FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 55–71

A D-PEARSON EQUATION FOR DUNKL-CLASSICAL ORTHOGONAL POLYNOMIALS

B. Bouras, J. Alaya, Y. Habbachi

Abstract. In this paper, we show that a monic orthogonal polynomial sequence is a Dunkl-classical sequence if and only if it belongs to a particular family of D-semiclassical polynomial sequence of class less or equal to two. In this case, the distributional equation fulfilled by the linear functionals corresponding to these polynomials is given. Some well known results in the literature are generalized.

Keywords: Orthogonal polynomials, Dunkl-classical polynomials, Regular linear functionals, D-semiclassical polynomials.

1. Introduction

A monic orthogonal polynomial sequence (MOPS, for shorter) $\{P_n\}_{n\geq 0}$ is called Dunkl-classical polynomial sequence (the associated linear functional is called Dunkl-classical linear functional) if $\{T_\mu P_n\}_{n\geq 1}$ is an orthogonal polynomial sequence, where T_μ is the Dunkl operator [6] : $T_\mu = D + 2\mu H_{-1}$, $\mu > -\frac{1}{2}$, D (resp. H_{-1}) denotes the derivative operator $D = \frac{d}{dx}$ (resp. the Hahn operator given by $(H_{-1}f)(x) = \frac{f(x)-f(-x)}{2x}$).

Y. Ben Cheikh and his coworker [1] introduced the notion of Dunkl-classical orthogonal polynomials and proved that the only symmetric Dunkl-classical orthogonal polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Note that both of them are D-semiclassical squences of class less or equal to two (see [2][4]). Later on, M. Sghaier [10] find a non-symmetric sequence of Dunkl-classical polynomials. This sequence is also D-semiclassical, since it is obtained by multiplying the generalized Gegenbauer linear functional by a polynomial of first degree [5].
It is natural, then, to ask if all Dunkl-classical orthogonal polynomials are D-semiclassical.

The aim of this paper is to answer this question. Namely, we prove the following result:

Received December 26, 2014; Accepted August 03, 2015
2010 *Mathematics Subject Classification.* Primary 33C45; Secondary 42C05

Theorem 1.1. *Let u_0 be a regular linear functional and let $\{P_n\}_{n\geq 0}$ be its corresponding MOPS. Then u_0 is a Dunkl-classical form if and only if there exist two polynomials Φ (monic) and B with $degB = 1 + deg\Phi \leq 3$ such that*

$$D\Big(\big(x^2\Phi(x) + 2\mu x B(x)\big)u_0\Big) + \Big((2\mu^2 - \mu - 2)x\Phi(x) - (2\mu^2 + 3\mu)B(x) + \frac{1-4\mu^2}{K}x^2\Psi(x)\Big)u_0 = 0, \tag{1.1}$$

$$\Psi'(0) + \frac{1}{2}\frac{K\Phi''(0)}{1-4\mu^2}(4\mu^2[n] - n) + \frac{1}{3}\frac{KB'''(0)}{(1-4\mu^2)}\mu([n] - n) \neq 0, \tag{1.2}$$

$$x\Phi(x)u_0 = h_{-1}(B(x)u_0), \tag{1.3}$$

where

$$\Psi(x) = \frac{1+2\mu}{\gamma_1}P_1, \tag{1.4}$$

$$K = \frac{1+2\mu}{< u_0, \Phi >}. \tag{1.5}$$

The structure of this paper is as follows: Section 2 is devoted to preliminary results and notations to be used in the sequel. In Section 3, we prove the main theorem. In Section 4, we illustrate 1.1 by analyzing some examples of D-semiclassical linear functionals which are Dunkl-classical.

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u. Let $h_a u$, gu and $Du = u'$ linear functionals defined by duality

$$\begin{aligned} \langle h_a u, f(x)\rangle &= \langle u, (h_a f)(x)\rangle = \langle u, f(ax)\rangle,\ f \in \mathcal{P},\ a \in \mathbb{C}\setminus\{0\}, \\ \langle gu, f(x)\rangle &= \langle u, g(x)f(x)\rangle,\ \langle Du, f(x)\rangle = -\langle u, f'(x)\rangle,\ f, g \in \mathcal{P}. \end{aligned}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$(uf)(x) =< u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} > .$$

The division of a linear functional by a polynomial of first degree is given by

$$\Big\langle (x-c)^{-1}u, f\Big\rangle = \langle u, \theta_c f\rangle, c \in \mathbb{C}, f \in \mathcal{P}, u \in \mathcal{P}',$$

where

$$(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.$$

It is easy to see that

$$fDu = D(fu) - f'u,\ f \in \mathcal{P},\ u \in \mathcal{P}' \tag{2.1}$$

and

$$x^{-1}(xu) = u - (u)_0\delta_0,\ u \in \mathcal{P}' \tag{2.2}$$

where $\delta_c,\ c \in \mathbb{C}$ is the Dirac linear functional defined by

$$\langle \delta_c, f\rangle = f(c),\ f \in \mathcal{P}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $degP_n = n,\ n \geq 0$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ and defined by $< u_n, P_m >= \delta_{n,m},\ n,\ m \geq 0$. Let us recall some result [7]

Lemma 2.1. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*
(*i*) $< u, P_{m-1} >\neq 0,\ < u, P_n >= 0,\ n \geq m.$

(*ii*) $\exists\, \lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$ *such that* $u = \displaystyle\sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$

