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TANGENT BUNDLES ENDOWED WITH SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Abstract. The differential geometry of the tangent bundle is an effective domain of differential geometry which reveals many new problems in the study of modern differential geometry. The generalization of connection on any manifold to its tangent bundle is an application of differential geometry. Recently a new type of semi-symmetric non-metric connection on a Riemannian manifold has been studied and a relationship between Levi-Civita connection and semi-symmetric non-metric connection has been established. The various properties of a Riemannian manifold with relation to such connection have also been discussed. The present paper aims to study the tangent bundle of a new type of semi-symmetric non-metric connection on a Riemannian manifold. The necessary and sufficient conditions for projectively invariant curvature tensors corresponding to such connection are proved and show many basic results on the Riemannian manifold in the tangent bundle. Furthermore, the properties of group manifolds of the Riemannian manifolds with respect to the semi-symmetric non-metric connection in the tangent bundle have been studied. Moreover, theorems on the symmetry property of Ricci tensor and Ricci soliton in the tangent bundle are established.
Keywords: Tangent bundle, Vertical and complete lifts, Riemannian manifold, semi-symmetric non-metric connection, Different curvature tensors.

1. Introduction

The concept of semi-symmetric linear connection on a differential manifold was introduced by Friedman and Schouten [8] in 1924. Hayden introduced the notion

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of metric connection on a Riemannian manifold in 1932 and known as Hayden connection [10].

Let M^n be a Riemannian manifold of n-dimensional with Riemannian metric g and ∇ be Levi-Civita connection on it. A linear connection $\tilde{\nabla}$ on M^n is said to be symmetric connection if its torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is of the form

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

is zero for all X and Y on M^n; otherwise it is non-symmetric . A linear connection $\tilde{\nabla}$ is said to be semi-symmetric connection if

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y \tag{1.2}$$

where

$$\pi(X) = g(P,X) \tag{1.3}$$

for all X and Y on M^n and π is 1-form and P is a vector field.

In 1969, Pak [18] studied the Hayden connection $\tilde{\nabla}$ and proved that it is a semi-symmetric metric and a linear connection $\tilde{\nabla}$ is said to be metric on M^n if $\tilde{\nabla}g = 0$ otherwise it is non-metric. In 1970, Yano [23] studied some curvature and derivational conditions for semi-symmetric connection in Riemannian manifolds. Agashe et al define a linear connection on a Riemannian manifold M^n which is semi-symmetric but non-metric in 1992 and studied some properties of the curvature tensor with respect to semi-symmetric non-metric connection [1]. In 1994, Liang [16] studied a type of semi-symmetric non-metric connection $\tilde{\nabla}$ which satiesfies $(\tilde{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, u is 1-form and such connection called a semi-symmetric recurrent metric connection. In 2019, Chaubey at el [3] defined and studied a new type of semi-symmetric non-metric connection on a Riemannian manifold. Studies of various types of semi-symmetric non-metric connection and their properties include [2, 4, 5, 6, 9, 12, 15, 17, 19] and others.

In a Riemannian manifold of dimension n, the curvature tensor $\tilde{R}$ corresponding to $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y) = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \tag{1.4}$$

for all X, Y, Z om M_n.

The Ricci tensor $\tilde{S}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ is given by [3]

$$\begin{aligned}\tilde{S}(Y,Z) &= S(Y,Z) + \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i,Z)g(Y,e_i)) - \theta(Y,Z)g(e_i,e_i) \\ &- (g(Ae_i,Y)g(Z,e_i)) + (g(AY,e_i)g(Z,e_i))\}\end{aligned} \tag{1.5}$$

where S is a Ricci tensor with respect to ∇.

The projective curvature $\tilde{P}$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ is defined as [7]

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{1.6}$$

for all vector fields X, Y and Z on M_n.

The conformal curvature tensor C [22] with respect to ∇ is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.7}$$

for arbitrary vector fields X, Y, Z on M_n.

The concircular curvature tensor $\breve{C}$ [3] on (M_n, g) with respect to ∇ is defined by

$$\begin{aligned} {}'\breve{C}(X,Y,Z,U) &= {}'R(X,Y,Z,U) \\ &- \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \end{aligned} \tag{1.8}$$

The conharmonic curvature tensor ${}'L$ of type (0,4) [3] is defined by

$$\begin{aligned} {}'L(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{r}{n-2}\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\} \end{aligned} \tag{1.9}$$

On the other hand, the differential geometry of tangent bundles is an important domain of the differential geometry because the theory provides many new problems in the study of modern differential geometry. The theory of vertical, complete and horizontal lifts of geometrical structures and connections from a manifold to its tangent bundles was developed by Yano and Ishihara [24]. They defined and studied prolongations called vertical, complete and horizontal lifts and connections. Tani [21] developed the theory of surfaces prolonged to tangent bundle with respect to the metric tensor of the original manifold.

Most recently, the author [13, 14] studied tangent bundle endowed with respect to semy-symmetric non-metric connection on Kähler manifold and tangent bundle of an almost Hermitian manifold and an almost Kähler manifold with respect to quarter symmetric non-metric connection. Motivated by the previously mentioned studies, we study the tangent bundles of a new type of semi-symmetric non-metric connection on a Riemannian manifold.

The main contributions are summarized as follows:

- A new type of semi-symmetric non-metric connection is defined and studied on a Riemannian manifold to the tangent bundle.

- To prove the existence of such a connection on the tangent bundle and some theorems on it.
- Various curvature tensors such as projective, conformal and concircular curvature tensors corresponding to semi-symmetric non-metric connection on the tangent bundle are calculated.
- Symmetric property of Ricci tensor are established.
- To define Ricci soliton on the tangent bundle and discuss shrinking, steady and expanding properties of it.

The paper is organized as follows: Section 2 deals with a brief account of tangent bundle, vertical lift, complete lift and a new class of semi-symmetric non-metric connection. Section 3 presents semi-symmetric non-metric connection in the tangent bundle TM_n over a Riemannian manifold M_n and proves some basic results. Section 4 discusses the relation between curvature tensors of the Levi-Civita and semi-symmetric non-metric connections in the tangent bundle and some basic properties of the curvature tensor of $\tilde{\nabla}^C$. It is proved that such connection on a Riemannian manifold is projectively invariant curvature tensors under certain conditions and also proves some results on the curvature, concircular curvature, and conharmonic curvature tensors in the tangent bundle. Finally, Section 5 devotes the study of a group manifold with respect to a semi-symmetric non-metric connection in the tangent bundle. The symmetric property of Ricci tensor and Ricci soliton in the tangent bundle are established.

2. Preliminaries

Let M_n be an n-dimensional differentiable manifold and TM_n its tangent bundle. The projection bundle $\pi_{M_n} : TM_n \to M_n$ which denotes the natural bundle structure of TM_n over M_n. Let $\{U; x^i\}$ be coordinate neighborhood in M_n where $\{x^i\}$ is a system of local coordinates in neighborhood U. Let $\{x^i, y^i\}$ be a system of local coordinates in $\pi^{-1}_{M_n}(U) \subset TM_n$ i.e. $\{x^i, y^i\}$ the induced coordinate in $\pi^{-1}_{M_n}(U)$. Let $\wp^r_s(M_n)$ be the set of all tensor fields of type (r, s) in M_n, namely contravariant of degree r and covariant of degree s. If we denote by $\wp(M_n)$ the tensor algebra associated with M_n i.e. $\wp(M_n) = \wp^r_s(M_n)$. The set of tensor fields in tangent bundle represented by $\wp^r_s(TM_n)$ and tensor algebra in the tangent bundle by $\wp(TM_n)$. The set of functions, vector fields, 1-forms and tensor fields of type (1,1) are denoted by $\wp^0_0(TM_n), \wp^1_0(TM_n), \wp^0_1(TM_n)$ and $\wp^1_1(TM_n)$ respectively.

