FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 1 (2022), 51–66
https://doi.org/10.22190/FUMI210105006M
Original Scientific Paper

A NEW NUMERICAL METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS IN THE SENSE OF CAPUTO-FABRIZIO DERIVATIVE

Leila Moghadam Dizaj Herik[1], Mohammad Javidi[1,2] and Mahmoud Shafiee[1]

[1] **Department of Mathematics, Rasht Branch**
Islamic Azad University, Rasht, Iran
[2] **Factulty of Mathematical Sciences, University of Tabriz, Tabriz, Iran**

Abstract. In this paper, fractional differential equations in the sense of Caputo-Fabrizio derivative are transformed into integral equations. Then a high order numerical method for the integral equation is investigated by approximating the integrand with a piecewise quadratic interpolant. The scheme is capable of handling both linear and nonlinear fractional differential equations. A detailed error analysis and stability region of the numerical scheme is rigorously established.
Key words: Fractional dierential equation, Caputo-Fabrizio fractional derivative, interpolation, non-singular kernel.

1. Introduction

The definition of the fractional derivative with a smooth kernel takes on two different representations for the temporal and spatial variables. The first works on the time variables; thus it is suitable to use the Laplace transform. The second definition is related to the spatial variables, by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform. The interest for this new approach with a regular kernel was born from the prospect that there is a

Received January 5, 2021, accepted: February 14, 2022
Communicated by Hari Mohan Srivastava
Corresponding Author: Mohammad Javidi, Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran | Factulty of Mathematical Sciences, University of Tabriz, Tabriz, Iran | E-mail: Mo_javidi@tabrizu.ac.ir
2010 *Mathematics Subject Classification.* Primary 34A08; Secondary 65D25

class of non-local systems, which can describe the material heterogeneities and the fluctuations of different scales, which cannot be well described by classical local theories or by fractional models with singular kernel [3].

Li et al in [14], applied the Simpson's rule instead of the trapezoidal quadrature formula to achieve higher order numerical algorithm for fractional differential equations. The authors of [6], established using the quadratic interpolation approximation using three points $(t_{j-2}, f(t_{j-2})), (t_{j-1}, f(t_{j-1}))$ and $(t_j, f(t_j))$ for the integrand $f(t)$ on each small interval $[t_{j-1}, t_j](j \geq 0)$, while the linear interpolation approximation is applied on the first small interval. In [24], The authors proposed a new fractional derivative without a singular kernel. Losada and Nieto [16], introduced the fractional integral corresponding to the new concept of fractional derivative recently presented by Caputo and Fabrizio and we study some related fractional differential equations. Mohammed and Kamal [1], considered classes of linear and nonlinear fractional differential equations involving the Caputo - Fabrizio fractional derivative of the non-singular kernel. They transformed the fractional problems to equivalent initial value problems with integer derivatives. Then illustrate the obtained results by presenting two mathematical models of fractional differential equations and their equivalent initial value problems. In [8], to bring a broader outlook on some unusual irregularities observed in wave motions and liquids movements, they explored the possibility of extending the analysis of Korteweg deVries Burgers equation with two perturbations levels to the concepts of fractional differentiation with no singularity. They made use of the newly developed Caputo - Fabrizio fractional derivative with no singular kernel to establish the model.

The authors of [2], proposed the idea of Caputo-Fabrizio time-fractional derivatives to magneto hydro dynamics (MHD) free convection flow of generalized Walters'-B fluid over a static vertical plate. Free convection is caused due to combined gradients of temperature and concentration. Hence, heat and mass transfers are considered together. The fractional model of Walters'-B fluid is used in the mathematical formulation of the problem. Garrappa and Roberto [7], described different approaches to generalize the trapezoidal method to fractional differential equations. A new definition for the fractional-order operator called the Caputo-Fabrizio (CF) fractional derivative operator without singular kernel has been numerically approximated using the two-point finite forward difference formula for the classical first-order derivative of the function $f(t)$ appearing inside the integral sign of the definition of the CF operator [18].

The monograph provides the most recent and up-to-date developments on fractional differential and fractional integro-differential equations involving many different potentially useful operators of fractional calculus. The subject of fractional calculus and its applications (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread [12].

In [19], the main purpose of this work is to study the dynamics of a fractional-order Covid-19 model. An efficient computational method, which is based on

the discretization of the domain and memory principle, is proposed to solve this fractional-order corona model numerically and the stability of the proposed method is also discussed. The efficiency of the proposed method is shown by listing the CPU time. It is shown that this method will work also for long-time behavior.
The authors of [13], studied two fractional models in the Caputo-Fabrizio sense and Atangana-Baleanu sense, in which the effects of malaria infection on mosquito biting behavior and attractiveness of humans are considered. Using Lyapunov theory, we prove the global asymptotic stability of the unique endemic equilibrium of the integer-order model, and the fractional models, whenever the basic reproduction number R_0 is greater than one.
In [5], a fractional-order mathematical model of the Caputo-Fabrizio type is presented for an alcoholism model. The existence and uniqueness of the alcoholism model were investigated by using a fixed-point theorem.