The linear functional u is called regular if there exists a polynomial sequence (PS, in short) $\{P_n\}_{n\geq 0}$ such that [4]:

$$< u, P_mP_n >= r_n\delta_{n,m},\ n,\ m \geq 0,\ r_n \neq 0,\ n \geq 0. \tag{2.3}$$

The sequence $\{P_n\}_{n\geq 0}$ is then called orthogonal with respect to u. In this case, we have $u_n = r_n^{-1}P_nu_0,\ n \geq 0$. According to the previous lemma, we have $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$. In what follows all regular linear functionals u will be taken normalized i.e, $(u)_0 = 1$.
According to Favard's theorem, a monic orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is characterized by the following three-term recurrence relation [4]:

$$\begin{aligned} &P_0(x) = 1,\ P_1(x) = x - \beta_0 \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{aligned} \tag{2.4}$$

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}\,, n \geq 0.$

The first associated of $\{P_n\}_{n\geq 0}$ is the MOPS $\{P_n^{(1)}\}_{n\geq 0}$ defined by

$$\begin{aligned} &P_0^{(1)}(x) = 1, P_1^{(1)}(x) = x - \beta_1, \\ &P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), n \geq 0. \end{aligned} \tag{2.5}$$

Definition 2.1. (see [4][8]) A linear functional u is called D-semiclassical of class s if it is regular and the following statement holds: There exist two polynomials Ψ of degree $p \geq 1$ and Φ of degree $t \geq 0$, such that

$$(\Phi u)' + \Psi u = 0, \tag{2.6}$$

$$\prod_{c\in Z_\Phi}\Big(|\Psi(c)+\Phi'(c)|+\Big|\Big\langle u,\theta_c\Psi+\theta_c^2\Phi\Big\rangle\Big|\Big)>0, \tag{2.7}$$

where Z_Φ is the set of zeros of Φ.
The class of u is given by $s = max(p-1, t-2)$. The sequence $\{P_n\}_{n\geq 0}$ corresponding to u is called D-semiclassical of class s.
When $s = 0$, the linear functional u (or the sequence $\{P_n\}_{n\geq 0}$) is said to be D-classical.

Let us introduce the Dunkl's operator

$$(T_\mu f)(x) = f'(x) + 2\mu(H_{-1}f)(x),\ f \in \mathcal{P},$$

where

$$(H_{-1}f)(x) = \frac{f(x)-f(-x)}{2x}.$$

We define the operator T_μ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows

$$< T_\mu u, f(x) >= -< u, (T_\mu f)(x) >,\ f \in \mathcal{P},\ u \in \mathcal{P}'.$$

In particular, this yields

$$(T_\mu u)_n = -\mu_n (u)_{n-1},\ n \geq 0,$$

where

$$(u)_{-1} = 0,\ \mu_n = n + 2\mu[n],\ [n] = \frac{1-(-1)^n}{2},\ n \geq 0.$$

It is easy to see that

$$T_\mu u = Du + 2\mu H_{-1}u,$$

where

$$< H_{-1}u, f(x) >= -< u, (H_{-1}f)(x) > .$$

Now, consider a MOPS $\{P_n\}_{n\geq 0}$ as above and let

$$P_n^{[1]}(x) = \frac{1}{\mu_{n+1}}(T_\mu P_{n+1})(x),\ \mu \neq -n-\frac{1}{2},\ n \geq 0.$$

Let denote by $\{u_n^{[1]}\}_{n\geq 0}$, the dual sequence of $\{P_n^{[1]}\}_{n\geq 0}$.

Lemma 2.2. *[10]*

$$T_\mu u_n^{[1]} = -\mu_{n+1}u_{n+1},\ n \geq 0. \tag{2.8}$$

3. Proof of the main theorem

For the proof, we need the following lemma:

Lemma 3.1. *The following formula holds*

$$(3.1)\qquad xT_\mu u = xDu - \mu(u + h_{-1}u),\ u \in \mathcal{P}'.$$

$$(3.2)\qquad T_\mu(fu) = fT_\mu u + f'u + 2\mu(H_{-1}f)(h_{-1}u),\ f \in \mathcal{P},\ u \in \mathcal{P}',$$

$$(3.3)\qquad T_\mu(fu) = fT_\mu u + (T_\mu f)u + 2\mu(H_{-1}f)(h_{-1}u - u),\ f \in \mathcal{P},\ u \in \mathcal{P}',$$

Proof of the lemma From the definition of the operator T_μ on $\mathcal{P}'$, we have

$$\begin{aligned} < xT_\mu u, f(x) > &= < xDu, f(x) > +2\mu < xH_{-1}u, f(x) > \\ &= < xDu, f(x) > -2\mu < u, \frac{xf(x) + xf(-x)}{2x} > \\ &= < xDu, f(x) > -\mu < u, f(x) + f(-x) > \\ &= < xDu, f(x) > -\mu\Big(< u, f(x) > + < h_{-1}u, f(x) > \Big) \\ &= < xDu, f(x) > -\mu < u + h_{-1}u, f(x) >, \end{aligned}$$

hence (3.1) follows.
For the proof of (3.2), let $g \in \mathcal{P}$. We have

$$\begin{aligned} < T_\mu(fu), g(x) > &= - < u, f(x)g'(x) + \mu f(x)\frac{g(x) - g(-x)}{x} > \\ &= - < u, (fg)'(x) - f'(x)g(x) + \mu f(x)\frac{g(x) - g(-x)}{x} > \\ &= - < u, T_\mu(fg)(x) - f'(x)g(x) - \mu g(-x)\frac{f(x) - f(-x)}{x} > \\ &= < fT_\mu u + f'u + 2\mu(H_{-1}f)(h_{-1}u), g(x) > . \end{aligned}$$