2.1. Vertical and complete lifts

The vertical and complete lifts of a function, a vector field, 1-form, tensor field of type (1,1) and affine connection ∇ are given by $f^V, X^V, \omega^V, F^V, \nabla^V$ and f^C, X^C, ω^C, F^C, ∇^C respectively [14, 24].

The following properties of complete and vertical lifts are given by

$$(fX)^V = f^V X^V, (fX)^C = f^C X^V + f^V X^C, \tag{2.1}$$

$$X^V f^V = 0, X^V f^C = X^C f^V = (Xf)^V, X^C f^C = (Xf)^C, \tag{2.2}$$

$$\omega^V(f^V) = 0, \omega^V(X^C) = \omega^C(X^V) = \omega(X)^V, \omega^C(X^C) = \omega(X)^C, \tag{2.3}$$

$$F^V X^C = (FX)^V, F^C X^C = (FX)^C, \tag{2.4}$$

$$[X, Y]^V = [X^C, Y^V] = [X^V, Y^C], [X, Y]^C = [X^C, Y^C]. \tag{2.5}$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \qquad \nabla^C_{X^C} Y^V = (\nabla_X Y)^V \tag{2.6}$$

We extend the vertical and complete lifts to a linear isomorphism of tensor algebra $\wp(M_n)$ into $\wp(TM_n)$ concerning constant coefficient. Let P^V and Q^V be vertical lift and P^C and Q^C be complete lift of arbitrary tensor fields P and Q of $\wp(M_n)$. Then by definition

$$(P \otimes Q)^V = P^V \otimes Q^V, (P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C$$

$$(P + Q)^V = P^V + Q^V, (P + Q)^C = P^C + Q^C.$$

2.2. Semi-symmetric non-metric connection

Let M_n be a Riemannian manifold of dimension n with Riemannian metric g. A linear connection $\tilde{\nabla}$ on M_n given by [3]

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\pi(Y)X - \pi(X)Y\} \tag{2.7}$$

where ∇ is a Levi-Civita connection, X, Y vector fields and π 1-form on M_n. The metric g have the relation

$$(\tilde{\nabla}_X g)(Y, Z) = \frac{1}{2}\{2\pi(X)g(Y, Z) - \pi(Y)g(X, Z) - \pi(Z)g(X, Y)\} \tag{2.8}$$

The connection $\tilde{\nabla}$ satisfying equations (1.2), (1.3), (2.7) and (2.8) is called a semi-symmetric non-metric connection.

3. Semi-symmetric non-metric connection of a Riemannian manifold in the tangent bundle

Let (M_n, g) be an n-dimensional Riemannian manifold with the Riemannian metric g and TM_n its tangent bundle. Then g^C is a Riemannian metric in TM_n. Taking complete lifts of equations (1.2), (1.3), (2.7) and (2.8), then obtained equations are [21]

$$\begin{aligned} \tilde{T}^C(X^C, Y^C) &= \pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \pi^C(X^C)Y^V - \pi^V(X^C)Y^C \end{aligned} \tag{3.1}$$

$$\pi^C(X^C) = g^C(X^C, P^C)$$

A linear connection $\tilde{\nabla}^C$ defined by

$$\begin{aligned}\tilde{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C &+ \frac{1}{2}\{\pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \pi^C(X^C)Y^V - \pi^V(X^C)Y^C\}\end{aligned} \tag{3.2}$$

is said to be a semi-symmetric non-metric connection if the torsion tensor $\tilde{T}^C$ of TM_n with respect to $\tilde{\nabla}^C$ satisfies equations (3.1) and (3.2) and the Riemannian metric g^C holds the relation

$$\begin{aligned}(\tilde{\nabla}^C_{X^C} g^C)(Y^C, Z^C) &= \frac{1}{2}\{2\pi^C(X^C)g^C(Y^V, Z^C) + 2\pi^V(X^C)g^C(Y^C, Z^C) \\ &- \pi^C(Y^C)g^C(X^V, Z^C) - \pi^V(Y^C)g^C(X^C, Z^C) \\ &- \pi^C(Z^C)g^C(X^V, Y^C) - \pi^V(Z^C)g^C(X^C, Y^C)\}\end{aligned} \tag{3.3}$$

where ∇^C is Levi-Civita connection on TM_n.

In order to prove the existence of such connection on tangent bundle TM_n, it suffices to prove the following theorem:

Theorem 3.1. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with the Levi-Civita connection ∇^C. Then there exists a unique linear connection $\tilde{\nabla}^C$ on TM_n, called a semi-symmetric non-metric connection, given by (3.2), and it satisfies equations* (3.1) *and* (3.3).

Proof. Let M_n be a Riemannian manifold of dimension n equipped with a linear connection $\tilde{\nabla}$. Then the relation between the linear connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ are are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y) \tag{3.4}$$

Operating complete lifts of both sides of equation (3.4), we get

$$\tilde{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C + U^C(X^C, Y^C) \tag{3.5}$$

for arbitrary vector fields X^C and Y^C on TM_n, where U^C is complete lift of a tensor field U of type (1, 2). Using equations (1.1) and (3.5), the obtained equation is

$$\tilde{T}^C(X^C, Y^C) = U^C(X^C, Y^C) - U^C(Y^C, X^C) \tag{3.6}$$

which gives

$$g^C(\tilde{T}(X^C, Y^C), Z^C) = g^C(U^C(X^C, Y^C)Z^C) - g^C(U^C(Y^C, X^C)Z^C) \tag{3.7}$$

In the view of equations (3.1) and (3.7), then

$$
\begin{aligned}
g^C(U^C(X^C,Y^C),Z^C) &- g^C(U^C(Y^C,X^C),Z^C)\\
&= \pi^V(Y^C)g^C(X^C,Z^C)+\pi^C(Y^C)g^C(X^V,Z^C)\\
&- \pi^V(X^C)g^C(Y^C,Z^C)-\pi^C(X^C)g^C(Y^V,Z^C) \qquad (3.8)
\end{aligned}
$$

Making use of (3.1), the obtained equation is

$$
\begin{aligned}
\tilde{\nabla}^C_{X^C}g^C(Y^C,Z^C) &= -g^C(\tilde{\nabla}^C_{X^C}Y^C-\nabla^C_{X^C}Y^C,Z^C)\\
&- g^C(Y^C,\tilde{\nabla}^C_{X^C}Z^C-\nabla^C_{X^C}Z^C)\\
&= -U'^C(X^C,Y^C,Z^C), \qquad (3.9)
\end{aligned}
$$

where $U'^C(X^C,Y^C,Z^C)=g^C(U^C(X^C,Y^C),Z^C)+g^C(U^C(X^C,Z^C)Y^C)$.

