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ON THE TRAJECTORIES OF STOCHASTIC FLOW GENERATED BY THE NATURAL MODEL IN MULTI-DIMENSIONAL CASE

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Abstract. Based on the same model stated in [3], we will study the differentiability of the stochastic flow generated by the natural model with respect to the initial data, based on an important idea of H-Kunita, R.M-Dudley and F-Ledrappier. This is the main motivation of our research.
Keywords: Sample path properties, stochastic flow, stochastic integrals

1. Introduction

The notion of the stochastic flow generated by a stochastic differential equation has been studied by several authors. For the differentiability of the stochastic flow, T-Fujiwara and H-Kunita [13] studied the differentiability of stochastic flows for stochastic differential equations with jumps then H-Kunita [6] demonstrated the differentiability of the stochastic flows with respect to the initial data for stochastic differential equations with smooth coefficients. Malliavin [14] demonstrated the differentiability of the solutions of stochastic differential equations according to the initial conditions for classical type equations on manifolds.

Recently, studies concerning the differentiability of the stochastic flow generated by the stochastic differential equations have been developed. A-Y-Pilipenko [15] demonstrated the differentiability of the solution of stochastic differential equations with reflection in the Sobolev space and he showed in [16] the same result

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but with Lipschitz continuous coefficients. In another work, he proved in collaboration with O-V-Aryasova [17] the differentiability of stochastic flow for stochastic differential equations with discontinuous drift in multidimensional case. In [18] K-Burdzy proved the differentiability of stochastic flow of reflected Brownian motions with respect to the initial data in a smooth multidimensional domain. A-Stefano [19] showed the differentiability of the solution for stochastic differential equations with discontinues drift in one-dimensional case. X-Zhang [20] obtained the differentiability of stochastic flow for stochastic differential equations without global Lipschitz coefficients. E-Fedrizzi and F-Flandoli [21] obtained weakly differentiable of solutions of stochastic differential equations with Non-regular drift. Qian Lin [22] studied the differentiability of the solutions of stochastic differential equations driven by G-Brownian motion with respect to the initial data and the parameter. S-Mohammed, T-Nilssen and F-Proske [23] demonstrated the differentiability of stochastic flow for stochastic differential equations with singular coefficients in the Sobolev sense.

In our paper, we consider a following stochastic differential equation:

$$(\natural_u) = \begin{cases} dX^x_{u,t} = X^x_{u,t}\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}N_t + f(X_t-(1-Z_t))dY_t\right), \ t \in [u,\infty[, \\ X^x_{u,u} = x, \end{cases}$$

where x is the initial condition.

This equation is called $\natural$-equation indicated in ([1],[3][5][24]),which is the priceless system in financial mathematics and it's one of the best ways to represent the evolution of a financial market after the default time, it's considered a prosperous system of parameters (Z, Y, f). the parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ.

Let's move to the multidimensional version of $\natural-$equation [3]. On a probability space $(\Omega, (\mathbb{F})_{t\geq 0}, \mathbb{P})$, we have:

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x)-(1-Z_t))dY_t\right), \ t \in [u,\infty[, \\ X_{u,u}(x) = x, \end{cases}$$

where $(\Lambda^1, ..., \Lambda^d)$ is d-dimensional is continuous increasing process null at the origin, $N_t = (N^1, ..., N^d)$ is a given $d-$dimensional continuous non-negative local martingale such that $0 < Z_t = N_t e^{-\Lambda_t} < 1, t > 0$ and $(Z(t,w) = (Z^1(t,w), ..., Z^d(t,w))$ presents the default intensity. $(Y(t,w) = (Y^1(t,w), ..., Y^n(t,w))$ is a given $n-$dimensional continuous local martingale and $F = (F_1, ..., F_n)$ on $\mathbb{R}^n$ is Lipschitz mapping null at the origin.

This equation has a unique solution $X_{u,t}(x)$ such as;

$$X^u_t = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s \sum_{i=1}^{d}\sum_{j=1}^{n} F^{ij}(X_s-(1-Z_s))dY^j_s, \ s \in [u,t]$$

where $X_u^u = x$ is the initial condition and F^{ij} is $i-th$ component of the vector function F^j.