Thus, we obtain (3.2). From which we derive (3.3)

□

Proof of the main theorem First of all, notice that for $\mu = 0$ we get the D-classical orthogonal polynomial sequences, which are D-semiclassical of class zero. Henceforth, we will suppose that $\mu \neq 0$.
From the assumption we have

$$(3.4)\qquad u_n = r_n^{-1}P_n u_0,\ n \geq 0$$

and

$$(3.5)\qquad u_n^{[1]} = (r_n^{[1]})^{-1}P_n^{[1]}u_0^{[1]},\ n \geq 0.$$

Substitution of (3.4) and (3.5) in (2.8) gives

$$(3.6)\qquad T_\mu(P_n^{[1]}u_0^{[1]}) = -\chi_n P_{n+1}u_0,\ n \geq 0,$$

where

$$(3.7)\qquad \chi_n = \mu_{n+1}\frac{r_n^{[1]}}{r_{n+1}},\ n \geq 0.$$

Using formula (3.3), equation (3.6) becomes

$$P_n^{[1]}T_\mu u_0^{[1]} + (T_\mu P_n^{[1]})u_0^{[1]} + \\ +2\mu(H_{-1}P_n^{[1]})\Big(h_{-1}u_0^{[1]} - u_0^{[1]}\Big) = -\chi_n P_{n+1}u_0,\ n \geq 0. \tag{3.8}$$

For $n = 0$, equation (3.8) becomes

$$T_\mu u_0^{[1]} = -\chi_0 P_1 u_0 = -\frac{1+2\mu}{\gamma_1}P_1 u_0. \tag{3.9}$$

For $n = 1$, equation (3.8) becomes

$$P_1^{[1]}T_\mu u_0^{[1]} + u_0^{[1]} + 2\mu h_{-1}u_0^{[1]} = -2\frac{r_1^{[1]}}{r_2}P_2 u_0. \tag{3.10}$$

Substitution of (3.9) in (3.10) gives

$$u_0^{[1]} + 2\mu h_{-1}u_0^{[1]} = K\Phi u_0, \tag{3.11}$$

where

$$K\Phi = \frac{1+2\mu}{\gamma_1}P_1 P_1^{[1]} - 2\frac{r_1^{[1]}}{r_2}P_2, \tag{3.12}$$

(K is a constant to make Φ monic).
Applying the operator h_{-1} to (3.11), we get

$$2\mu u_0^{[1]} + h_{-1}u_0^{[1]} = Kh_{-1}(\Phi u_0). \tag{3.13}$$

Multiplying (3.13) by 2μ and subtracting the result from (3.11), we get

$$u_0^{[1]} = \frac{K}{1-4\mu^2}(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)). \tag{3.14}$$

Substitution of (3.14) in (3.9) gives

$$T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \frac{1-4\mu^2}{K}\Psi u_0 = 0. \tag{3.15}$$

From (1.4), (3.15) and the regularity of u_0, we have

$$\begin{aligned} 0 &= < T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \frac{1-4\mu^2}{K}\Psi u_0, P_1 > \\ &= \frac{1-4\mu^2}{K}(- < u_0, \Phi > + \frac{1+2\mu}{K}). \end{aligned}$$

Thus, (1.5) follows.

Now, putting $n = 2$ in (3.8), we obtain

$$(3.16)\quad P_2^{[1]} T_\mu u_0^{[1]} + (T_\mu P_2^{[1]}) u_0^{[1]} + \\ +2\mu H_{-1} P_2^{[1]} \Big(h_{-1} u_0^{[1]} - u_0^{[1]} \Big) = -\chi_2 P_3 u_0.$$

Taking into account (3.9) and (3.14), we get

$$\frac{-2\mu K}{1-4\mu^2}\Big(T_\mu P_2^{[1]} - (1+2\mu) H_{-1} P_2^{[1]} \Big) h_{-1}(\Phi u_0) = \\ \Big(\frac{1+2\mu}{\gamma_1} P_1 P_2^{[1]} - \frac{K}{1-4\mu^2} \Phi T_\mu P_2^{[1]} + \frac{2\mu K}{1-2\mu} \Phi H_{-1} P_2^{[1]} - \chi_2 P_3 \Big) u_0.$$

Applying the operator h_{-1} to the last equation and taking into account the fact that

$$(T_\mu P_2^{[1]})(x) - ((1+2\mu) H_{-1} P_2^{[1]})(x) = 2x$$

and the formulas:

$$h_{-1}(xv) = -x h_{-1} v$$

and

$$h_{-1}(h_{-1} v) = v,\ v \in \mathcal{P}',$$

we obtain (1.3), where

$$(3.17)\quad B(x) = \frac{1-4\mu^2}{4\mu K}\Big(\frac{1+2\mu}{\gamma_1} P_1(x) P_2^{[1]}(x) - \frac{K}{1-4\mu^2} \Phi(x) (T_\mu P_2^{[1]})(x) \\ + \frac{2\mu K}{1-2\mu} \Phi(x) (H_{-1} P_2^{[1]})(x) - \chi_2 P_3(x) \Big).$$