Using equations (3.6), (3.7), and (3.9), the obtained equation is

$$
\begin{aligned}
g^C(\tilde{T}^C(X^C,Y^C),Z^C) &+ g^C(\tilde{T}^C(Z^C,X^C),Y^C)+g^C(\tilde{T}'^C(Z^C,Y^C),X^C)\\
&= 2g^C(U^C(X^C,Y^C),Z^C)-U'^C(X^C,Y^C,Z^C)\\
&- U'^C(X^C,Y^C,Z^C)+U'^C(Z^C,X^C,Y^C)\\
&- U'^C(Y^C,X^C,Z^C)
\end{aligned}
$$

From equations (3.3) and (3.9), the above equation becomes

$$
\begin{aligned}
2g^C(U^C(X^C,Y^C),Z^C) &= g^C(\tilde{T}^C(X^C,Y^C),Z^C)+g^C(\tilde{T}'^C(Z^C,X^C),Y^C)\\
&+ g^C(\tilde{T}'^C(Z^C,Y^C),X^C)-\pi^V(X^C)g^C(Y^C,Z^C)\\
&- \pi^C(X^C)g^C(Y^V,Z^C)-\pi^V(Y^C)g^C(X^C,Z^C)\\
&- \pi^C(Y^C)g^C(X^V,Z^C)+2\pi^V(Z^C)g^C(X^C,Y^C)\\
&+ \pi^C(Z^C)g^C(X^V,Y^C) \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
g^C(\tilde{T}'^C(X^C,Y^C),Z^C) &= g^C(\tilde{T}^C(Z^C,X^C),Y^C)=\pi^V(X^C)g^C(Z^C,Y^C)\\
&+ \pi^C(X^C)g^C(Z^V,Y^C)-\pi^V(Z^C)g^C(X^C,Y^C)\\
&- \pi^C(Z^C)g^C(X^V,Y^C) \qquad (3.11)
\end{aligned}
$$

for all vector fields X^C,Y^C and Z^C on TM_n.

Making use of equation (3.11), then equation (3.10) becomes

$$
\begin{aligned}
2U^C(X^C,Y^C) &= (\pi^C(Y^C))(X^V)+(\pi^V(Y^C))(X^C)\\
&- (\pi^C(X^C))(Y^V)+(\pi^V(X^C))(Y^C) \qquad (3.12)
\end{aligned}
$$

and thus equations (3.5) and (3.12) give (3.1).

Conversely, it is easy to show that if the affine connection $\tilde{\nabla}^C$ satisfies (3.2) then it will also satisfy equations (3.1) and (3.3). Hence, the theorem is proved. □

The covariant derivative of equation (3.2) with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ on TM_n , then the obtained equation is

$$\begin{aligned}(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) &= (\nabla^C_{X^C}\pi^C)(Y^C) + \pi^C(P^C)g^C(X^V, Y^C)\\ &+ \pi^V(P^C)g^C(X^C, Y^C) - \pi^V(Y^C)\pi^C(X^C)\\ &- \pi^C(Y^C)\pi^V(X^C)\end{aligned} \tag{3.13}$$

for arbitrary vector fields X^C and Y^C on TM_n. Using equation (3.13), then

$$\begin{aligned}(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) &- (\tilde{\nabla}^C_{Y^C}\pi^C)(X^C) = (\nabla^C_{X^C}\eta^C)(Y^C)\\ &- (\nabla^C_{Y^C}\eta^C)(X^C).\end{aligned} \tag{3.14}$$

Hence, the following theorem is obtained:

Theorem 3.2. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, and then the necessary and sufficient condition for the 1-form π^C to be closed with respect to $\tilde{\nabla}^C$ is that it is also closed corresponding to the Levi-Civita connection ∇^C.*

Theorem 3.3. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then*

$$'\tilde{T}^C(X^C, Y^C, Z^C) +' \tilde{T}^C(Y^C, X^C, Z^C) = 0,$$
$$'\tilde{T}^C(X^C, Y^C, Z^C) +' \tilde{T}^C(Y^C, Z^C, X^C) +' \tilde{T}^C(Z^C, X^C, Y^C) = 0.$$

Proof: Let $\tilde{T}$ be the torsion tensor on TM_n and define
$'\tilde{T}^C(X^C, Y^C, Z^C) = g^C(\tilde{T}^C(X^C, Y^C), Z^C)$ on TM_n.

In the view of equation (3.1), then obtained equation is

$$\begin{aligned}'\tilde{T}^C(X^C, Y^C, Z^C) &= \pi^V(Y^C)g^C(X^C, Z^C) + \pi^C(Y^C)g^C(X^V, Z^C)\\ &- \pi^V(X^C)g^C(Y^C, Z^C)\\ &- \pi^C(X^C)g^C(Y^V, Z^C)\end{aligned} \tag{3.15}$$

Making use of equation (3.15), it can easily prove theorem.

Theorem 3.4. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then $\tilde{T}^C$ is cyclic parallel if and only if the 1-form π^C is closed.*

Proof. Operating the covariant derivative of (3.1) with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$, the obtained equation is

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C, Z^C) = (\tilde{\nabla}_X\pi)^C(Z^C)Y^V + (\tilde{\nabla}_X\pi)^V(Z^C)Y^C$$

$$- \quad (\tilde{\nabla}_X\pi)^C(Y^C)Z^V - (\tilde{\nabla}_X\pi)^V(Y^C)Z^C \tag{3.16}$$

The cyclic sum of (3.16) for vector fields X^C, Y^C and Z^C gives

$$\begin{aligned}
(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) &+ (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C)\\
&= (\tilde{\nabla}_X\pi)^C(Z^C)Y^V + (\tilde{\nabla}_X\pi)^V(Z^C)Y^C\\
&- (\tilde{\nabla}_X\pi)^C(Y^C)Z^V - (\tilde{\nabla}_X\pi)^V(Y^C)Z^C\\
&+ (\tilde{\nabla}_Y\pi)^C(X^C)Z^V + (\tilde{\nabla}_Y\pi)^V(X^C)Z^C\\
&- (\tilde{\nabla}_Y\pi)^C(Z^C)X^V - (\tilde{\nabla}_Y\pi)^V(Z^C)X^C\\
&+ (\tilde{\nabla}_Z\pi)^C(Y^C)X^V + (\tilde{\nabla}_Z\pi)^V(Y^C)X^C\\
&- (\tilde{\nabla}_Z\pi)^C(X^C)Y^V - (\tilde{\nabla}_Z\pi)^V(X^C)Y^C
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) &+ (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C)\\
&= \{(\tilde{\nabla}_X\pi)^C(Z^C) - (\tilde{\nabla}_Z\pi)^C(X^C)\}Y^V\\
&+ \{(\tilde{\nabla}_X\pi)^V(Z^C) - (\tilde{\nabla}_Z\pi)^V(X^C)\}Y^C\\
&+ \{(\tilde{\nabla}_Y\pi)^C(X^C) - (\tilde{\nabla}_X\pi)^C(Y^C)\}Z^V\\
&+ \{(\tilde{\nabla}_Y\pi)^V(X^C) - (\tilde{\nabla}_X\pi)^V(Y^C)\}Z^C\\
&+ \{(\tilde{\nabla}_Z\pi)^C(Y^C) - (\tilde{\nabla}_Y\pi)^C(Z^C)\}X^V\\
&+ \{(\tilde{\nabla}_Z\pi)^V(Y^C) - (\tilde{\nabla}_Y\pi)^V(Z^C)\}X^C
\end{aligned} \tag{3.17}$$

From equation (3.17) and Theorem 3.3, it can easily show that

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) + (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C) = 0$$

if and only if the 1-form π^C is closed. Hence, the theorem is proved. □

Theorem 3.5. *Let M_n, g be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then for any arbitrary vector fields X^C, Y^C and the vector field P^C defined as (3.2), the following relation holds:*

$$\begin{aligned}
(\tilde{\pounds}_P g)^C(X^C,Y^C) &= (\pounds_P g)^C(X^C,Y^C) + 2\{\pi^C(P^C)g^C(X^V,Y^C)\\
&+ \pi^V(P^C)g^C(X^C,Y^C) - \pi^V(Y^C)\pi^C(X^C)\\
&- \pi^C(Y^C)\pi^V(X^C)\}
\end{aligned} \tag{3.18}$$

where $\tilde{\pounds}^C_P$ and $\pounds^C_P$ denote the Lie derivatives along the vector field P^C corresponding to $\tilde{\nabla}^C$ and ∇^C, respectively.