The authors of [9], considered a new fractional derivative with nonsingular kernel introduced by Caputo-Fabrizio (CF) and propose a finite difference method for computing the CF fractional derivatives.
In [17], recently a new fractional differentiation was introduced to get rid of the singularity in the Riemann-Liouville and Liouville-Caputo fractional derivative. The new fractional derivative has then generated a new class of ordinary differential equations. These class of fractional ordinary differential equations cannot be solved using conventional Adams-Bashforth numerical scheme, thus, in this paper, a new three-step fractional Adams-Bashforth scheme with the Caputo-Fabrizio derivative is formulated for the solution linear and nonlinear fractional differential equations.

Recently [15], introduced Caputo and Fabrizio operator, which this new operator was derived by replacing the singular kernel in the classical Liouville-Caputo derivative with the regular kernel. We introduce some useful properties based on the definition by Caputo and Fabrizio for a general order $n < \alpha < n+1, n \in N$.

In [23], new results related to the Marichev-Saigo-Maeda a fractional integral and fractional derivative operators are proposed. Izadi and Srivastava [11], proposed a numerical approximation to the nonlinear fractional-order logistic population model with fractional-order Bessel and Legendre bases. The primary focus of [10], is to propose a computationally effective approximation algorithm to find the numerical solution of the so-called a new design of second-order Lane–Emden pantograph delayed problem with singularity and nonlinearity.
In [25], the authors proposed a new numerical technique based on a certain two-dimensional extended differential transform via local fractional derivatives and derive its associated basic theorems and properties.

In [21], a potentially useful new method based on the Gegenbauer wavelet expansion, together with operational matrices of fractional integral and block-pulse functions, is proposed in order to solve the Bagley–Torvik equation.
The authors of [20], introduced a numerical algorithm for the solution of the fractional vibration equation (FVE).

In [4], the Bernoulli wavelet method for the numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order. The main object of this survey-cum-expository article is to present a brief elementary and introductory overview of the theory of the integral and derivative operators of fractional calculus and their applications especially in developing solutions of certain interesting families of ordinary and partial fractional "differintegral" equations [22].

The paper is organized as follows. Section 2 contains some basic definitions. Derivation of the new numerical method for solving fractional differential equations is presented in Section3. Truncation error analysis is presented in Section 4, which contains the new an analysis of linear stability and stability regions an analysis method is presented in Section 5. Finally, in Section 6 the results of some numerical tests are presented to compare the methods under investigation and some concluding remarks is presented at the end of the paper.

2. Mathematical Preliminaries

In this section, we mainly recall some definitions which will be used later.

Definition 2.1. [12] The fractional derivative of order $\alpha > 0$ for $y(t)$ in the classical Liouville-Caputo sense is defined as

$$ {}_{0}^{LC}D_t^{\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} y^{(m)}(x) dx, \quad t > 0, \tag{2.1} $$

where $m-1 < \alpha \leq m$, $m \in N$, $t > 0$, $y \in C^{m-1}[0,t]$ and $k(t) = t^{m-\alpha-1}$ is the singular kernel.

To get rid of this singular kernel, a new definition was introduced in [3] that facilitates solving various natural and physical laws without being caught by the convoluted integrals.

Definition 2.2. [3]. The new operator called the Caputo-Fabrizio operator for fractional derivatives of order $\alpha > 0$ is defined as follows:

$$ {}_{0}^{CF}D_t^{\alpha} y(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t y^{'}(x) \exp\Big(-\alpha \frac{t-x}{1-\alpha}\Big) dx, \quad t > 0, \tag{2.2} $$

where $M(\alpha)$ is the normalization function (any smooth positive function) such that $M(0) = M(1) = 1$, and $b > a$. Furthermore, note the absence of any singular kernel in the definition above. Also Losada and Nieto suggested the new fractional Caputo-Fabrizio derivative operator [16]

$$ {}_{0}^{CF}D_t^{\alpha} y(t) = \frac{1}{1-\alpha} \int_0^t y^{'}(x) \exp\big(\frac{-\alpha(t-x)}{1-\alpha}\big) dx. \tag{2.3} $$

Formula (2.3) forms most of the work presented in the subsequent sections.