Multiplying (3.14) by x and taking into account (1.3), we get

$$(3.18)\quad x u_0^{[1]} = \frac{K}{1-4\mu^2} (x\Phi(x) + 2\mu B(x)) u_0.$$

Applying the operator h_{-1} to the last equation and using again (1.3), we obtain

$$(3.19)\quad x h_{-1} u_0^{[1]} = -\frac{K}{1-4\mu^2} (B(x) + 2\mu x \Phi(x)) u_0.$$

On the other hand, from (3.2) we have

$$(3.20)\quad x T_\mu (u_0^{[1]}) = T_\mu (x u_0^{[1]}) - u_0^{[1]} - 2\mu h_{-1} u_0^{[1]}.$$

Multiplying (3.20) by x and taking into account (3.18) and (3.19), we get

$$(3.21)\quad x^2 T_\mu (u_0^{[1]}) = \frac{K}{1-4\mu^2}\Big(x T_\mu \Big((x\Phi(x) + 2\mu B(x)) u_0 \Big) - (1-4\mu^2) x \Phi(x) u_0 \Big).$$

From (3.1) and (1.3) we get

$$x^2 T_\mu(u_0^{[1]}) = \frac{K}{1-4\mu^2}\Big(xD\Big((x\Phi(x)+2\mu B(x))u_0\Big) + ((2\mu^2-\mu-1)x\Phi(x)-(2\mu^2+\mu)B(x))u_0\Big), \tag{3.22}$$

or, equivalently,

$$x^2 T_\mu(u_0^{[1]}) = \frac{K}{1-4\mu^2}\Big(D\Big(x(x\Phi(x)+2\mu B(x))u_0\Big) + ((2\mu^2-\mu-2)x\Phi(x)-(2\mu^2+3\mu)B(x))u_0\Big). \tag{3.23}$$

According to (3.9) and (1.4), from (3.23) we get (1.1).
Notice that $x^2\Phi(x)+2\mu xB(x)\neq 0$. Indeed, if not then $x\Phi(x)+2\mu B(x)=0$. Therefore, (3.18) becomes $xu_0^{[1]}=0$. This contradicts the regularity of $u_0^{[1]}$. Thus, u_0 (or $\{P_n\}_{n\geq 0}$) is D-semiclassical. Furthermore, by examination of the degrees of polynomials Φ, Ψ and B in (3.12), (1.4) and (3.17) respectively, we can easily see that the class of $\{P_n\}_{n\geq 0}$ is less or equal to two.

Conversely, suppose that u_0 is a linear functional such that (1.1)-(1.5) hold. Using Lemma 2.2, we get (3.9). Substituting (3.9) in (1.1), we obtain (3.22). Putting

$$v = \frac{K}{1-4\mu^2}(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)) \tag{3.24}$$

and using (3.1) and (1.3), we obtain

$$x^2 T_\mu v = \frac{K}{1-4\mu^2}\Big(xT_\mu\Big((x\Phi(x)+2\mu B(x))u_0\Big) - (1-4\mu^2)x\Phi(x)u_0\Big).$$

Therefore, equation (3.22) becomes

$$x^2 T_\mu u_0^{[1]} = x^2 T_\mu v. \tag{3.25}$$

Multiplying (3.25) by x^{-1} and using (2.2), we get

$$xT_\mu u_0^{[1]} - \Big(xT_\mu u_0^{[1]}\Big)_0 \delta_0 = xT_\mu v - \Big(xT_\mu v\Big)_0 \delta_0. \tag{3.26}$$

But, on the one hand we have

$$\Big(xT_\mu u_0^{[1]}\Big)_0 = -(1+2\mu)$$

and, on the other hand, from (3.24) and (1.5), we have

$$\Big(xT_\mu v\Big)_0 = -(1+2\mu)$$

then, (3.26) becomes

$$xT_\mu u_0^{[1]} = xT_\mu v. \tag{3.27}$$

In a similar way, multiplying (3.27) by x^{-1} and using (2.2), we get

$$T_\mu u_0^{[1]} = T_\mu v. \tag{3.28}$$

Hence,

$$v = u_0^{[1]}.$$

Therefore, equation (3.9) becomes

$$T_\mu v + \Psi u_0 = 0. \tag{3.29}$$

Let us prove that the sequence $\{P_n^{[1]}\}_{n\geq 0}$ is orthogonal with respect to v. Let $m \leq n-1$. From (3.2), we have

$$\begin{aligned}\left\langle v, P_m(x)P_n^{[1]}\right\rangle &= -\frac{1}{\mu_{n+1}}\left\langle T_\mu(P_m v), P_{n+1}(x)\right\rangle \\ &= -\frac{1}{\mu_{n+1}}\left\langle P_m T_\mu v + P_m' v + 2\mu H_{-1}P_m h_{-1}v, P_{n+1}\right\rangle.\end{aligned}$$

Taking into account (3.29) and the fact that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0, we get

$$\left\langle v, P_m P_n^{[1]}\right\rangle = -\frac{1}{\mu_{n+1}}\left\langle v, P_{n+1}(x)P_m'(x) + 2\mu(H_{-1}P_m)(x)P_{n+1}(-x)\right\rangle.$$