Proof. The Lie derivative along P [3],

$$\pounds_P g(X,Y) = g(\nabla_X P, Y) + g(X, \nabla_Y P) \tag{3.19}$$

Taking complete lifts on both sides, then

$$(\pounds_P g)^C(X^C, Y^C) = g^C(\nabla^C_{X^C} P^C, Y^C) + g^C(X^C, \nabla^C_{Y^C} P^C) \tag{3.20}$$

holds for arbitrary vector fields X^C and Y^C on TM_n. From equations (2.7) and (3.18) and the definition of the Lie derivative, the obtained equation is

$$\begin{aligned}(\tilde{\pounds}_P g)^C(X^C, Y^C) &= (P^C)g^C(X^V, Y^C) + (P^V)g^C(X^C, Y^C)\\ &- g^C(\tilde{\nabla}^C_{P^C} X^C - g^C(\tilde{\nabla}^C_{X^C} P^C, Y^C)\\ &- g^C(Y^C, \tilde{\nabla}^C_{P^C} Y^C - g^C(\tilde{\nabla}^C_{Y^C} P^C)\\ &= (\pounds_P g)^C(X^C, Y^C) + 2\{\pi^C(P^C)g^C(X^V, Y^C)\\ &+ \pi^V(P^C)g^C(X^C, Y^C) - \pi^V(Y^C)\pi^C(X^C)\\ &- \pi^C(Y^C)\pi^V(X^C)\}\end{aligned} \tag{3.21}$$

Hence, the theorem is proved. □

If the vector field P^C is Killing on (TM_n, g^C), then $(\pounds_P g)^C = 0$. From theorem 3.5, the following corollary is obtained:

Corollary 3.1. *If the vector field P^C defined as in (3.2) is Killing on TM_n equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then*

$$\begin{aligned}(\tilde{\pounds}_P g)^C(X^C, Y^C) &= 2\{\pi^C(P^C)g^C(X^V, Y^C) + \pi^V(P^C)g^C(X^C, Y^C)\\ &- \pi^V(Y^C)\pi^C(X^C) - \pi^C(Y^C)\pi^V(X^C)\}\end{aligned} \tag{3.22}$$

where X^C and Y^C are vector fields and π^C 1-form on TM_n.

4. Curvature tensor with respect to the semi-symmetric non-metric connection in the tangent bundle

Let M_n be an n-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection $\tilde{\nabla}$. If the curvature tensor $\tilde{R}$ corresponding to $\tilde{\nabla}$ then there exists the curvature tensor $\tilde{R}^C$ corresponding to $\tilde{\nabla}^C$ in TM_n is defined by

$$\tilde{R}^C(X^C, Y^C)Z^C = \tilde{\nabla}^C_{X^C}\tilde{\nabla}^C_{Y^C} Z^C - \tilde{\nabla}^C_{Y^C}\tilde{\nabla}^C_{X^C} Z^C - \tilde{\nabla}^C_{[X^C, Y^C]} Z^C$$

for arbitrary vector fields X^C, Y^C and Z^C on (TM_n, g^C), then the Riemannian curvature tensor R^C of the Levi-Civita connection ∇^C is defined by

$$R^C(X^C, Y^C)Z^C = \nabla^C_{X^C}\nabla^C_{Y^C} Z^C - \nabla^C_{Y^C}\nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C$$

for arbitrary vector fields X^C, Y^C, and Z^C on (TM_n, g^C).

Making use of equation (3.2), we have

$$\tilde{R}^C(X^C, Y^C)Z^C = \tilde{\nabla}^C_{X^C}\{\nabla^C_{Y^C} Z^C + \frac{1}{2}(\pi^C(Z^C)(Y^V) + \pi^V(Z^C)(Y^C)$$

$$
\begin{aligned}
&- \quad \pi^C(Y^C)(Z^V) - \pi^V(Y^C)(Z^C)\})\\
&- \quad \tilde{\nabla}^C_{Y^C}\{\nabla^C_{X^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(X^V) + \pi^V(Z^C)(X^C)\\
&- \quad \pi^C(X^C)(Z^V) - \pi^V(X^C)(Z^C))\}\\
&- \quad \{\nabla^C_{[X^C,Y^C]}Z^C + \frac{1}{2}(\pi^C(Z^C)([X,Y]^V) + \pi^V(Z^C)([X,Y]^C)\\
&- \quad \pi^C([X,Y]^C)(Z^V) - \pi^V([X,Y]^C)(Z^C))\}\\
&= \quad \nabla^C_{X^C}\{\nabla^C_{Y^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(Y^V) + \pi^V(Z^C)(Y^C)\\
&- \quad \pi^C(Y^C)(Z^V) - \pi^V(Y^C)(Z^C)\})\\
&- \quad \nabla^C_{Y^C}\{\nabla^C_{X^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(X^V) + \pi^V(Z^C)(X^C)\\
&- \quad \pi^C(X^C)(Z^V) - \pi^V(X^C)(Z^C))\}\\
&- \quad \nabla^C_{[X^C,Y^C]}Z^C + \frac{1}{2}(\pi^C(\nabla^C_{Y^C}Z^C)(X^V) + \pi^V(\nabla^C_{Y^C}Z^C)(X^C)\\
&- \quad \pi^C(X^C)(\nabla_Y Z)^V - \pi^V(X^C)(\nabla_Y Z)^C\}\\
&- \quad \frac{1}{4}\{\pi^V(X^C)\pi^C(Z^C)Y^C) + \pi^C(X^C)\pi^V(Z^C)Y^C)\\
&+ \quad \pi^C(X^C)\pi^C(Z^C)Y^V) - \pi^V(Y^C)\pi^C(Z^C)X^C)\\
&- \quad \pi^C(Y^C)\pi^V(Z^C)X^C) - \pi^C(Y^C)\pi^C(Z^C)X^V)\\
&- \quad \pi^C(Z^C)([X,Y]^V) - \pi^V(Z^C)([X,Y]^C)\\
&- \quad \pi^C([X,Y]^C)(Z^V) - \pi^V([X,Y]^C)(Z^C)\\
&= \quad R^C(X^C,Y^C)Z^C + \frac{1}{2}\{\theta^C(X^C,Z^C)Y^C - \theta^C(Y^C,Z^C)X^C\\
&- \quad (\theta^C(X^C,Y^C) - \theta^C(Y^C,X^C))Z^C\} \qquad (4.1)
\end{aligned}
$$

for arbitrary vector fields X^C, Y^C and Z^C on TM_n, where θ^C is a complete lift of a tensor field θ of type (0, 2) and is defined by

$$
\begin{aligned}
\theta^C(X^C,Y^C) &= g^C(AX,Y)^C = (\nabla^C_{X^C}\pi^C)(Y^C)\\
&- \pi^V(X^C)\pi^C(Y^C) - \pi^C(X^C)\pi^V(Y^C) \qquad (4.2)
\end{aligned}
$$

and

$$
(AX)^C = (\nabla_X P)^C - \frac{1}{2}\{\pi^V(X^C)(P^C) - \pi^C(X^C)(P^V)\} \qquad (4.3)
$$

for arbitrary vector fields X^C and Y^C on TM_n.