3. Derivation of the new numerical method for solving fractional differential equations

In this section, we consider and investigate the numerical solution for the following initial value problem:

$$\begin{cases} {}_0^{CF}D^{\alpha_t}y(t)\mid_{t=t_k}=f(t,y(t))=u(t), \quad 0\le t\le T, \quad 0<\alpha\le 1, \\ y(t_0)=y_0. \end{cases} \tag{3.1}$$

Throughout the forthcoming analysis, it is assumed that $f(t,y(t))$ is a continuous function that satisfies a Lipschitz condition with respect to the second argument, that is, $|f(t,y)-f(t,x)|\le L|y-x|$, which $L>0$. Notice that continuity and Lipschitz conditions are sufficient to ensure the existence of a unique solution to the problem (3.1) on the interval $[0,T]$ [14].
From the definition of the Caputo-Fabrizio fractional derivative (2.3), for any $\alpha(0<\alpha<1)$, we have

$$\begin{aligned} {}_0^{CF}D_t^{\alpha}y(t) &= \tfrac{1}{1-\alpha}\int_0^t y^{'}(x)\exp\left(\tfrac{-\alpha(t-x)}{1-\alpha}\right)dx \\ \tfrac{1}{1-\alpha}\int_0^t y^{'}(x)\exp\left(\tfrac{-\alpha(t-x)}{1-\alpha}\right)dx &= u(t) \end{aligned}$$

therefor

$$\int_0^t y^{'}(x)\exp(-\mu x)dx=\frac{-\alpha}{\mu}u(t)\exp(-\mu t).$$

Where $\mu=\frac{-\alpha}{1-\alpha}$ By derivating from both sides we have

$$y^{'}(t)\exp(-\mu t)=\tfrac{-\alpha}{\mu}[-\mu\exp(-\mu t)u(t)+\exp(-\mu t)u^{'}(t)]$$

and by eliminating $\exp(-\mu t)$ from both sides we gain

$$\begin{gathered} y^{'}(t)=\tfrac{-\alpha}{\mu}[u^{'}(t)-\mu u(t)], \\ \int_0^t y^{'}(x)dx=\tfrac{-\alpha}{\mu}\int_0^t[u^{'}(x)-\mu u(x)]dx, \\ y(t)-y(0)=\tfrac{-\alpha}{\mu}[u(t)-u(0)]+\alpha\int_0^t u(x)dx, \\ y(t)=y(0)-(\alpha-1)[f(t,y(t))-f(0,y(0))]+\alpha\int_0^t f(x,y(x))dx. \end{gathered}$$

For $t=t_k$, one obtains

$$\begin{aligned} y(t_k)&=y(0)-(\alpha-1)[f(t_k,y(t_k))-f(0,y(0))] \\ &+\alpha\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}f(x,y(x))dx. \end{aligned} \tag{3.2}$$

The main problem is to solve the integral on the right-hand side of (3.2) by numerical method. To construct the high order scheme, For solving Eq.(3.2) on $[0,T]$ the interval is divided into N subintervals. Let $\Delta t = T/N$, $t_j = j\Delta t, j = 0, 1, \ldots, N$. For notational convenience, $F(t_j) = f(t_j, y(t_j))$ and $F_j = f(t_j, y_j)$, where y_j is the numerical approximation to $y(t_j)$. On each small interval $[t_{j-1}, t_j](1 \leq j \leq k)$, the piecewise Lagrange interpolation polynomial of degree one will be used to approximate of $F(t)$ as $\gamma_{1,j}F(t)$,

$$\gamma_{1,j}F(t) = F(t_{j-1})\frac{t_j - t}{\Delta t} + F(t_j)\frac{t - t_{j-1}}{\Delta t}.$$

For $j \geq 2$, we make a quadratic interpolation function $\gamma_{2,j}F(t)$ of $F(t)$ using three points $(t_{j-1}, F(t_{j-1})), (t_{j-2}, F(t_{j-2}))$ and $(t_j, F(t_j))$ and obtaining a constraint of the result onto small interval $[t_j, F(t_j)]$, we get

$$\begin{aligned}\gamma_{2,j}F(t) = F(t_{j-2})&\frac{(t - t_{j-1})(t - t_j)}{2\Delta t^2} \\ &+ F(t_{j-1})\frac{(t - t_{j-2})(t_j - t)}{\Delta t^2} + F(t_j)\frac{(t - t_{j-1})(t - t_{j-2})}{2\Delta t^2} \\ &= \gamma_{1,j}F(t) + \frac{1}{2}(\delta_t^2 F_{j-1})(t - t_{j-1})(t - t_j), \quad t \in [t_{j-1}, t_j].\end{aligned}$$

In(3.2), we use $\gamma_{1,1}F(t)$ to approximate $F(t)$ on the first interval $[t_0, t_1]$ and $\gamma_{2,j}F(t)$ to approximate $F(t)$ on the first interval $[t_{j-1}, t_j](j \geq 2)$. We have

$$\begin{aligned}\int_{t_0}^{t_1} \gamma_{1,1}F(x)dx &= \int_{t_0}^{t_1} \left(F(t_0)\frac{t_1 - x}{\Delta t} + F(t_1)\frac{x - t_0}{\Delta t}\right)dx \\ &= \frac{1}{2}\Delta t(F(t_0) + F(t_1))\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \gamma_{2,j}F(x)dx &= \int_{t_{j-1}}^{t_j} (\gamma_{1,j}F(x) + \frac{1}{2}\delta_t^2 F_{j-1}(x - t_{j-1})(x - t_j))dx \\ &= \frac{1}{2}\Delta t(F(t_{j-1}) + F(t_j)) \\ &- \frac{1}{12}\Delta t^3 \delta_t^2 F_{j-1}.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we can obtain a new numerical method for solving the Caputo-Fabrizio fractional differential equations of order $\alpha(0 < \alpha < 1)$ in the following formula to calculate $y(t_k)$:

$$y(t_k) = y_0 - (\alpha-1)(F_k - F_0) + \alpha \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} F(x)dx$$

$$\approx y_0 - (\alpha-1)(F_k - F_0) + \alpha\Big[\int_{t_0}^{t_1} \gamma_{1,1} F(x)dx + \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \Pi_{2,j} F(x)dx \Big]$$

$$= y_0 - (\alpha-1)(F_k - F_0) + \alpha\left[\frac{1}{2}\Delta t(F_0 + F_1) + \sum_{j=2}^{k}(\frac{1}{2}\Delta t(F_{j-1} + F_j) - \frac{1}{12}\Delta t^3 \delta_t^2 F_{j-1})\right]$$

$$= y_0 - (\alpha-1)(F_k - F_0) + \alpha\left[\frac{1}{2}\Delta t \sum_{j=1}^{k}(F_{j-1} + F_j) - \frac{1}{12}\Delta t^3 \sum_{j=2}^{k} \delta_t^2 F_{j-1}\right]$$

$$= y_0 - (\alpha-1)(F_k - F_0) + \alpha\left[\frac{1}{2}\Delta t \sum_{j=1}^{k}(F_{j-1} + F_j) - \frac{1}{12}\Delta t^2 \sum_{j=2}^{k}(\delta_t F_{j-\frac{1}{2}} - \delta_t F_{j-\frac{3}{2}})\right]$$

$$= y_0 - (\alpha-1)(F_k - F_0) + \alpha\Delta t\left[\frac{1}{2} \sum_{j=1}^{k}(F_{j-1} + F_j) - \frac{1}{12} \sum_{j=2}^{k}(F_{j-2} - 2F_{j-1} + F_j)\right]$$

then

$$\begin{aligned} y(t_k) &= y_0 + ((\alpha-1) + \frac{5}{12}\alpha\Delta t)F_0 \\ &+ (\frac{5}{12}\alpha\Delta t - (\alpha-1))F_k \\ &+ \left[\frac{13}{12}F_1 + \sum_{j=2}^{k-2} F_j + \frac{13}{12}F_{k-1}\right]\alpha\Delta t \end{aligned} \tag{3.5}$$

Finally, the new numerical method for solving Eq (3.1), is completely described by (3.5).

4. Truncation error analysis

Now, truncation errors of the new formula (3.5) are explained in the following theorem.

Theorem 4.1. *Assume that $F(t) \in C^3[0, t_k]$. and for any $\alpha (0 < \alpha < 1)$, $y(t_k)$ is defined by (3.5). Define $\hat{R}(y(t_k)) := y(t_k) - \tilde{y}(t_k)$. Then we have*

$$|\hat{R}(y(t_1))| \le \frac{\alpha}{12} \max_{t_0 \le t \le t_1} |F''(t)| \Delta t^3 \tag{4.1}$$

and

$$\begin{aligned} |\hat{R}(y(t_k))| &\le \frac{\alpha}{12} \max_{t_0 \le t \le t_1} |F''(t)| \Delta t^3 \\ &+ \frac{\alpha}{3} \max_{t_0 \le t \le t_k} |F'''(t)| (t_k - t_1) \Delta t^3 \end{aligned} \tag{4.2}$$

Proof. According to [7, Theorem 2.1.4.1]

$$F(t)-\gamma_{1,j}F(t)=\frac{F^{''}(\zeta_j)}{2}(t-t_{j-1})(t-t_j),\quad t\in[t_{j-1},t_j],\zeta_j\in(t_{j-1},t_j),\quad 1\le j\le k. \tag{4.3}$$

and for $t\in[t_{j-1},t_j]$

$$F(t)-\gamma_{2,j}F(t)=\frac{F^{'''}(\eta_j)}{6}(t-t_{j-2})(t-t_{j-1})(t-t_j), \tag{4.4}$$

where $\eta_j\in(t_{j-2},t_j)$, $2\le j\le k$.
From (4.3), we have

$$\begin{aligned}\hat{R}(y(t_1))&=\alpha\int_{t_0}^{t_1}(F(x)-\gamma_{1,1}F(x))dx\\&=\alpha\int_{t_0}^{t_1}\frac{F^{''}(\zeta_1)}{2}(x-t_0)(x-t_1)dx\\&=\frac{\alpha}{2}F^{''}(\eta_1)\int_{t_0}^{t_1}(x-t_0)(x-t_1)dx\\&=\frac{-\alpha}{12}F^{''}(\eta_1)\Delta t^3\end{aligned}$$

where $\zeta_1\in(t_0,t_1)$. Hence, (4.1) holds.
For $k\ge 2$, from (3.2), we get

$$\begin{aligned}\hat{R}(y(t_k))=\alpha\Big(&\int_{t_0}^{t_1}[F(x)-\gamma_{1,1}F(x)]dx\\&+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}[F(x)-\gamma_{2,j}F(x)]dx\Big)\end{aligned} \tag{4.5}$$