Using (3.24), the orthogonality of $\{P_n\}_{n\geq 0}$ with respect to u_0 and the fact that $deg\Phi \leq 2$, we obtain

$$\left\langle v, P_m(x)P_n^{[1]}\right\rangle = \frac{2\mu K}{(1-4\mu^2)\mu_{n+1}}\left\langle u_0, \Phi(x)P_{n+1}(-x)\Big(P_m'(-x) - (H_{-1}P_m)(-x)\Big)\right\rangle$$

Writing $P_m(x) = \theta_0 + \theta_1 x + ... + \theta_{m-1}x^{m-1} + x^m$, we can easily see that

$$P_m'(-x) - (H_{-1}P_m)(-x) = xQ(x),$$

where Q is a polynomial of degree less than or equal to $m-1$ (with the convention that the degree of the zero polynomial is $-\infty$). Then,

$$\left\langle v, P_m(x)P_n^{[1]}\right\rangle = \frac{2\mu K}{(1-4\mu^2)\mu_{n+1}}\left\langle u_0, x\Phi(x)Q(x)P_{n+1}(-x)\right\rangle.$$

Application of (1.3) gives

$$\left\langle v, P_m(x)P_n^{[1]}\right\rangle = \frac{2\mu K}{(1-4\mu^2)\mu_{n+1}}\left\langle u_0, B(x)Q(-x)P_{n+1}(x)\right\rangle.$$

Since B is a polynomial of degree less or equal to three then, from the orthogonality of $\{P_n\}_{n\geq 0}$ with respect to u_0, we get

$$\left\langle v, P_m(x)P_n^{[1]}\right\rangle = 0.$$

For $m = n$, a second use of (3.2) gives

$$\left\langle v, P_n(x)P_n^{[1]}\right\rangle = -\frac{1}{\mu_{n+1}}\left\langle P_n T_\mu v + P_n' v + 2\mu H_{-1}P_n h_{-1}v, P_{n+1}\right\rangle. \tag{3.30}$$

Using (3.29) and the fact that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the linear functional u_0, we get

$$\left\langle P_n T_\mu v, P_{n+1}\right\rangle = -\Psi'(0) r_{n+1}, \tag{3.31}$$

where r_{n+1} is given in (2.3).
From (3.24), we obtain

$$\left\langle P_n' v, P_{n+1}\right\rangle = \frac{K}{1-4\mu^2}\left(\frac{1}{2}n\Phi''(0)r_{n+1} - 2\mu\left\langle u_0, \Phi(x)P_n'(-x)P_{n+1}(-x)\right\rangle\right) \tag{3.32}$$

and

$$\left\langle 2\mu H_{-1}P_n h_{-1}v, P_{n+1}\right\rangle = \frac{2\mu K}{1-4\mu^2}\left(\left\langle u_0, \Phi(x)(H_{-1}P_n)(-x)P_{n+1}(-x)\right\rangle - \mu\Phi''(0)[n]r_{n+1}\right). \tag{3.33}$$

Substitution of (3.31), (3.32) and (3.33) in (3.30) gives

$$\begin{aligned}\left\langle v, P_n(x)P_n^{[1]}\right\rangle &= \left(\Psi'(0) + \frac{1}{2}\frac{K\Phi''(0)}{1-4\mu^2}(4\mu^2[n]-n)\right)\frac{r_{n+1}}{\mu_{n+1}} \\ &-\frac{2\mu K}{\mu_{n+1}(1-4\mu^2)}\left\langle u_0, \Phi(x)\Big((H_{-1}P_n)(-x) - P_n'(-x)\Big)P_{n+1}(-x)\right\rangle.\end{aligned} \tag{3.34}$$

Writing $(H_{-1}P_n)(-x) - P_n'(-x) = xQ(x)$, where Q is a polynomial of degree $n-2$ with leading coefficient $(-1)^{n-1}([n]-n)$ for $n \geq 2$ and $Q = 0$ for $n \in \{0,1\}$, and using (1.3), we get

$$\left\langle u_0, \Phi(x)\Big((H_{-1}P_n)(-x) - P_n'(-x)\Big)P_{n+1}(-x)\right\rangle = -\frac{1}{6}B'''(0)([n]-n)r_{n+1},\ n \geq 0.$$

Therefore, (3.34) becomes

$$\left\langle v, P_n(x)P_n^{[1]}\right\rangle = \left(\Psi'(0) + \frac{1}{2}\frac{K\Phi''(0)}{1-4\mu^2}(4\mu^2[n]-n) + \frac{1}{3}\frac{\mu K B'''(0)}{(1-4\mu^2)}([n]-n)\right)\frac{r_{n+1}}{\mu_{n+1}}.$$

On account of condition (1.2), the last equation implies that

$$\left\langle v, P_n(x)P_n^{[1]}\right\rangle \neq 0.$$

So, the sequence $\{P_n^{[1]}\}_{n\geq 0}$ is orthogonal with respect to the linear functional v. □

4. Examples

In order to illustrate Theorem 1.1, we present three Dunkl-classical linear functionals: the generalized Hermite, the generalized Gegenbauer and a non-symmetric Dunkl-classical linear functional.