The aim of this paper is to show the differentiability of the process X_t^u with respect to the initial value, this property was studied for several stochastic differential equations under different conditions like H-Kunita [6], Bismut [14], Malliavin [14], K.D.Elworthy and Z.Brzezniak [9]. Our paper is based mainly on an idea of R.M-Dudley, H-Kunita and F-Ledrappier [12], such that:

- We demonstrate the existence of the partial derivative for any s, t, x a.s if our stochastic flow generated by the $\natural$-equation in multidimensional case, has a continuous extension at $y = 0$ for any s, t, x a.s and this follows from the estimate given by the proposition of H.Kunita and also the Kolmogorov's theorem. Without forgetting the use of the usual estimation inequalities: Hölder Inequality, BDG inequality, and Gronwall's lemma. This means that the solution is continuously differentiable and the derivative is Hölder continuous.
- We assume the following hypothesis: the coefficients of $\natural-$equation are continuous and the processes represented in this equation take real values.

The rest of the paper is organized as follows: the second section contains generalities which we will need in what follows, the third section represents the obtained results about the differentiability of stochastic flow and the last section gives the main result of this paper.

2. Generalities

Theorem 2.1. *(BDG Inequality)*[11]. *Let $T > 0$ and ξ be a continuous local martingale such that $\xi_0 = 0$. For any $1 \leq p < \infty$ there exists positive constants c_p, C_p independent of T and $(\xi_t)_{0\leq t\leq T}$ such that,*

$$c_p\mathbb{E}[< \xi >_T^{p/2}] \leq \mathbb{E}[(\xi_t^*)^p] \leq C_p\mathbb{E}[< \xi >_T^{p/2}]$$

where $\xi_t^* = \sup_{0\leq t\leq T} |\xi_t|$.

Theorem 2.2. *(Hölder Inequality)*[11]. *Let $1 \leq p, q \leq \infty$ so that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ are Lebesgue measurable. Then*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

Proposition 2.1. [6] *Let $2 \leq p < \infty$. There exists a constant R such that, for any $(s, x), (s', x')$ belonging to $[0, T] \times \mathbb{R}^n$,*

$$\mathbb{E}\left[\sup_{s\leq t\leq T} |\xi_{s,t}^x - \xi_{s',t}^{x'}|^p\right] \leq R\left(|x - x'|^p + |s - s'|^{\frac{p}{2}}(1 + |x'|^p)\right) \tag{2.1}$$

Theorem 2.3. *(Kolmogorov's theorem)*[11]. *Let $\xi_\lambda(w)$ be a real valued random field with parameter $\lambda = (\lambda_1, ..., \lambda_d) \in [0,1]^d$. Suppose that there are constants $\gamma > 0$, $\alpha_i > d$, $i = 1, ..., d$ and $C > 0$ such that*

$$\mathbb{E}\left[|\xi_\lambda - \xi_\mu|^\gamma\right] \leq C \sum_{i=1}^{d} |\lambda_i - \mu_i|^{\alpha_i} \ \forall \lambda, \mu \in [0,1]^d. \tag{2.2}$$

Then ξ_λ has a continuous modification $\widetilde{\xi_\lambda}$.

We need also the following importan lemma.

Lemma 2.1. *(Gronwall's lemma)*[11]. *Let $(a,b) \in \mathbb{R}^2$ with $a < b$, φ, β and $\phi : [a,b] \to \mathbb{R}$ non-negative continuous functions, such that $\forall t \in [a,b]$,*

$$\varphi(t) \leq \beta(t) + \int_a^t \varphi(s)\phi(s)ds$$

Then,

$$\forall t \in [a,b], \varphi(t) \leq \beta(t) \exp\left(\int_a^t \phi(s)ds\right)$$

Lemma 2.2. [11] *Let $T > 0$ and p be any real number. Then there is a positive constant $C_{p,T}$ such that $\forall x, y \in \mathbb{R}^d$ and $\forall t \in [0,T]$,*

$$\mathbb{E}\left|J_t(x) - J_s(y)\right|^p \leq C_{p,T} \left|x - y\right|^p$$

3. The Found Results on the differentiability of the Solutions of SDE in multi-dimensional case

3.1. The case studied by Olga.V. Aryasova and Andrey.Yu. Pilipenko

This subsection is borrowed from [10]. We consider an SDE of the form:

$$\begin{cases} d\zeta_t(x) = a(\zeta_t(x))dt + dw_t, \\ \zeta_0(x) = x, \end{cases}$$

Where $x \in \mathbb{R}^d$,$d \geq 1$,$(w_t)_{t\geq 0}$ is a d-dimensional Wiener process, $a = (a^1, ..., a^d)$ is a bounded measurable mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$, this equation has a unique strong solution. The differentiability of this solution with respect to initial data is given in the following theorem.