From (4.3), it follows

$$\begin{aligned}\Big|\int_{t_0}^{t_1}[F(x)-\gamma_{1,1}F(x)]dx\Big|&=\Big|\int_{t_0}^{t_1}\frac{F''(\zeta_1)}{2}(x-t_0)(x-t_1)dx\Big|\\&=\Big|\frac{F''(\eta_1)}{2}\int_{t_0}^{t_1}(x-t_0)(t_1-x)dx\Big|\\&\le\frac{1}{12}|F''(\eta_1)|\Delta t^3,\end{aligned} \tag{4.6}$$

where $\zeta_1 \in (t_0, t_1)$. From (4.4), we know

$$\begin{aligned}
\Big|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}[F(x)-\Pi_{2,j}F(x)]dx\Big| &= \Big|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}\frac{F'''(\eta_j)}{6}(x-t_{j-2})(x-t_{j-1})(x-t_j)dx\Big| \\
&= \frac{1}{6}\Big|\sum_{j=2}^{k-1}F^{'''}(\vartheta_j)\int_{t_{j-1}}^{t_j}(x-t_{j-2})(x-t_{j-1})(x-t_j)dx\Big| \\
&= \frac{1}{6}|F'''(\vartheta)|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}(x-t_{j-2})(x-t_{j-1})(t_j-x)dx \\
&\le \frac{1}{3}|F'''(\vartheta)|\Delta t^3\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}dx \\
&\le \frac{1}{3}|F'''(\vartheta)|\Delta t^3\sum_{j=2}^{k-1}(t_j-t_{j-1}) \\
&\le \frac{1}{3}|F'''(\vartheta)|(t_{k-1}-t_1)\Delta t^3
\end{aligned}$$

then

$$\Big|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}[F(x)-\Pi_{2,j}F(x)]dx\Big| \le \frac{1}{3}|F'''(\vartheta)|(t_{k-1}-t_1)\Delta t^3 \tag{4.7}$$

where $\vartheta_j \in (t_{j-2}, t_j)$, $2 \le j \le k-1$, $\vartheta \in (t_0, t_{k-1})$. In addition, For $j = k$

$$\begin{aligned}
\int_{t_{k-1}}^{t_k}[F(x)-\gamma_{2,k}F(x)]dx &= \int_{t_{k-1}}^{t_k}\frac{F'''(\eta_k)}{6}(x-t_{k-2})(x-t_{k-1})(x-t_k)dx \\
&= \frac{-1}{6}F'''(\vartheta_k)\int_{t_{k-1}}^{t_k}(x-t_{k-2})(x-t_{k-1})(t_k-x)dx \qquad (4.8)\\
&= \frac{-1}{3}F'''(\vartheta_k)(t_k-t_{k-1})\Delta t^3,
\end{aligned}$$

where $\vartheta_k \in (t_{k-2}, t_k)$. The substitution of (4.6), (4.7) and (4.8) into (4.5) will lead to (4.2). The proof ends. □

5. Linear stability analysis

To study linear stability of new method, consider the linear test problem

$$ {}_0^{CF}D_t^{\alpha}y(t)\,|_{t=t_k} = \lambda y(t), \quad y(t_0) = y_0, \quad 0 < \alpha < 1. \tag{5.1}$$

Theorem 5.1. [7] *The steady-state $y = 0$ of* (5.1) *is stable if and only if*

$$\lambda \in \{\nu \in \mathbb{C} : |arg(\nu)| > \alpha\frac{\pi}{2}\}.$$

We consider test problem (5.1), to investigate the stability region of the presented numerical method. The new method gives the following iteration formula for solving the test problem:

$$\begin{aligned} y_k &= y_0 + ((\alpha-1) + \frac{5}{12}\alpha h)F_0 + (\frac{5}{12}\alpha h - (\alpha-1))F_k \\ &+ \left[\frac{13}{12}F_1 + \sum_{j=2}^{k-2} F_j + \frac{13}{12}F_{k-1}\right]\alpha h \\ &= y_0 + ((\alpha-1) + \frac{5}{12}\alpha h)\lambda y_0 + (\frac{5}{12}\alpha h - (\alpha-1))\lambda y_k \\ &+ \left[\frac{13}{12}\lambda y_1 + \sum_{j=2}^{k-2} \lambda y_j + \frac{13}{12}\lambda y_{k-1}\right]\alpha h. \end{aligned} \tag{5.2}$$

Suppose that $z = \lambda h$, then we have

$$z = \frac{(1+(\alpha-1)\lambda)(y_k - y_0)}{\alpha\left[\frac{5}{12}(y_0+y_k) + \frac{13}{12}(y_1+y_{k-1}) + \sum_{j=2}^{k-2} y_j\right]}. \tag{5.3}$$

Let $y_j = \xi^j$, then by assumming $\xi = e^{i\theta}$ with $0 \le \theta \le 2\pi$ we get the stability region

$$S = \{z : z = \frac{(1+(\alpha-1)\lambda)(\xi^k - \xi^0)}{\alpha[\frac{5}{12}(\xi^0+\xi^k) + \frac{13}{12}(\xi^1+\xi^{k-1}) + \sum_{j=2}^{k-2}\xi^j]}\}. \tag{5.4}$$

6. Numerical results and discussion

Now, with an example, let's examine the accuracy of the obtained formulas. Take a positive integer N. Let $T_0 = 1$, $\Delta t = T_0/N = 1/N$.

