|Q'_{2^n}| = 2$. Since $|Q'_{2^n}| = 2^{n-2}$, we have $n = 3$ and $G \cong C_m \times Q_8$.

Conversely Q_8 is a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Therefore $C_m \times Q_8$ is also a CC-group and by Lemma 1.7, $[x]_\sim = xZ(G)$ for all $x \in G \cong C_m \times Q_8$. □

Proposition 2.1. *Let G be a non-abelian group and $G' \leq Z(G)$. Then if $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ then $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$.*

Proof. Let $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Since $G' \leq Z(G)$, so $xG' \leq xZ(G)$. By Lemma 1.3, $xZ(G) \subseteq [x]_\sim$, for all $x \in G$. Hence $xZ(G) \subseteq [x]_\sim = x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. This implies that $[x]_\sim = x^G = xG' = xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|xG'| = |xZ(G)|$ we have $G' = Z(G)$ and the proof is complete. □

Example 2.2. Let G be an extra especial group of order 32. Then $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$.

Theorem 2.3. *Let G be a* CA*-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Then G is a 2-group, $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$.*

Proof. Since G is a CA-group, by Lemma 2.1, $C_G(x) = [x]_\sim \cup Z(G)$, for all $x \in G \setminus Z(G)$. Therefore $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{|Z(G)|+|[x]_\sim|} = \frac{|G|}{|Z(G)|+|x^G|}$ which implies that $|x^G|^2 + |Z(G)||x^G| - |G| = 0$. So $|x^G|$ is a constant and $\Gamma(G)$ is a regular graph. By Lemma 1.2, $G = A \times P$ where A is an abelian group and P is a p-group (p is a prime) and by Lemma 1.3, $xZ(G) \subseteq [x]_\sim$, for all $x \in G \setminus Z(G)$. Therefore $xZ(G) \subseteq [x]_\sim = x^G \subseteq xG'$ which implies that $xZ(G) \subseteq xG'$. Thus $Z(G) \leq G'$ and $Z(G) = A \times Z(P) \leq G' = 1 \times P'$. Hence $A \cong 1$ and $Z(P) \leq P'$. So G is a p-group and $G \cong P$ and there exist positive integers m, n, t so that $|P| = p^n, |Z(P)| = p^t, |x^P| = p^m$ and $p^m = \frac{p^n}{(p^t+p^m)}$. This implies that $p^{2m} + p^{t+m} = p^n$ and $p^{m-t} + 1 = p^{n-m-t}$. Since p is a prime, by discussing the different states of the prime numbers, we obtain $p = 2$ and $m = t$. Since $xZ(P) \subseteq [x]_\sim = x^P$ and $|x^P| = |Z(P)|$, so $[x]_\sim = x^P = xZ(P)$, for all $x \in P \setminus Z(P)$. By Lemma 1.3, $|\mathrm{Cent}(P)| = \frac{|P|}{|Z(P)|}$. This implies by Lemma 1.8, that $\frac{P}{Z(P)}$ is an elementary abelian 2-group and $P' \leq Z(P)$. Hence $Z(P) = P'$. □

Corollary 2.1. *Let G be a* CC*-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Then $G \cong Q_8$.*

Proof. By Theorem 2.3, $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$. and by Theorem 2.2, $G \cong C_m \times Q_8$ where m is an odd positive integer. Since $G' = Z(G)$, so $1 \times Q_8' \cong C_m \times Z(Q_8)$. Therefore $C_m \cong 1$. Hence $G \cong Q_8$. $\square$

Lemma 2.3. *A group G is a* CA*-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ if and only if $|G| = 2|Z(G)|^2$ and $k(G) = 3|Z(G)| - 1$.*

Proof. Let G be a CA-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. By Theorem 2.3, $[x]_\sim = xZ(G)$ for all $x \in G \setminus Z(G)$ and by Lemma 2.1, $C_G(x) = [x]_\sim \cup Z(G)$, for all $x \in G \setminus Z(G)$. Hence $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{|Z(G)|+|[x]_\sim|} = \frac{|G|}{2|Z(G)|}$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |xZ(G)|$, for all $x \in G \setminus Z(G)$ we have $|Z(G)| = \frac{|G|}{2|Z(G)|}$ which implies that

$$|G| = 2|Z(G)|^2. \tag{2.1}$$

Since $[x]_\sim = xZ(G)$, for all $x \in G \setminus Z(G)$, by Lemma 2.2,

$$|G| = \frac{2|Z(G)|^2}{(3|Z(G)| - k(G))}. \tag{2.2}$$

From Equations 2.1 and 2.2 we have $k(G) = 3|Z(G)| - 1$.