Theorem 3.1. *Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be such that for all $1 \leq i \leq d$, a^i is a function of bounded variation on $\mathbb{R}^d$. Put $\mu^{ij} = \dfrac{\partial a^i}{\partial x_j}$, and assume that the measures $|\mu^{ij}|, 1 \leq$*

$i, j \leq d$, belong to Kato's class. Let $\phi_t(x), t \geq 0$, be a solution to the integral equation

$$\phi_t(x) = E + \int_0^t dA_s(\zeta(x))\phi_s(x), \tag{3.1}$$

where E is $d \times d-$identity matrix, the integral on the right-hand side of (3.1) is the Lebesgue-Stieltjes integral with respect to the continuous function of bounded variation $t \to A_t(\zeta(x))$. Then $\phi_t(x)$ is the derivative of $\zeta_t(x)$ in L^p-sense, for all $p > 0$, $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d$, $t > 0$:

$$\mathbb{E}\left\|\frac{\zeta_t(x+h) - \zeta_t(x)}{\epsilon} - \phi_t(x)h\right\|^p \to 0,\ \epsilon \to 0, \tag{3.2}$$

where $\|.\|$ is a norm in the space $\mathbb{R}^d$. Moreover:

$$\mathbb{P}\{\forall t \geq 0 : \zeta_t(.) \in W^1_{p,loc}(\mathbb{R}^d, \mathbb{R}^d),\ \nabla\zeta_t(x) = \phi_t(x)\, for\ \lambda - a.a.x\} = 1,$$

where λ is the Lebesgue measure on $\mathbb{R}^d$.

3.2. The case studied by Philip E. Protter

This subsection is borrowed from [11]. Consider a following system:

$$D : \begin{cases} \varphi_t^i = x_i + \displaystyle\sum_{\alpha=1}^m \int_0^t f_\alpha^i(\varphi_{s^-})dZ_s^\alpha \\ D_{kt}^i = \delta_k^i + \displaystyle\sum_{\alpha=1}^m \sum_{j=1}^n \int_0^t \frac{\partial f_\alpha^i}{\partial x_j}(\varphi_{s^-})D_{ks}^j dZ_s^\alpha \end{cases}$$

($1 \leq i \leq n$) where D denotes an $n \times n$ matrix-valued process and $\delta_k^i = 1$ if $i = k$ and 0 otherwise (Kronecker's delta). A convenient convention, sometimes called the Einstein convention, is to leave the summations implicit. Thus, the system of equations (D) can be alternatively written as:

$$D : \begin{cases} \varphi_t^i = x_i + \displaystyle\int_0^t f_\alpha^i(\varphi_{s^-})dZ_s^\alpha \\ D_{kt}^i = \delta_k^i + \displaystyle\int_0^t \frac{\partial f_\alpha^i}{\partial x_j}(\varphi_{s^-})D_{ks}^j dZ_s^\alpha \end{cases}$$

Theorem 3.2. [11] *Let Z be as in (H_1) and let the functions (f_α^i) in (H_2) have locally Lipschitz first partial derivatives. Then for almost all w there exists a function $\varphi(t, w, x)$ which is continuously differentiable in the open set $\{x : \rho(x, w) > t\}$, where ρ is the explosion time. If (f_α^i) are globally Lipschitz then $\rho = \infty$. Let $D_k(t, w, x) \equiv \frac{\partial}{\partial x_k}\varphi(t, w, x)$. Then for each x the process $(\varphi(., w, x), D(., w, x))$ is identically càdlàg, and it is the solution of equations (D) on $[0, \rho(x, .)]$.*