From equation (4.2), it is obvious that the tensor field θ^C is symmetric if and only if the 1-form π^C is closed. Taking the inner product of (4.1) with W^C and then setting $X^C = W^C = e_i^C, 1 \le i \le n$, where e_i^C is complete lift of $\{e_i, i = 1, 2, 3, ..., n\}$ which is an orthonormal basis of the tangent space at each point of the Riemannian manifold M_n, then obtained equation is

$$\begin{aligned}
\tilde{S}^C(Y^C,Z^C) &= S^C(Y^C,Z^C)+\frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i,Z)g(Y,e_i))^C-\theta^C(Y^C,Z^C)g^C(e_i,e_i)\\
&- (g(Ae_i,Y)g(Z,e_i))^C+(g(AY,e_i)g(Z,e_i))^C\}\\
&= S^C(Y^C,Z^C)+\frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i,Z)^C g(Y,e_i))^V\\
&+ (g(Ae_i,Z)^V g(Y,e_i))^C-\theta^C(Y^C,Z^C)g^C(e_i,e_i)\\
&- (g(Ae_i,Y))^C(g(Z,e_i))^V-(g(Ae_i,Y))^V(g(Z,e_i))^C\\
&+ (g(AY,e_i))^C(g(Z,e_i))^V+(g(AY,e_i))^V(g(Z,e_i))^C\}\\
&= S^C(Y^C,Z^C)-\frac{n-1}{2}\theta^C(Y^C,Z^C)\\
&+ \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i,e_i))^C g(Z,e_i))^C g(Y,e_i))^V\\
&+ (g(Ae_i,e_i))^C g(Z,e_i))^V g(Y,e_i))^C\\
&+ (g(Ae_i,e_i))^V g(Z,e_i))^C g(Y,e_i))^C\\
&- (g(Ae_i,e_i))^C g(Z,e_i))^C g(Y,e_i))^V\\
&- (g(Ae_i,e_i))^C g(Z,e_i))^V g(Y,e_i))^C\\
&- (g(Ae_i,e_i))^V g(Z,e_i))^C g(Y,e_i))^C\}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\tilde{S}^C(Y^C,Z^C) &= S^C(Y^C,Z^C)-\frac{n-1}{2}\theta^C(Y^C,Z^C)\\
&\Leftrightarrow \tilde{Q}^C(Y^C)=Q^C(Y^C)-\frac{n-1}{2}(AY)^C
\end{aligned}\tag{4.4}$$

for all vector fields Y^C and Z^C on TM_n. Here $\tilde{Q}^C$ and Q^C are the complete lift of Ricci operators corresponding to the Ricci tensors $\tilde{Q}^C$ and Q^C complete lifts $\tilde{S}^C$ and S^C Ricci tensors $\tilde{S}$ and S of the connections $\tilde{\nabla}^C$ and ∇^C, respectively; that is, $\tilde{S}^C(Y^C,Z^C)=g^C(\tilde{Q}^CY^C,Z^C)$ and $S^C(Y^C,Z^C)=g^C(Q^CY^C,Z^C)$

Again contracting eqution (4.4) along the vector field Y^C, then

$$\tilde{r}=r-(n-1)a,\tag{4.5}$$

where $\tilde{r}$ and r denote the scalar curvatures corresponding to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ and the Levi-Civita connection ∇^C, respectively, and

$$a\overset{def}{=}\frac{1}{2}trA$$

Here trA represents the trace of A. From equation (4.5), the following Theorem is obtained:

Theorem 4.1. *Let (M_n, g) be n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. Then the necessary and sufficient condition for the scalar curvatures $\tilde{r}$ and r to coincide is that a be zero; that is, $trA = 0$.*

Interchanging Y^C and Z^C in equation (4.4), the obtained equation is

$$\tilde{S}^C(Z^C, Y^C) = S^C(Z^C, Y^C) - \frac{n-1}{2}\theta^C(Z^C, Y^C). \tag{4.6}$$

Subtracting equation (4.6) from equation (4.4) and then using equation (4.2) and the symmetric property of the Ricci tensor in it, the obtained equation is

$$\begin{aligned} \tilde{S}^C(Y^C, Z^C) - \tilde{S}^C(Z^C, Y^C) &= \frac{n-1}{2}\{\theta^C(Z^C, Y^C) - \theta^C(Y^C, Z^C)\} \\ &= -\frac{n-1}{2} d\pi^C(Y^C, Z^C), \end{aligned} \tag{4.7}$$

where d denotes the exterior derivative. In view of equation (4.7) and Theorem 3.2, the following theorem is obtained:

Theorem 4.2. *If an $n(>1)$-dimensional Riemannian manifold (M_n, g) and TM_n its tangent bundle admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then the Ricci tensor $\tilde{S}^C$ corresponding to the connection $\tilde{\nabla}^C$ is symmetric if and only if the 1-form π^C is closed.*

Theorem 4.3. *Let (M_n, g) be an $n(>1)$-dimensional Riemannian manifold and TM_n its tangent bundle equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ defined as in equation (3.1). Then the connection $\tilde{\nabla}^C$ is projectively invariant; that is, the projective curvature tensors with respect to $\tilde{\nabla}^C$ and ∇^C coincide if and only if the 1-form π^C is closed.*

Proof. If the 1-form π^C is closed and from equation (4.2) θ^C is symmetric. Using these in equation (4.1), then equation (4.1) becomes

$$\tilde{R}^C(X^C, Y^C)Z^C = R^C(X^C, Y^C)Z^C + \frac{1}{2}\{\theta^C(X^C, Z^C)Y^C - \theta^C(Y^C, Z^C)X^C \tag{4.8}$$

Contracting equation (4.8) along the vector field X^C, then

$$\tilde{S}^C(Y^C, Z^C) = S^C(Y^C, Z^C) - \frac{n-1}{2}\theta^C(Y^C, Z^C) \tag{4.9}$$

which gives

$$\tilde{Q}^C(Y^C) = Q^C(Y^C) - \frac{n-1}{2}(AY)^C \tag{4.10}$$

and

$$\tilde{r} = r - (n-1)a. \tag{4.11}$$

The projective curvature tensor $\tilde{P}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ is given in equation (1.6). Taking complete lift of equation (1.6), then

$$\begin{aligned} \tilde{P}^C(X^C, Y^C)Z^C &= \tilde{R}^C(X^C, Y^C)Z^C - \frac{1}{n-1}\{\tilde{S}^C(Y^C, Z^C)X^V \\ &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\ &- \tilde{S}^V(X^C, Z^C)Y^C\} \end{aligned} \tag{4.12}$$

for all vector fields X^C, Y^C and Z^C on TM_n, where $\tilde{P}^C$ is the complete lift the projective curvature $\tilde{P}$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$. In view of equations (4.8) and (4.9), equation (4.12) becomes

$$\tilde{P}^C(X^C, Y^C)Z^C = P^C(X^C, Y^C)Z^C, \tag{4.13}$$

where P^C denotes the complete lift of the projective curvature tensor P with respect to ∇^C and is defined by

$$\begin{aligned} P^C(X^C, Y^C)Z^C &= R^C(X^C, Y^C)Z^C - \frac{1}{n-1}\{\tilde{S}^C(Y^C, Z^C)X^V \\ &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\ &- \tilde{S}^V(X^C, Z^C)Y^C\} \end{aligned} \tag{4.14}$$

for arbitrary vector fields X^C, Y^C and Z^C on TM_n and P is given in (1.6). Conversely, suppose that (TM_n, g^C) equipped with $\tilde{\nabla}^C$ satisfies (4.13). Thus, use of equations (4.1), (4.4), (4.10), (4.12), and (4.14) in equation (4.13) gives

$$\{\theta^C(X^C, Y^C) - \theta^C(Y^C, X^C)\}Z^C = 0$$

Contracting the last equation along the vector field X^C, we find

$$\theta^C(X^C, Y^C) - \theta^C(Y^C, X^C) = 0$$

which shows that $\theta^C(Y^C, Z^C) = \theta^C(Z^C, Y^C)$.