Conversely suppose $|G| = 2|Z(G)|^2$ and $k(G) = 3|Z(G)| - 1$. This implies that $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k(G))}$ and by Lemma 2.2, G is a CA-group and $[x]_\sim = xZ(G)$ for all $x \in G \setminus Z(G)$. Also by Lemma 2.1, $|C_G(x)| = 2|Z(G)|$. This implies that $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{2|Z(G)|^2}{2|Z(G)|} = |Z(G)|$. Since $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemma 1.3, $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$. Hence by Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G)$ and $x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |Z(G)|$, for all $x \in G \setminus Z(G)$, we have $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Hence we conclude that $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. $\square$

Lemma 2.4. *Let G be a* CA*-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ if and only if $|G| = 2|Z(G)|^2$.*

Proof. Let G be a CA-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. By Lemma 2.3, $|G| = 2|Z(G)|^2$. Conversely let $|G| = 2|Z(G)|^2$. Since G is a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemma 2.1, $C_G(x) = Z(G) \cup [x]_\sim = Z(G) \cup xZ(G)$, for all $x \in G \setminus Z(G)$. Therefore $|C_G(x)| = 2|Z(G)|$, for all $x \in G \setminus Z(G)$. This implies that $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{2|Z(G)|} = \frac{2|Z(G)|^2}{2|Z(G)|} = |Z(G)|$, for all $x \in G \setminus Z(G)$. Since $[x]_\sim = xZ(G)$, for all $x \in G \setminus Z(G)$, by Lemma 1.3 and Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G)$. Hence $x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |Z(G)| = |xZ(G)$, we have $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and finally $[x]_\sim = x^G = xZ(G)$ for all $x \in G \setminus Z(G)$. $\square$

Example 2.3. Let G be a non-abelian CA-group and assume that $[x]_\sim = x^G$ for all $x \in G \setminus Z(G)$ and $|G| \leq 100$. Then $G \cong Q_8$ or D_8.

Lemma 2.5. *Let G be a non-abelian group. Then $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ if and only if $G' = Z(G)$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$.*

Proof. Let $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Since $x^G \subseteq xG'$, so $Z(G) \leq G'$. Now we show that $G' \leq Z(G)$. Let $1 \neq t \in G'$. Then there exist $x, y \in G$ so that $[x, y] = t$. Hence $t = y^{-1}x^{-1}yx = y^{-1}y^x$. Since $y^G = yZ(G)$, there exists $z \in Z(G)$ such that $y^x = yz$. Therefore $t = y^{-1}y^x = y^{-1}yz = z$. This implies that $t \in Z(G)$. Thus $G' \leq Z(G)$ and we have $G' = Z(G)$. Moreover $|G| = |Z(G)| + (k(G) - |Z(G)|)|x^G|$ because $|x^G| = |xZ(G)|$ for all $x \in G \setminus Z(G)$. Hence $\frac{|G|}{|Z(G)|} = k(G) - |Z(G)| + 1$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$.

Conversely, suppose $G' = Z(G)$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$. Then $x^G \subseteq xG' = xZ(G)$, for all $x \in G \setminus Z(G)$. Hence $|x^G| \leq |xZ(G)|$, for all $x \in G \setminus Z(G)$. Since $k(G) - |Z(G)| = \frac{|G|}{|Z(G)|} - 1$ we have $|x^G| = |xZ(G)|$, for all $x \in G \setminus Z(G)$. Therefore $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. □

Lemma 2.6. *Let G be a non-abelian group and $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Then G is a p-group where p is a prime.*

Proof. Since $|x^G| = |Z(G)|$, for all $x \in G \setminus Z(G)$, so $\Gamma(G)$ is a regular graph. By Lemma 1.2, $G \cong A \times P$ where A is an abelian group and P is a p-group (p is a prime). By Lemma 2.5, $G' = Z(G)$ which implies that $A \cong 1$ and G is a p-group. □

Theorem 2.4. *Let G be a* CC*-group and $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Then $G \cong Q_8$.*

Proof. By Lemma 2.6, G is a p-group. So G is a nilpotent group. By Lemma 1.9, $G \cong C_m \times Q_{2^n}$ where n is positive integer and m is an odd positive integer. By Lemma 2.5, $G' = Z(G)$, so $1 \times Q'_{2^n} \cong C_m \times C_2$. Hence $Q'_{2^n} \cong C_2$ and $|Q'_{2^n}| = 2$. Since $|Q'_{2^n}| = 2^{n-2}$ we have $n = 3$. Hence $G \cong Q_8$ and the proof is complete. □

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REFERENCES

1. A. Abdollahi, S. Akbari and H. R. Maimani: *Non commuting graph of group* J. Algebra. **28** (2006), 468–492.
2. A. Abdollahi, S. M. Jafarian Amiri and A. M. Hassanabadi: *Groups with specific number of centralizers* Houston J. Math., **33(1)** (2007), 43–57.
3. A. Ashrafi: *On finite groups with a given number of centralizers* Algebra Colloq. **7(2)** (2000), 139–146.

4. S. M. Belcastro and G. J. Sherman: *Counting centralizers in finite groups* Math. Mag. **5** (1994), 111–114.

5. M. A. Iranmanesh and M. H. Zareian: *On n-centralizer* CA-*groups* submitted.

6. S. M. Jafarian Amiri, H. Madadi and H. Rostami: *Finite groups with certain number of centralizers* Third Biennial International Group Theory Conference., (2015).

7. S. M. Jafarian Amiri and H. Rostami: *Finite groups all of whose proper centralizers are cyclic* B. Iran. Math. Soc. **43(3)** (2017), 755-762.

8. K. Parattu and A. Wingerter: *Tribimaximal mixing from small groups, additonal material* Phys. Rev. D, **84(1)** 013011.