3.3. The case studied by R.M.Dudley, H.Kunita and F.Ledrappier

This subsection is borrowed from [12]. We shall consider an Itô's stochastic differential equation:

$$d\chi_t = \overline{\xi}_0(t,\chi_t)dt + \sum_{k=1}^{m} \overline{\xi}_k(t,\chi_t)dB_t^k \tag{3.3}$$

has a solution χ_t, $t \in [s,T]$ such that for all $x \in \mathbb{R}^d$

$$\chi_t = x + \int_s^t \overline{\xi}_0(t,\chi_t)dt + \sum_{k=1}^{m}\int_s^t \overline{\xi}_k(t,\chi_t)dB_t^k$$

For the convenience of notations, we will often write dt as dB_t^0 and write the last equation as:

$$\chi_t = x + \sum_{k=1}^{m}\int_s^t \overline{\xi}_k(t,\chi_t)dB_t^k$$

where $x = \chi_s$ be initial condition.
The following theorem give the smoothness property of this solution.

Theorem 3.3. [12] *suppose that coefficients $\overline{\xi}^0,...,\overline{\xi}^m$ of an Itô's stochastic differential equation, are globally Lipschitz continuous ($C_g^{1,\alpha}$) functions for some $\alpha > 0$ and their first derivatives are bounded. Then the solution $\chi_{s,t}(x)$ is a $C^{1,\beta}$ of x for any β less than α for each $s < t$ a.s.*

4. Main result

This section contains the main result which is concerning the differentiability of the solution of the natural equation with respect to the initial value. But before that we give a detailed description of the natural equation in multidimensional case, we have:

$$(\natural_u) = \begin{cases} dX_{u,t}^1(x) = X_{u,t}^1(x)\left(-\dfrac{e^{-\Lambda_t^1}}{1-Z_t^1}dN_t^1 + F_{11}dY_t^1 + ... + F_{1d}dY_t^n\right) \\ . \qquad\qquad . \\ . \qquad\qquad . \\ . \qquad\qquad . \\ dX_{u,t}^d(x) = X_{u,t}^d(x)\left(-\dfrac{e^{-\Lambda_t^d}}{1-Z_t^d}dN_t^d + F_{n1}dY_t^1 + ... + F_{nd}dY_t^n\right) \end{cases}$$

Then

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x) - (1-Z_t))dY_t\right), \, t \in [u,\infty[, \\ X_{u,u}(x) = x, \end{cases}$$

where $X_{u,t}(x) = (X^1_{u,t}(x),...,X^d_{u,t}(x))^T, -\frac{e^{-\Lambda_t}}{1-Z_t} = \left(-\frac{e^{-\Lambda^1_t}}{1-Z^1_t},...,-\frac{e^{-\Lambda^d_t}}{1-Z^d_t}\right)^T$, $x = (x^1,...,x^d)^T$ the initial condition and:

$$F = \begin{pmatrix} F_{11} & . & . & . & F_{1d} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F_{n1} & . & . & . & F_{nd} \end{pmatrix}$$

Then we can write the solution X^u_t for $u \leq s \leq t$ in this form:

$$X^u_t = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s\sum_{i=1}^d\sum_{j=1}^n F^{ij}\left(X_s-(1-Z_s)\right)dY^j_s,\ s\in[u,t]$$

We introduce the stopping time $\tau_n = \inf\left\{t, 1-Z_t < \frac{1}{n}\right\}$ on the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ (because we don't know if it's finite or not). Therefore, we assume the process $\widetilde{X}^x_{u,t}$ instead of $X^x_{u,t}$:

$$\widetilde{X^u_t} = x + \int_u^t \widetilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \widetilde{X}_s\sum_{i=1}^d\sum_{j=1}^n F^{ij}(\widetilde{X}_s-(1-Z_s))dY^j_s,\ s\in[u,t]$$