Hence, the proof is completed. □

Theorem 4.4. *Let (M_n, g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ whose curvature tensor $\tilde{R}^C$ vanishes identically, then (TM_n, g^C) is projectively flat if and only if θ^C is a symmetric tensor.*

Proof. Suppose that the curvature tensor with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ vanishes on (TM_n, g^C) i.e., $\tilde{R}^C = 0$, and the tensor field θ^C is symmetric. Then equation (4.8) takes the form

$$R^C(X^C, Y^C)Z^C = \frac{1}{2}\{\theta^C(Y^C, Z^C)X^C - \theta^C(X^C, Z^C)Y^C\} \tag{4.15}$$

which implies that

$$S^C(Y^C, Z^C) = \frac{n-1}{2}\theta^C(Y^C, Z^C), \qquad r = (n-1)a. \tag{4.16}$$

Using of equations (4.14), (4.15) and (4.16), then $P^C = 0$. Conversely, if the projective curvature tensor of ∇^C is zero and the curvature tensor $\tilde{R}^C$ is also zero, then equations (4.1) and (4.14) take the form

$$\begin{aligned} R^C(X^C, Y^C)Z^C &= \frac{1}{n-1}\{\tilde{S}^C(Y^C, Z^C)X^V + \tilde{S}^V(Y^C, Z^C)X^C \\ &- \tilde{S}^C(X^C, Z^C)Y^V - \tilde{S}^V(X^C, Z^C)Y^C\} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} R^C(X^C, Y^C)Z^C &= \frac{1}{2}\{\theta^C(Y^C, Z^C)X^C - \theta^C(X^C, Z^C)Y^C \\ &- (\theta^C(Y^C, X^C) - \theta^C(X^C, Y^C))Z^C\}. \end{aligned} \tag{4.18}$$

Equating equations (4.17) and (4.18) and then using equation (4.4), obtained equation is

$$\{\theta^C(X^C, Y^C) - \theta^C(Y^C, X^C)\}Z^C = 0$$

Contracting the above equation along the vector field Z^C, then

$$\theta^C(X^C, Y^C) = \theta^C(Y^C, X^C)$$

Hence, the proof is completed. □

Theorem 4.5. *Let (M_n, g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. If the curvature tensor with respect to $\tilde{\nabla}^C$ vanishes, then the tensor field θ^C is symmetric if and only if*

$$\begin{aligned} (n-2)\{'C^C(X^C, Y^C, Z^C, U^C) &+ {}'\breve{C}^C(X^C, Y^C, Z^C, U^C)g^C \\ &= -2'R^C(X^C, Y^C, Z^C, U^C) \end{aligned}$$

Proof. Let the curvature tensor $\tilde{R}^C$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ vanish on TM_n. For necessary part, consider the tensor field θ^C is symmetric i.e., $\theta^C(X^C, Y^C) = \theta^C(Y^C, X^C)$. The conformal curvature tensor C

given in equation (1.7) with respect to ∇. Taking complete lift of equation (1.7), the obtained equation is

$$\begin{aligned}
C^C(X^C,Y^C)Z^C &= R^C(X^C,Y^C)Z^C - \frac{1}{n-2}\{\tilde{S}^C(Y^C,Z^C)X^V \\
&+ \tilde{S}^V(Y^C,Z^C)X^C - \tilde{S}^C(X^C,Z^C)Y^V \\
&- \tilde{S}^V(X^C,Z^C)Y^C + g^C(Y^C,Z^C)(QX)^V \\
&+ g^V(Y^C,Z^C)(QX)^C - g^C(X^C,Z^C)(QY)^V \\
&- g^V(X^C,Z^C)(QY)^C\} \\
&+ \frac{r}{(n-1)(n-2)}\{g^C(Y^C,Z^C)X^V \\
&+ g^V(Y^C,Z^C)X^C - g^C(X^C,Z^C)Y^V \\
&- g^V(X^C,Z^C)Y^C\} \qquad (4.19)
\end{aligned}$$

for arbitrary vector fields X^C, Y^C, Z^C on TM_n, where C^C is the complete lift of the conformal curvature tensor C with respect to ∇^C.

The inner product of equation (4.19) with U^C gives

$$\begin{aligned}
{}'C^C(X^C,Y^C,Z^C,U^C) &= {}'R^C(X^C,Y^C,Z^C,U^C) \\
&- \frac{1}{n-1}\{\tilde{S}^C(Y^C,Z^C)g^V(X^V,U^C) \\
&+ \tilde{S}^V(Y^C,Z^C)g^V(X^C,U^C) \\
&- \tilde{S}^C(X^C,Z^C)g^V(Y^V,U^C) \\
&- \tilde{S}^V(X^C,Z^C)g^V(Y^C,U^C) \\
&+ g^C(Y^C,Z^C)\tilde{S}^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)\tilde{S}^C(X^C,U^C) \\
&- g^C(X^C,Z^C)\tilde{S}^V(Y^C,U^C) \\
&- g^V(X^C,Z^C)\tilde{S}^C(Y^C,U^C)\} \\
&+ \frac{r}{(n-1)(n-2)}\{g^C(Y^C,Z^C)g^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)g^C(X^C,U^C) \\
&- g^C(X^C,Z^C)g^V(Y^C,U^C) \\
&- g^V(X^C,Z^C)g^C(Y^C,U^C)\} \qquad (4.20)
\end{aligned}$$

where ${}'C^C(X^C,Y^C),Z^C,U^C) = g^C(C^C(X^C,Y^C)Z^C,U^C)$. Using equations (4.15) and (4.16) in equation (4.20), the obtained equation is

$$\begin{aligned}
{}'C^C((X^C,Y^C,Z^C,U^C) &= \frac{n}{(n-2)}{}'R^C(X^C,Y^C,Z^C,U^C) \\
&- \frac{a}{(n-2)}\{g^C(Y^C,Z^C)g^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)g^C(X^C,U^C)
\end{aligned}$$

$$
\begin{aligned}
& - \; g^C(X^C, Z^C)g^V(Y^C, U^C) \\
(4.21) \qquad & - \; g^V(X^C, Z^C)g^C(Y^C, U^C)\}
\end{aligned}
$$

The concircular curvature tensor $\breve{C}$ is given in equation (1.8) with respect to ∇. Taking complete lift of equation (1.8), then

$$
\begin{aligned}
{}'\breve{C}^C((X^C, Y^C, Z^C, U^C) &= {}'R^C((X^C, Y^C, Z^C, U^C) \\
&- \frac{r}{(n-1)(n-2)}\{g^C(Y^C, Z^C)g^V(X^C, U^C) \\
&+ g^V(Y^C, Z^C)g^C(X^C, U^C) \\
&- g^C(X^C, Z^C)g^V(Y^C, U^C) \\
&- g^V(X^C, Z^C)g^C(Y^C, U^C)\} \qquad (4.22)
\end{aligned}
$$

for arbitrary vector fields X^C, Y^C, Z^C, U^C on TM_n, where $\breve{C}^C$ is complete lift of the concircular curvature tensor $\breve{C}$ and

$$
{}'\breve{C}^C(X^C, Y^C Z^C, U^C) = g^C(\breve{C}^C(X^C, Y^C)Z^C, U^C).
$$

Using equations (4.16) and (4.22) in equation (4.21), the obtained equation is

$$
\begin{aligned}
& (n-2)\{{}'C^C(X^C, Y^C Z^C, U^C){}'\breve{C}^C(X^C, Y^C Z^C, U^C)\} \\
= \; & -{}'R^C(X^C, Y^C Z^C, U^C). \qquad (4.23)
\end{aligned}
$$

For the sufficient part, Suppose that the Riemannian manifold (TM_n, g^C) equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ satisfies relation (4.23). Using equations (4.1), (4.20), (4.22), and (4.23), the obtained equation is:

$$
\begin{aligned}
\tilde{S}^C(Y^C, Z^C)X^V &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\
&- \tilde{S}^V(X^C, Z^C)Y^C + g^C(Y^C, Z^C)(QX)^V \\
&+ g^V(Y^C, Z^C)(QX)^C - g^C(X^C, Z^C)(QY)^V \\
&- g^V(X^C, Z^C)(QY)^C\} \\
&= (n-10\{\theta^C(Y^C, Z^C)X^C - \theta^C(X^C, Z^C)Y^C \\
&- (\theta^C(Y^C, X^C) - \theta^C(X^C, Y^C))Z^C\}. \qquad (4.24)
\end{aligned}
$$