4.1. Generalized Hermite linear functional

The generalized Hermite linear functional denoted by $\mathcal{H}(\mu)$ satisfies (see [4]):

$$D\Big(x\mathcal{H}(\mu)\Big) + \Big(2x^2 - (2\mu+1)\Big)\mathcal{H}(\mu) = 0. \tag{4.1}$$

The sequence of generalized Hermite polynomials $\{H_n^{(\mu)}\}_{n\geq 0}$ satisfies (2.4) with

$$\beta_n = 0,\ \gamma_{n+1} = \frac{1}{2}\mu_{n+1},\ n \geq 0, \tag{4.2}$$

where the regularity condition is

$$\mu \neq -n - \frac{1}{2},\ n \geq 0. \tag{4.3}$$

The weight function for generalized Hermite polynomials in the positive definite case is given by

$$w(x) = |x|^{2\mu} e^{-x^2},\ -\infty < x < \infty.$$

We will show that $\mathcal{H}(\mu)$ satisfies conditions (1.1) - (1.5).
Multiplying (4.1) by $(1-2\mu)x$ and using (2.1), we get (1.1)

$$\Phi(x) = 1, \tag{4.4}$$

$$B(x) = -x, \tag{4.5}$$

$$\Psi(x) = 2x, \tag{4.6}$$

$$K = 1 + 2\mu. \tag{4.7}$$

On the other hand, since $\mathcal{H}(\mu)$ is a symmetric linear functional, we have

$$\mathcal{H}(\mu) = h_{-1}(\mathcal{H}(\mu)).$$

Multiplying the last equation by x, we get (1.3).
Finally, if we substitute (4.4)-(4.7) in the left hand side of (1.2), then we get

$$\Psi'(0) + \frac{1}{2}\frac{K\Phi''(0)}{1-4\mu^2}(4\mu^2[n]-n) + \frac{1}{3}\frac{KB'''(0)}{(1-4\mu^2)}\mu([n]-n) = 2 \neq 0.$$

Therefore, Theorem 1.1 implies that $\mathcal{H}(\mu)$ is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15) and (1.3), $\mathcal{H}(\mu)$ satisfies the following T_μ-distributional equation:

$$T_\mu\Big(\mathcal{H}(\mu)\Big) + 2x\mathcal{H}(\mu) = 0. \tag{4.8}$$

Notice that Y. Ben Cheikh and M. Gaied [1] have proved differently that $\mathcal{H}(\mu)$ is a Dunkl classical linear functional. But they did not give a T_μ-distributional equation for $\mathcal{H}(\mu)$.

4.2. Generalized Gegenbauer polynomials

The generalized Gegenbauer linear functional denoted by $\mathcal{G}(\alpha,\beta)$ satisfies (see [2]):

$$D\Big(x(x^2-1)\mathcal{G}(\alpha,\beta)\Big)+\Big(-2(\alpha+\beta+2)x^2+2(\beta+1)\Big)\mathcal{G}(\alpha,\beta)=0. \tag{4.9}$$

This linear functional is regular for

$$\alpha\neq -n,\ \beta\neq -n,\ \alpha+\beta\neq -n,\ n\geq 1. \tag{4.10}$$

The weight function for generalized Gegenbauer polynomials in the positive definite case is given by

$$w(x)=|x|^{2\beta+1}(1-x^2)^{\alpha},\ -1<x<1.$$

Putting $\beta=\mu-\frac{1}{2}$ in (4.9, we get

$$D\Big(x(x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})\Big)+\Big(-2(\alpha+\mu+\frac{3}{2})x^2+2\mu+1\Big)\mathcal{G}(\alpha,\mu-\frac{1}{2})=0. \tag{4.11}$$

As in the previous example, multiplying (4.11) by $(1-2\mu)x$ and using the fact that $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is symmetric, we obtain (1.1) and (1.3), where

$$\Phi(x) = x^2-1, \tag{4.12}$$

$$B(x) = -x(x^2-1), \tag{4.13}$$

$$\Psi(x) = (2\alpha+2\mu+3)x, \tag{4.14}$$

$$K = -\frac{(1+2\mu)(\alpha+\mu+\frac{3}{2})}{\alpha+1}. \tag{4.15}$$

The condition (1.2) follows, immediately, from (4.10). Indeed:

$$\Psi'(0)+\frac{1}{2}\frac{K\Phi''(0)}{1-4\mu^2}(4\mu^2[n]-n)+\frac{1}{3}\frac{KB'''(0)}{(1-4\mu^2)}\mu([n]-n)=$$

$$\frac{2\alpha+2\mu+3}{2\alpha+2}(2\alpha+2+2\mu[n]+n)\neq 0,\ n\geq 0.$$

Hence, Theorem 1.1 follows that $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15), the linear functional $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ satisfies

$$T_\mu\Big((x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})\Big)-2(\alpha+1)x\mathcal{G}(\alpha,\mu-\frac{1}{2})=0. \tag{4.16}$$

4.3. An example of non-symmetric Dunkl-classical linear functional

In this subsection, we will construct a non-symmetric Dunkl-classical linear functional by using the following result stated in [9]:
Let L be a regular linear functional and let c and λ be two complex numbers. The linear functional u defined by

$$u = \lambda(x-c)^{-1}L + \delta_c \tag{4.17}$$

is regular, for every complex λ such that the following condition:

$$\lambda \neq 0,\ P_n(c) + \lambda P^{(1)}_{n-1}(c) \neq 0,\ n \geq 1, \tag{4.18}$$

where $\{P_n\}_{n\geq 0}$ is the MOPS corresponding to L. Moreover, if L is a D-semiclassical linear functional satisfying (2.6, then u satisfies

$$(\widetilde{\Phi}u)' + \widetilde{\Psi}u = 0, \tag{4.19}$$

where,

$$\widetilde{\Phi}(x) = (x-c)\Phi(x),\ \widetilde{\Psi}(x) = (x-c)\Psi(x). \tag{4.20}$$

Let us apply this result and take

$$L = \mathcal{G}(\alpha, \mu - \frac{1}{2}),\ c = 1,\ \lambda = \frac{-2\alpha}{2\alpha + 2\mu + 1},\ \alpha \neq 0.$$

We will show that the obtained linear functional u satisfies the conditions (1.1)-(1.5). But, first we will study the regularity of u.