In order to prove the differentibility property, it's enough to apply the idea of R.M.Dudley, H.Kunita and F.Ledrappier [12]: For $y \in \mathbb{R}\setminus 0$, we define

$$\theta_{u,t}(x,y) = \frac{\partial \widetilde{X}^x_{u,t}}{\partial x_k} = \frac{1}{y}\left[\widetilde{X}^{x+ye_k}_{u,t} - \widetilde{X}^x_{u,t}\right]$$

where e_k is the unit vector $(0,...,0,1,0,...,0)$ for $k = 1...d$.
So we will demonstrate that $\theta_{u,t}(x,y)$ has a continuous extension at $y = 0$ for any (u,t,x). Depending on the following estimate and Kolmogorov's theorem, for any $p > 2$, there exits a positive constant C^p such that:

$$\begin{aligned} & \mathbb{E}|\theta_{u,t}(x,y) - \theta_{u',t'}(x',y')|^p \\ \text{(4.1)} \quad \leq\ & C^p\Big[|x-x'|^{\alpha p} + |y-y'|^{\alpha p} + (1+|x|+|x'|)^{\alpha p}(|u-u'|^{\frac{\alpha p}{2}} \\ \text{(4.2)} \quad +\ & |t-t'|^{\frac{\alpha p}{2}})\Big] \end{aligned}$$

Proof: Firstly we show the boundedness of $\mathbb{E}|\theta_{u,t}(x,y)|^p$, we have:

$$\theta_{u,t}(x,y) = \frac{1}{y}\left[\widetilde{X}^{x+ye_k}_{u,t} - \widetilde{X}^x_{u,t}\right]$$

we denote

$$M_t = -\frac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}}$$

$$\widetilde{F}^{ij}(\widetilde{X}_t^{x+ye_k}) = \widetilde{X}_t^{x+ye_k} F^{ij}\left(\widetilde{X}_t^{x+ye_k} - (1-Z_t)\right)$$
$$\widetilde{F}^{ij}(\widetilde{X}_t^{x}) = \widetilde{X}_t^{x} F^{ij}\left(\widetilde{X}_t^{x} - (1-Z_t)\right)$$

So

$$\begin{aligned} \theta_{u,t}(x,y) &= e_k + \frac{1}{y}\left[\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s\right] \\ &+ \frac{1}{y}\left[\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right] \end{aligned} \tag{4.3}$$

Then

$$\begin{aligned} \mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + \frac{1}{y}\mathbb{E}\left|\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s\right|^p \\ &+ \frac{1}{y}\sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left|\int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right|^p \end{aligned} \tag{4.4}$$

Using BDG's inequality, we have:

$$\begin{aligned} \mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + C_1^p\mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^2|M_s|^2 ds\right]^{\frac{p}{2}} \\ &+ C_1^p\frac{1}{y}\sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})|^2 ds\right]^{\frac{p}{2}} \end{aligned} \tag{4.5}$$

Now we apply the hölder inequality, noting q the conjugate of $\frac{p}{2}$:

$$\begin{aligned} &\mathbb{E}|\theta_{u,t}(x,y)|^p \\ \leq\ & 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty}|\theta_{u,t}(x,y)|^p\int_u^t |M_s|^p ds\right] + (t-u)^{\frac{p}{2q}} C_1^p\frac{1}{y} \\ &\times\sum_{i=1}^{d}\sum_{n}^{j=1}\mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})|^p ds\right] \end{aligned} \tag{4.6}$$

And as $\widetilde{F}^{ij}$ is Lipschitz, we have:

$$\left|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s)\right| \leq k_1\left|\widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x}\right|$$

Therefore

$$\begin{aligned} \mathbb{E}\,|\theta_{u,t}(x,y)|^p &\leq 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty}|\theta_{u,t}(x,y)|^p\int_u^t |M_s|^p ds\right] \\ &+ (t-u)^{\frac{p}{2q}} k_1 C_1^p\mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^p ds\right] \end{aligned} \tag{4.7}$$

and by following, we have $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, so:

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^{2p}\right] + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \tag{4.8}$$

Then the proposition(2.1), yields for any $x \in \mathbb{R}^d$ and a constant c':

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}c' + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \tag{4.9}$$

Furthermore, we have the quantity $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] < \infty$. Next, note that $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] = \overline{R}$, then:

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq C_2^p + C_3^p \overline{R}^2 \tag{4.10}$$

where $\frac{1}{2}c'(t-u)^{\frac{p}{2q}} C_1^p = C_2^p$ and $\frac{1}{2}(t-u)^{\frac{p}{2q}} C_1^p = C_3^p$. As a result:

$$\mathbb{E}\,|\theta_{u,t}(x,y)|^p \leq C_4^p + C_5^p \int_u^t \mathbb{E}|\theta_{r,s}(x,y)|^p ds \tag{4.11}$$