Contracting equation (4.24) along the vector field Z^C, implies that $\theta^C(X^C, Y^C) = \theta^C(Y^C, X^C)$. Hence, the proof is completed. □

Corollary 4.1. *Let (M_n, g) be an $n > 2$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. The Riemannian manifold (TM_n, g^C) admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$ whose curvature tensor vanishes and whose 1-form π^C is closed, then*

$$
(n-2)'L^C(X^C, Y^C, Z^C, U^C) + nR^C((X^C, Y^C, Z^C, U^C) = 0.
$$

Proof. The relation among $'C,'\breve{C},'L$ and $'R$ on a Riemannian manifold M_n is given by [3]

$$\begin{aligned} 'C(X,Y,Z,U) &+ \ '\breve{C}(X,Y,Z,U) \\ &= \ 'L(X,Y,Z,U) \\ &+ \ 'R((X,Y,Z,U). \end{aligned} \tag{4.25}$$

Taking complete lifts on both sides of above equation , the obtained equation is

$$\begin{aligned} 'C^C(X^C,Y^C,Z^C,U^C) &+ \ '\breve{C}^C(X^C,Y^C,Z^C,U^C) \\ &= \ 'L^C(X^C,Y^C,Z^C,U^C) \\ &+ \ 'R^C((X^C,Y^C,Z^C,U^C). \end{aligned} \tag{4.26}$$

where $'L^C$ is the complete lift of a conharmonic curvature tensor $'L$ of type (0, 4), which is obtain by taking complete lift of equation (1.9)

$$\begin{aligned} 'L^C(X^C,Y^C,Z^C,U^C) &= \ 'R^C(X^C,Y^C,Z^C,U^C) \\ &- \ \frac{1}{n-1}\{\tilde{S}^C(Y^C,Z^C)g^V(X^V,U^C) \\ &+ \ \tilde{S}^V(Y^C,Z^C)g^V(X^C,U^C) \\ &- \ \tilde{S}^C(X^C,Z^C)g^V(Y^V,U^C) \\ &- \ \tilde{S}^V(X^C,Z^C)g^V(Y^C,U^C) \\ &+ \ g^C(Y^C,Z^C)\tilde{S}^V(X^C,U^C) \\ &+ \ g^V(Y^C,Z^C)\tilde{S}^C(X^C,U^C) \\ &- \ g^C(X^C,Z^C)\tilde{S}^V(Y^C,U^C) \\ &- \ g^V(X^C,Z^C)\tilde{S}^C(Y^C,U^C)\} \end{aligned} \tag{4.27}$$

From equations (4.23) and (4.26), the statement of Corollary 4.1 is obtained. □

5. Group manifolds with respect to the semi-symmetric non-metric connection in the tangent bundle

Let (M_n,g) be n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is said to be a group manifold [23] if

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C)=0 \quad \text{and} \quad \tilde{R}^C(X^C,Y^C)Z^C=0 \tag{5.1}$$

for arbitrary vector fields X^C,Y^C and Z^C on TM_n.

Making use of equations (3.17) and (5.1), the obtained equation is

$$\begin{aligned} (\tilde{\nabla}_X\pi)^C(Z^C)Y^V &+ \ (\tilde{\nabla}_X\pi)^V(Z^C)Y^C-(\tilde{\nabla}_X\pi)^C(Y^C)Z^V \\ &- \ (\tilde{\nabla}_X\pi)^V(Y^C)Z^C=0 \end{aligned}$$

Using equation (3.13) and $n > 1$, then above equation gives

$$
\begin{aligned}
(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) = 0 \quad \Leftrightarrow \quad & (\nabla^C_{X^C}\pi^C)(Y^C) = \pi^V(Y^C)\pi^C(X^C) \\
+ \quad & \pi^C(Y^C)\pi^V(X^C) - \pi^C(P^C)g^C(X^V, Y^C) \\
- \quad & \pi^V(P^C)g^C(X^C, Y^C),
\end{aligned} \tag{5.2}
$$

Using equations (4.1) and (5.1), the curvature tensor R^C on TM_n is given by

$$
\begin{aligned}
R^C(X^C, Y^C)Z^C \quad = \quad & \frac{1}{4}\{\pi^V(Y^C)\pi^C(Z^C)X^C) + \pi^C(Y^C)\pi^V(Z^C)X^C) \\
+ \quad & \pi^C(Y^C)\pi^C(Z^C)X^V)\} - \pi^V(X^C)\pi^C(Z^C)Y^C) \\
- \quad & \pi^C(X^C)\pi^V(Z^C)Y^C) - \pi^C(X^C)\pi^C(Z^C)Y^V) \\
- \quad & \frac{\pi^C(P^C)}{2}\{g^C(Y^C, Z^C)X^V \\
+ \quad & g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
- \quad & g^V(X^C, Z^C)Y^C\} \\
- \quad & \frac{\pi^V(P^C)}{2}\{g^C(Y^C, Z^C)X^V \\
+ \quad & g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
- \quad & g^V(X^C, Z^C)Y^C\}
\end{aligned} \tag{5.3}
$$

Contracting equation (5.3) along the vector field X^C, then

$$
\begin{aligned}
S^C(Y^C, Z^C) \quad = \quad & \frac{n-1}{4}[\pi^V(Y^C)\pi^C(Z^C) + \pi^C(Y^C)\pi^V(Z^C) \\
- \quad & 2\pi^C(P^C)g^C(Y^V, Z^C) \\
- \quad & 2\pi^V(P^C)g^C(Y^C, Z^C)
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
Q^C(Y^C) \quad = \quad & \frac{n-1}{4}\{\pi^V(Y^C)(P^C) + \pi^C(Y^C)(P^V \\
- \quad & \pi^V(P^C)(Y^C) - \pi^C(P^C)(Y^V)\}
\end{aligned} \tag{5.5}
$$

Changing Z^C with P^C in equation (5.4) and using equation (3.2) in it, obtained equation is

$$
\begin{aligned}
S^C(Y^C, Z^C) \quad = \quad & -\frac{n-1}{4}[\pi^C(P^C)g^C(Y^V, P^C) \\
- \quad & 2\pi^V(P^C)g^C(Y^C, P^C).
\end{aligned}
$$

The following theorem is obtained:

Theorem 5.1. *Let (M_n, g) be $n(>1)$-dimensional group manifold and TM_n its tangent bundle with Riemannian metric g^C admit a semi-symmetric non-metric connection $\tilde{\nabla}^C$. Then $-\frac{n-1}{4}\pi^C(P^C)$ is an eigenvalue of S^C is the complete lift of the Ricci tensor S^C corresponding to the eigenvector P^C.*

Also contracting equation (5.5) along Y^C, then

$$r = -\frac{(n-1)(2n-1)\pi^C(P^C)}{4} \tag{5.6}$$

Using equations (5.3) and (5.4) in equation (4.14), then $P^C = 0$. Hence, the following theorem is obtained:

Theorem 5.2. *Let (M_n, g) be an $n > 1$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Every group manifold (M_n, g) in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is projectively flat.*

Theorem 5.3. *Let (M_n, g) be an $n > 2$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is P^C-conformally flat.*