Example 6.1. $y(t) = \exp(t)$

$${}_0^{CF}D_t^{\alpha} y(t) \quad = \exp(t) - \exp(\tfrac{\alpha}{\alpha-1})t$$

Taking different temporal stepsizes $\Delta t = 1/10,\ 1/20,\ 1/40,\ 1/80,\ 1/160,\ 1/320,\ 1/640,$ $1/1/1280$, we compute the example by the formula (3.5), respectively.
Table 1 lists the computational results with different parameters $\alpha = 0.9, 0.5, 0.1$. From the results presented in Table 1, the accuracy of the approximate solution increases by increasing the number of nodes points t_k.

Example 6.2. $y(t) = sin(t)$

$${}_0^{CF}D_t^{\alpha} y(t) \quad = \frac{-\alpha\exp(\frac{\alpha}{\alpha-1})t + \alpha cos(t) + (1-\alpha)sin(t)}{1-2\alpha+2\alpha^2}$$

Example 6.3. $y(t) = t\exp(t)$

$${}_0^{CF}D_t^{\alpha} y(t) \quad = \exp(\tfrac{\alpha}{\alpha-1})t[-\alpha + (\alpha+t)\exp(\tfrac{1}{\alpha-1})t]$$

Example 6.4. $y(t) = sin(t)cos(t)$

$$ {}_0^{CF}D_t^{\alpha} y(t) = \frac{-\alpha \exp(\frac{\alpha}{\alpha-1})t + \alpha cos(2t) - 2(\alpha-1)sin(2t)}{4-8\alpha+5\alpha^2} $$

The absolute error of [18] and presented scheme are shown in Tables 2–4 and they are compared for different values of h and α for $t = 1$. From Tables 2–4, it can be seen that the errors of the presented scheme are improved significantly compared with the literature. It is noteworthy that error of the presented scheme is always smaller than the error of literature in all given cases. So the new formula is more accurate. The codes are written in Matlab software.

7. Conclusion

In this paper, we have discussed a new numerical method for solving fractional differential equations in the sense of Caputo- Fabrizio derivative. The integral equation of the new method is solved using the quadratic interpolation approximation using three points $(t_{j-2}, y(t_{j-2})), (t_{j-1}, y(t_{j-1}))$ and $(t_j, y(t_j))$ for the integrand $F(t)$ on each interval $[t_{j-1}, t_j](j \geq 0)$, while the linear interpolation approximation is applied on the first interval $[t_0, t_1]$. We demonstrate the efficiency and accuracy of the proposed method by applying it to four typical examples. The stability region of the new numerical method of fractional order $0 < \alpha < 1$ has been addressed. Moreover, because of its simplicity, our method applies to a wide class of initial-boundary value problems occurring in applied sciences.

Table 1: Absolute errors with different temporal stepsizes for Example 6.1.

α	Δt	E^N (3.5)
0.9	1/10	$1.3719e-04$
	1/20	$1.7285e-05$
	1/40	$2.1696e-06$
	1/80	$2.7176e-07$
	1/160	$3.4006e-08$
	1/320	$4.2530e-09$
	1/640	$5.3176e-10$
	1/1280	$6.6479e-11$
0.5	1/10	$7.6216e-05$
	1/20	$9.6030e-06$
	1/40	$1.2053e-06$
	1/80	$1.5098e-07$
	1/160	$1.8892e-08$
	1/320	$2.3628e-09$
	1/640	$2.9543e-10$
	1/1280	$3.6933e-11$
0.1	1/10	$1.5243e-05$
	1/20	$1.9206e-06$
	1/40	$2.4106e-07$
	1/80	$3.0196e-08$
	1/160	$3.7784e-09$
	1/320	$4.7255e-10$
	1/640	$5.9085e-11$
	1/1280	$7.3861e-12$

Table 2: Absolute errors of the present scheme (3.5) and the numerical method of [18] for Example 6.2.