Where $C_4^p = C_2^p + C_3^p \overline{R}^2$ and $C_5^p = (t-u)^{\frac{p}{2q}} k_1 C_1^p$, therefore by Gronwall's lemma, we get:

$$\mathbb{E}\,|\theta_{u,t}(x,y)|^p \leq C_4^p \exp\left(C_5^p (t-u)\right) \tag{4.12}$$

Consequently $\mathbb{E}|\theta_{u,t}(x,y)|^p$ is bounded. Secondly we prove the estimate (4.1). In case $t = t'$, we suppose that $u < u' < t$. Other cases will be proven in the same way. Then we have:

$$\begin{aligned}
&\theta_{u,t}(x,y) - \theta_{u',t}(x',y') \\
= &\int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s + \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) \\
- &\widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j
\end{aligned} \tag{4.13}$$

Noting

$$\widetilde{I}_1 = \int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s$$

$$\widetilde{I}_2 = \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j$$

So

$$\mathbb{E}|\widetilde{I}_1|^p = \mathbb{E}\left|\int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y')M_s dN_s\right|^p \tag{4.14}$$

The BDG's inequality leads to:

$$\mathbb{E}|\widetilde{I}_1|^p \leq C_6^p \mathbb{E}\left[\int_u^{u'} |\theta_{r,s}(x,y) - \theta_{r',s}(x',y')|^2 |M_s|^2 ds\right]^{\frac{p}{2}} \tag{4.15}$$

using Hölder's inequality, noting q^* the conjugate of $\frac{p}{2}$:

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_6^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^p \int_u^{u'} |M_s|^p ds\right] \tag{4.16}$$

and by following, we have $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$:

$$\begin{aligned} \mathbb{E}|\widetilde{I}_1|^p &\leq (u'-u)^{\frac{p}{2q^*}} C_7^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^{2p}\right] \\ &+ (u'-u)^{\frac{p}{2q^*}} C_7^p \left[\int_u^{u'} \mathbb{E}|M_s|^p ds\right]^2 \end{aligned} \tag{4.17}$$

Then the proposition (2.1), gives:

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_7^p \left[R_1|y-y'|^{2p} + \overline{R}_1^2\right] \tag{4.18}$$

where $C_7^p = \frac{1}{2}C_6^p$.
it remains to study the term $\widetilde{I}_2$:

$$\begin{aligned} |\widetilde{I}_2| &\leq \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)| \\ &+ |-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})| dY_s^j \end{aligned} \tag{4.19}$$

Using again the BDG's inequality, we obtain:

$$\begin{aligned} &\mathbb{E}|\widetilde{I}_2|^p \\ &\leq \frac{1}{y} C_8^p \sum_{i=1}^d \sum_{j=1}^n \mathbb{E}\left[\int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)|^2 \right. \\ &\left. + |-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})|^2 ds\right]^{\frac{p}{2}} \end{aligned} \tag{4.20}$$

applying Hölder's inequality, noting q^* the conjugate of $\frac{p}{2}$, we have:

$$\begin{aligned}\mathbb{E}|\widetilde{I}_2|^p &\leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^{u'}\mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}_s^x)|^p \\ &+ \mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x'})-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k})|^p ds \end{aligned} \tag{4.21}$$

We have always $\widetilde{F}$ is Lipschitz:

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}k_1\int_u^{u'}\mathbb{E}|\widetilde{X}_s^{x+ye_k}-\widetilde{X}_s^x|^p+\mathbb{E}|\widetilde{X}_s^{x'}-\widetilde{X}_s^{x'+y'e_k}|^p ds \tag{4.22}$$

Thus, by lemma (2.2), we have:

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}K_{p,T}^1 C_8^p\,(u'-u)^{\frac{p}{2q^*}+1}\,k_1\,(|y|^p+|y'|^p) \tag{4.23}$$

From (4.18)and (4.23), we obtain:

$$\mathbb{E}\,|\theta_{u,t}(x,y)-\theta_{u',t}(x',y')|^p \leq C_9^p\,(u'-u)^{\frac{p}{2q^*}} \tag{4.24}$$