Let $\{S_n\}_{n\geq 0}$ be the MOPS associated with $\mathcal{G}(\alpha, \mu - \frac{1}{2})$. It satisfies the recurrence relation (2.4) with [2]:

$$\beta_n = 0,\ \gamma_{n+1} = \frac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{4(n+\alpha+\mu+\frac{1}{2})(n+\alpha+\mu+\frac{3}{2})},\ \delta_n = \mu(1+(-1)^n),$$
$$n \geq 1, \tag{4.21}$$

where the regularity conditions are

$$\alpha + n \neq 0,\ 2\mu + 2n - 1 \neq 0,\ 2\alpha + 2\mu + 2n - 1 \neq 0,\ n \geq 1. \tag{4.22}$$

Putting

$$\lambda_{2n} = 2^n n! \frac{(1+2\mu)(3+2\mu)...(2n-1+2\mu)}{(1+2\mu+2\alpha)(3+2\mu+2\alpha)...(4n-1+2\mu+2\alpha)}, \tag{4.23}$$

$$\lambda_{2n+1} = 2^n n! \frac{(1+2\mu)(3+2\mu)...(2n+1+2\mu)}{(1+2\mu+2\alpha)(3+2\mu+2\alpha)...(4n+1+2\mu+2\alpha)},\ n \geq 0.$$

From the regularity condition (4.22), we have

$$\lambda_n \neq 0. \tag{4.24}$$

On the other hand, simple computations show that $(\lambda_n)_{n\geq 1}$ satisfies the recurrence:

$$\lambda_{n+2} = \lambda_{n+1} - \gamma_{n+1}\lambda_n,\ n \geq 1. \tag{4.25}$$

Let us prove, by recurrence on n, that

$$\lambda_n = S_n(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_{n-1}(1),\ n \geq 1. \tag{4.26}$$

Using (2.4), (2.5), (4.21) and (4.23), we get

$$\begin{aligned} S_1(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_0(1) &= 1 - \frac{2\alpha}{2\alpha+2\mu+1} \\ &= \frac{1+2\mu}{1+2\mu++2\alpha} \\ &= \lambda_1. \end{aligned}$$

Hence, (4.26) is true for $n = 1$.

Using (2.4), (2.5), (4.21) and (4.23), we obtain

$$\begin{aligned} S_2(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_1(1) &= S_1(1) - \gamma_1 S_0(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_1(1) \\ &= 1 - \frac{1+2\mu}{3+2\mu+2\alpha} - \frac{2\alpha}{2\alpha+2\mu+1} \\ &= 2^1 1! \frac{1+2\mu}{(2\alpha+2\mu+1)(2\alpha+2\mu+3)} \\ &= \lambda_2. \end{aligned}$$

Hence, (4.26) is true for $n = 2$.

Suppose that (4.26) is true until $n+1$, $n \geq 1$ and let us prove it for $n+2$. From (4.25) and the recurrence hypothesis, we have

$$\begin{aligned} \lambda_{n+2} &= \lambda_{n+1} - \gamma_{n+1}\lambda_n \\ &= \left(S_{n+1}(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_n(1)\right) - \gamma_{n+1}\left(S_n(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_{n-1}(1)\right) \\ &= \left(S_{n+1}(1) - \gamma_{n+1} S_n(1)\right) - \frac{2\alpha}{2\alpha+2\mu+1}\left(S^{(1)}_n(1) - \gamma_{n+1} S^{(1)}_{n-1}(1)\right),\ n \geq 1. \end{aligned}$$

Taking into account the recurrences (2.4) and (2.5), we get

$$\lambda_{n+2} = S_{n+2}(1) - \frac{2\alpha}{2\alpha+2\mu+1} S^{(1)}_{n+1}(1),\ n \geq 1.$$

So, (4.26) is true for every nonnegative integer n. Then, according to (4.24), condition (4.18) is fulfilled. Therefore, u is regular.
According to (4.11), (4.19) and (4.20), u satisfies the following D-Pearson equation:

$$D\Big(x(x^2-1)(x-1)u\Big)+(x-1)\Big(-2(\alpha+\mu+\frac{3}{2})x^2+2\mu+1\Big)u=0. \tag{4.27}$$

Let us see when (4.27) can be simplified by $x-1$. From (4.17), we have

$$(u)_2=(u)_1=\frac{1+2\mu}{1+2\mu+2\alpha}.$$

Then,

$$<u,\theta_1^2(x(x^2-1)(x-1))+\theta_1((x-1)(-2(\alpha+\mu+\frac{3}{2})x^2+2\mu+1))>=0.$$