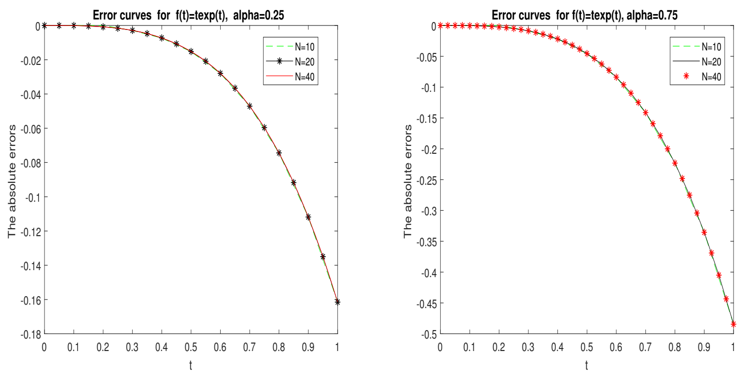
	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]
10^{-2}	$8.7828e-09$	$4.1345e-03$	$1.7566e-08$	$7.4921e-04$	$2.6348e-08$	$6.0877e-03$
10^{-3}	$8.7672e-12$	$4.0793e-04$	$1.7534e-11$	$6.7530e-05$	$2.6302e-11$	$6.1385e-04$

Table 3: Absolute errors of the present scheme (3.5) and the numerical method of [18] for Example 6.3.

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]
10^{-2}	$1.0548e-07$	$6.1334e-02$	$2.1096e-07$	$6.9383e-02$	$3.1644e-07$	$7.5564e-0$
10^{-3}	$1.0575e-10$	$6.1716e-03$	$2.1150e-10$	$6.9755e-03$	$3.1725e-10$	$7.5842e-03$

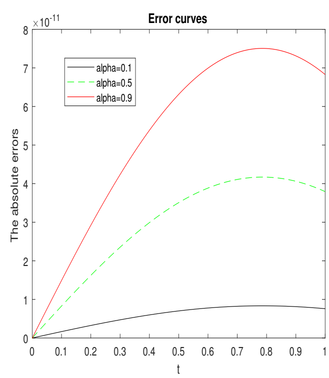
Table 4: Absolute errors of the present scheme (3.5) and the numerical method of [18] for Example 6.4.

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]	Our scheme (3.5)	E[18]
10^{-2}	$1.9159e-08$	$7.0012e-03$	$3.8319e-08$	$1.2357e-02$	$5.7478e-08$	$2.0529e-02$
10^{-3}	$1.8965e-11$	$7.0960e-04$	$3.7931e-11$	$1.2451e-03$	$5.6896e-11$	$2.0530e-03$



(a) (b)

FIG. 1: The absolute errors with ($\alpha = 0.25$ by the scheme (a) and $\alpha = 0.75$ by the scheme (b)) for different N for Example 6.3.



(a)

FIG. 2: The absolute errors for different α and $N = 1000$ for Example 6.4.

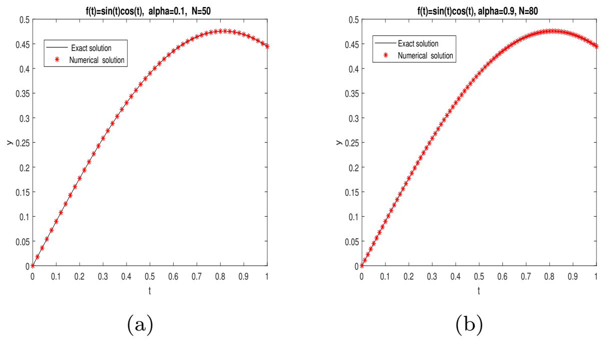


FIG. 3: Numerical and Exact solution with ($\alpha = 0.1, N = 50$ by the scheme (a) and $\alpha = 0.9, N = 90$ by the scheme (b)) for Example 6.4.

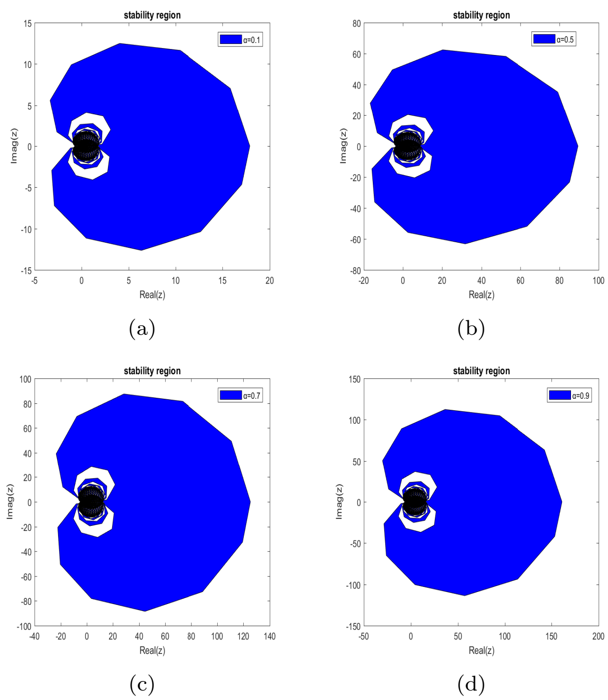


FIG. 4: Stability regions of the new method CF with $\alpha = 0.1, 0.5, 0.7, 0.9, N = 160$.