Proof. From equations (5.3), (5.4), (5.5), and (5.6), then equation (4.19) takes the form

$$\begin{aligned}
C^C(X^C,Y^C)Z^C &= \frac{\pi^C(P^C)}{4(n-2)}\{g^C(Y^C,Z^C)X^V \\
&+ g^V(Y^C,Z^C)X^C - g^C(X^C,Z^C)Y^V \\
&- g^V(X^C,Z^C)Y^C\} \\
&- \frac{\pi^V(P^C)}{4(n-2)}\{g^C(Y^C,Z^C)X^V \\
&+ g^V(Y^C,Z^C)X^C - g^C(X^C,Z^C)Y^V \\
&- g^V(X^C,Z^C)Y^C\} \\
&- \frac{1}{4(n-2)}\{\pi^V(X^C)\pi^C(Z^C)Y^C) \\
&+ \pi^C(X^C)\pi^V(Z^C)Y^C) + \pi^C(X^C)\pi^C(Z^C)Y^V) \\
&- \pi^V(Y^C)\pi^C(Z^C)X^C) - \pi^C(Y^C)\pi^V(Z^C)X^C) \\
&- \pi^C(Y^C)\pi^C(Z^C)X^V)\}
\end{aligned}$$

$$\begin{aligned}
&- \frac{n-1}{4(n-2)}\{\pi^C(Y^C)g^C(X^V,Z^C) \\
&+ \pi^V(Y^C)g^C(X^C,Z^C) - \pi^C(X^C)g^C(Y^V,Z^C) \\
&- \pi^V(X^C)g^C(Y^C,Z^C)\}P^C
\end{aligned}$$

$$
\begin{aligned}
& - \frac{n-1}{4(n-2)}\{\pi^C(Y^C)g^C(X^V, Z^C) \\
& + \pi^V(Y^C)g^C(X^C, Z^C) \\
& - \pi^C(X^C)g^C(Y^V, Z^C) \\
& - \pi^V(X^C)g^C(Y^C, Z^C)\}P^V. \qquad (5.7)
\end{aligned}
$$

Let (M_n, g) be an $n(> 2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Then (TM_n, g^C) is said to be P^C-conformally flat [3] if its nonvanishing conformal curvature tensor C^C satisfies $C^C(X^C, Y^C)P^C = 0$ for all vector fields X^C and Y^C on TM_n. Replacing Z^C by P^C in equation (5.7), it can easily show that $C^C(X^C, Y^C)P^C = 0$. Hence, Theorem 5.2 is verified. □

Theorem 5.4. *Let (M_n, g) be an $n(> 2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Every Ricci-symmetric group manifold (M_n, g) in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ satisfies $\pi^C(P^V) = 0$ and $\pi^V(P^C) = 0$.*

Proof. Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. The covariant derivative of equation (5.4) gives

$$
\begin{aligned}
(\nabla^C_{X^C} S^C)(Y^C, Z^C) = & \frac{n-1}{4}[(\nabla^C_{X^C}\pi^V)(Y^C)\pi^C(Z^C) \\
& + \pi^V(Y^C)(\nabla^C_{X^C}\pi^C)(Z^C) \\
& + (\nabla^C_{X^C}\pi^C)(Y^C)\pi^V(Z^C) \\
& + \pi^C(Y^C)(\nabla^C_{X^C}\pi^V)(Z^C) \\
& - 2g^C(Y^V, Z^C)(\nabla^C_{\pi^C}(P^C) \\
& - 2g^C(Y^V, Z^C)\pi^C(\nabla^C_{X^C}P^C) \\
& - 2g^C(Y^C, Z^C)(\nabla^C_{\pi^V X^C}(P^C) \\
& - 2g^C(Y^C, Z^C)\pi^V(\nabla^C_{X^C}P^C)] \qquad (5.8)
\end{aligned}
$$

which becomes

$$
\begin{aligned}
(\nabla^C_{X^C} S^C)(Y^C, Z^C) = & \frac{n-1}{4}\{2\pi^V(X^C)\pi^C(Y^C)\pi^C(Z^C) \\
& + 2\pi^C(X^C)\pi^V(Y^C)\pi^C(Z^C) \\
& + 2\pi^C(X^C)\pi^C(Y^C)\pi^V(Z^C) \\
& - [\pi^C(Y^C)g^C(X^V, Z^C) \\
& + \pi^V(Y^C)g^C(X^C, Z^C) \\
& - \pi^C(X^C)g^C(Y^V, Z^C) \\
& - \pi^V(X^C)g^C(Y^C, Z^C)]\} \qquad (5.9)
\end{aligned}
$$

where equation (5.2) is used.

A Riemannian manifold (M_n, g) of dimension n and TM_n its tangent bundle. Then tangent bundle TM_n is said to be Ricci symmetric if and only if $\nabla^C S^C=0$. If possible, we suppose that the group manifold (M_n, g) in TM_n is Ricci-symmetric, and then the last equation gives $\pi^C(P^V) = 0$ and $\pi^V(P^C) = 0$. Hence, the Theorem 5.3 is proved. □

Theorem 5.5. *Let (M_n, g) be an $n(> 1)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Suppose (M_n, g) is a group manifold in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. A Ricci soliton (g^C, P^C, λ) on (TM_n, g^C) to be shrinking, steady, and expanding according as $\pi^C(P^V)$ and $\pi^V(P^C)$ are $<, =,$ and > 0, respectively.*

Proof. If (M_n, g) is a group manifold in TM_n equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then equation (5.2) and Theorem 3.5 give

$$\begin{aligned}(\pounds_P g)^C(X^C, Y^C) &= 2\{\pi^V(Y^C)\pi^C(X^C) \\ &+ \pi^C(Y^C)\pi^V(X^C) - \pi^C(P^C)g^C(X^V, Y^C) \\ &- \pi^V(P^C)g^C(X^C, Y^C)\}\end{aligned} \tag{5.10}$$

for arbitrary vector fields X^C and Y^C on TM_n. A triplet (g^C, P^C, λ) on an n-dimensional Riemannian manifold (M_n, g) in TM_n is said to be a Ricci soliton if it satisfies the relation

$$(\pounds_V g)^C + 2S^C + 2\lambda g^C = 0, \tag{5.11}$$

where $\pounds_V g + 2S + 2\lambda g = 0$ and V is a complete vector field on M_n and λ is a real constant [11]. A Ricci soliton (g^C, P^C, λ) on (TM_n, g^C) is said to be shrinking, steady, and expanding if λ is negative, zero, and positive, respectively. Changing V with P^C in equation (5.11) and then using equations (5.4) and (5.10), then the obtained equation is

$$\begin{aligned}(n-3)\{\pi^V(X^C)\pi^C(Y^C) &+ \pi^C(X^C)\pi^V(Y^C)\} \\ &- 2(n+1)\{\pi^C(P^C)g^C(X^V, Y^C) \\ &+ \pi^V(P^C)g^C(X^C, Y^C)\} \\ &+ 4\lambda g^C(X^C, Y^C) = 0\end{aligned} \tag{5.12}$$

for arbitrary vector fields X^C and Y^C on TM_n.

Setting $Y^C = P^C$ in equation (5.12), then

$$\{\lambda - \frac{n-1}{4}\pi^C(P^C)\}\pi^C(X^V) + \{\lambda - \frac{n-1}{4}\pi^V(P^C)\}\pi^C(X^C) = 0,$$

$$\{\lambda - \frac{n-1}{4}(\pi(BP))^C\}\pi^C(X^V) + \{\lambda - \frac{n-1}{4}(\pi(BP))^V\}\pi^C(X^C) = 0,$$

which shows that $\lambda = \frac{n-1}{4}(\pi(BP))^C$ and $\lambda = \frac{n-1}{4}(\pi(BP))^V$, because $\pi^C(X^V) \neq 0$ and $\pi^C(X^C) \neq 0$ on TM_n (in general). In view of the last expression, it can easily observe that the Ricci soliton (g^C, P^C, λ) on TM_n is shrinking, steady, and expanding if $\pi(BP) <, =$ and > 0, respectively. Thus, Theorem 5.4 is satisfied. □

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