Therefore, we can divide both hand sides of (4.27) by $(x-1)$ taking into account (2.7) does not hold. Thus, we obtain

$$D\Big(x(x^2-1)u\Big)+\Big(-2(\alpha+\mu+1)x^2+x+2\mu+1\Big)u=0. \tag{4.28}$$

Multiplication of the last equation by $(1+2\mu)x$ gives (1.1), where

$$\Phi(x) = (x-1)(x+\frac{1+2\mu}{1-2\mu}), \tag{4.29}$$

$$B(x) = x(x-1)(x-\frac{2\mu+1}{1-2\mu}), \tag{4.30}$$

$$\Psi(x) = \frac{1+2\mu}{\gamma_1}(x-\beta_0), \tag{4.31}$$

$$K = \frac{2\mu-1}{\gamma_1}\beta_0, \tag{4.32}$$

$$\beta_0 = \frac{1+2\mu}{1+2\mu+2\alpha}. \tag{4.33}$$

On the other hand, from (4.17), we have

$$(x-1)u=\frac{-2\alpha}{2\alpha+2\mu+1}\mathcal{G}(\alpha,\mu-\frac{1}{2}). \tag{4.34}$$

Taking into account the fact that $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is symmetric, we get

$$(x-1)u=h_{-1}((x-1)u). \tag{4.35}$$

Multiplying (4.35) by $x(x+\dfrac{1+2\mu}{1-2\mu})$, we obtain (1.3).
According to (4.29)-(4.33), (1.2) is equivalent to the following condition:

$$2\mu+2\alpha+1\neq 2\mu[n]-n,\ n\geq 0.$$

Table 4.1: Coefficients of the T_μ-distributional equation (4.36)

Linear functional	Ω	φ	Restriction
Generalized Hermite $\mathcal{H}(\mu)$	1	$2x$	$\mu \neq -n - \frac{1}{2},\ n \geq 0.$
Generalized Gegenbauer $\mathcal{G}(\alpha, \mu - \frac{1}{2})$	$x^2 - 1$	$-2(\alpha + 1)x$	$\alpha \neq -n, \alpha + \mu - \frac{1}{2} \neq -n,$ $\mu - \frac{1}{2} \neq -n,\ n \geq 1.$
Modified Generalized Gegenbauer $u = \lambda(x-1)^{-1}\mathcal{G}(\alpha, \mu - \frac{1}{2}) + \delta_1,$ $\lambda = \dfrac{-2\alpha}{2\alpha + 2\mu + 1}$	$x^2 - 1$	$-(2\alpha + 2\mu + 1)x$ $+1 + 2\mu$	$\alpha \neq -n, \alpha + \mu - \frac{1}{2} \neq -n,$ $\mu - \frac{1}{2} \neq -n,\ n \geq 1, \alpha \neq 0.$

This last condition is an immediate consequence of (4.22). So, according to Theorem 1.1, u is a Dunkl-classical linear functional. Furthermore, by virtue of (3.15) and (4.35), the linear functional u satisfies

$$T_\mu\Big((x^2 - 1)u\Big) + \Big(- (2\alpha + 2\mu + 1)x + 1 + 2\mu\Big)u = 0.$$

Notice that, for all Dunkl-classical linear functionals discussed before, the T_μ-distributional equation (3.15) is reduced to another one of type

$$T_\mu(\Omega u) + \varphi u = 0. \tag{4.36}$$

where Ω and φ are two polynomials such that $deg\Omega \leq 2,\ deg\varphi \leq 1$. To conclude this paper, we will present a table where we give polynomials Ω and φ in (4.36) for each example of D-semiclassical linear functional previously studied.

REFERENCES

1. Y. Ben Cheikh and M. Gaied: *Characterizations of the Dunkl-classical symmetric orthogonal polynomials*, App. Math. Comput. **187** (2007), 105–114.
2. S. Belmehdi: *Generalized Gegenbauer polynomials*, J. Comput. Appl. Math. **133** (2001), 195–205.
3. B. Bouras: *Some characterizations of Dunkl-classical orthogonal polynomials*, J. Difference Equ. Appl. **20** (2014), no.8, 1240-1257.
4. T. S. Chihara: *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, (1978).
5. J. Dini and P. Maroni: *Sur la multiplication d'une forme linéaire par une fraction rationnelle. Applications aux formes de Laguerre-Hahn*, Ann. Polon. Math. **52** (1990), 175-185.
6. C. F. Dunkl: *Integral kernels reflection group invariance*, Canad. J. Math. **43** (1991), 1213-1227.

7. F. Marcellán, A. Branquinho and J. Petronilho: *Classical orthogonal polynomials: A functional approach*, Acta. Appl. Math. **34** (1994), 283-303.

8. P. Maroni: *Variation around Classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. **48** (1993), 133-155.

9. P. Maroni: *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hungar. **21** (1990), 223-248.

10. M. Sghaier: *A note on Dunkl-classical orthogonal polynomials*, Integral Transforms Spec. Funct. **24** (2012), 753-760.

Belgacem Bouras
Faculty of Science
Department of Mathematics
Qassim University
P.O. Box 6644
51452, Buraidah
Kingdom of Saudi Arabia
Belgacem.Bouras@issatgb.rnu.tn

Jilani Alaya
Faculty of Science
Department of Mathematics
Qassim University
P.O. Box 6644
51452, Buraidah
Kingdom of Saudi Arabia
alayajilani@yahoo.fr

Yahia Habbachi
Faculty of Science
Department of Mathematics
Gabès University
Tunisia
habbachiyahia@yahoo.fr