REFERENCES

1. M. Al-Refai and K. Pal. *New aspects of Caputo–Fabrizio fractional derivative.* Progress in Fractional Differentiation and Applications, **5**:157-166, 04 2019.
2. F. Ali, M. Saqib, I. Khan and N. A. Sheikh. *Application of Caputo–Fabrizio derivatives to MHD free convection flow of generalized Walters-B fluid model.* The European Physical Journal Plus, **131(10)**:377, 2016.
3. M. Caputo and M. Fabrizio. *A new definition of fractional derivative without singular kernel.* Progr. Fract. Differ. Appl, **1(2)**:1-13, 2015.
4. D. Chouhan, V. Mishra and H. Srivastava. *Bernoulli wavelet method for numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order.* Results in Applied Mathematics, **10**:100146, 2021.
5. M. A. Dokuyucu. *A fractional order alcoholism model via Caputo–Fabrizio derivative.* AIMS Mathematics, **5(2)**:781-797, 2020.
6. G.-h. Gao, Z.-z. Sun and H.-w. Zhang. *A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications.* Journal of Computational Physics, **259**:33-50, 2014.
7. R. Garrappa. *Trapezoidal methods for fractional differential equations: Theoretical and computational aspects.* Mathematics and Computers in Simulation, **110**:96-112, 015.
8. E. F. D. Goufo. *Application of the Caputo–Fabrizio Fractional Derivative without Singular Kernel to Korteweg-de Vries-Burgers Equation.* Mathematical Modelling and Analysis, **21(2)**:188-198, 2016.
9. X. Guo, Y. Li and T. Zeng. *A Finite Difference Scheme For Caputo-Fabrizio Fractional Differential Equations.* International Journal of Numerical Analysis & Modeling, **17(2)**, 2020.
10. M. Izadi and H. Srivastava. *An efficient approximation technique applied to a non-linear Lane–Emden pantograph delay differential model.* Applied Mathematics and Computation, **401**:126123, 2021.
11. M. Izadi and H. Srivastava. *Numerical approximations to the nonlinear fractional-order Logistic population model with fractional-order Bessel and Legendre bases.* Chaos, Solitons & Fractals, **145**:110779, 2021.
12. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume **204**. elsevier, 2006.
13. P. Kumar, N. A. Rangaig, H. Abboubakar and S. Kumar. *A Malaria Model with Caputo–Fabrizio and Atangana-Baleanu Derivatives.* International Journal of Modeling, Simulation, and Scientific Computing, 2020.
14. C. Li, A. Chen and J. Ye. *Numerical approaches to fractional calculus and fractional ordinary differential equation.* Journal of Computational Physics, **230(9)**:3352-3368, 2011.
15. J. R. Loh, A. Isah, C. Phang and Y. T. Toh. *On the new properties of Caputo–Fabrizio operator and its application in deriving shifted Legendre operational matrix.* Applied Numerical Mathematics, **132**:138-153, 2018.
16. J. Losada and J. J. Nieto. *Properties of a new fractional derivative without singular kernel.* Progr. Fract. Differ. Appl, **1(2)**:87-92, 2015.

17. K. M. Owolabi and A. Atangana. *Analysis and application of new fractional Adams–Bashforth scheme with Caputo–Fabrizio derivative*. Chaos, Solitons & Fractals, **105**:111-119, 2017.

18. S. Qureshi, N. A. Rangaig and D. Baleanu. *New numerical aspects of Caputo–Fabrizio fractional derivative operator*. Mathematics, **7(4)**:374, 2019.

19. H. Singh, H. M. Srivastava, Z. Hammouch and K. S. Nisar. *Numerical simulation and stability analysis for the fractional-order dynamics of COVID-19*. Results in Physics, **20**:103722, 2021.

20. H. Singh, H. M. Srivastava and D. Kumar. *A reliable numerical algorithm for the fractional vibration equation*. Chaos, Solitons & Fractals, **103**:131-138, 2017.

21. H. Srivastava, F. Shah and R. Abass. *An application of the Gegenbauer wavelet method for the numerical solution of the fractional Bagley-Torvik equation*. Russian Journal of Mathematical Physics, **26(1)**:77-93, 2019.

22. H. M. Srivastava. *Fractional-order derivatives and integrals: Introductory overview and recent developments*. Kyungpook Mathematical Journal, **60(1)**:73-116, 2020.

23. H. M. Srivastava, E. S. AbuJarad, F. Jarad, G. Srivastava and M. H. AbuJarad. *The Marichev-Saigo-Maeda Fractional-Calculus Operators Involving the (p, q)-Extended Bessel and Bessel-Wright Functions*. Fractal and Fractional, **5(4)**:210, 2021.

24. J. Stoer and R. Bulirsch. *Introduction to numerical analysis*, volume **12**. Springer Science & Business Media, 2013.

25. X.-J. Yang, J. T. Machado and H. Srivastava. *A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach*. Applied Mathematics and Computation, **274**:143-151, 2016.