Where $C_9^p = C_7^p(R_1|y-y'|^{2p}+\overline{R}_1^2)+\frac{1}{y}K_{p,T}^1C_8^p(u'-u)k_1(|y|^p+|y'|^p)$.
It remains Kolmogorov's theorem, we denote $G=\theta_{u,t}(x,y)-\theta_{u',t'}(x',y')$ and simply applying It$\widehat{o}$'s formula to the function $f(G)=|G|^p$ for $t=t'$, we obtain

$$|G|^p = \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s+\frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

noting

$$\widehat{I} = \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s$$

$$\overline{I} = \frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

such that

$$\begin{aligned}\widehat{I} &= \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)\Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}^{ij}(\widetilde{X}^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}^x) \\ &\quad -\widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^{ij}(\widetilde{X}^{x'})dY_s^j\Big]\end{aligned}$$

Then

$$\begin{aligned}\overline{I} &= \sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)\Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}_l^i(\widetilde{X}^{x+ye_k})-\widetilde{F}_l^i(\widetilde{X}^x) \\ &\quad -\widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k})+\widetilde{F}_l^i(\widetilde{X}^{x'})dY_s^l\Big] \\ &\times \Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}_h^j(\widetilde{X}^{x+ye_k})-\widetilde{F}_h^j(\widetilde{X}^x)-\widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k})+\widetilde{F}_h^j(\widetilde{X}^{x'})dY_s^h\Big]\end{aligned}$$

For $\widehat{I}$, we denote:

$$\frac{\partial f}{\partial G_i}(G) = |p||G|^{P-1}$$

$$\widehat{I}_1 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G) G_s M_s dN_s$$

$$\widehat{I}_2 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G)\frac{1}{y}\widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'})$$

So, we have

$$\sum_i \left|\frac{\partial f}{\partial G_i}(G) G_s\right| \leq d|p||G|^{P-1}|G_s| \tag{4.25}$$

Then

$$|\widehat{I}_1| \leq d|p| \int_u^{u'} |G_s|^P ds \times \int_u^{u'} M_s dN_s \tag{4.26}$$

noting $\varphi_t = \int_u^{u'} M_s dN_s$, it's a local martingale (see [2]):

$$|\widehat{I}_1| \leq d|p|\varphi_t \int_u^{u'} |G_s|^P ds \tag{4.27}$$

And we have $\widetilde{F}^{ij}(\widetilde{X}^x)$ is Lipschitz function, therefore:

$$\sum_i \left|\frac{\partial f}{\partial G_i}(G)\frac{1}{y}\widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'})\right| \leq d\, k_1\, |p|\, |G|^P \tag{4.28}$$

Then

$$|\widehat{I}_2| \leq d\, n\, k_1\, |p| \int_u^{u'} |G_s|^P ds \tag{4.29}$$

From (4.27) and (4.29), we get:

$$|\widehat{I}| \leq d\, |p|(\varphi_t + n\, k_1) \int_u^{u'} |G_s|^P ds \tag{4.30}$$

For $\overline{I}$, we denote

$$\overline{I}_1 = \sum_{i,j,h,l} \int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2 (M_s)^2 dN_s dN_s \tag{4.31}$$

$$\begin{aligned}\overline{I}_2 \;&=\; \frac{1}{y}\sum_{i,j,h,l} \int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G) G_s M_s \widetilde{F}^i_l(\widetilde{X}^{x+ye_k}) \\ &\quad -\widetilde{F}^i_l(\widetilde{X}^{x}) - \widetilde{F}^i_l(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^i_l(\widetilde{X}^{x'}) dN_s dY^l_s\end{aligned} \tag{4.32}$$

$$\overline{I}_3 = \frac{1}{y}\sum_{i,j,h,l}\int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G)G_s M_s \widetilde{F}_h^j(\widetilde{X}^{x+ye_k}) - \widetilde{F}_h^j(\widetilde{X}^x) - \widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_h^j(\widetilde{X}^{x'})dN_s dY_s^h \tag{4.33}$$

$$\begin{aligned}\overline{I}_4 &= \frac{1}{y^2}\sum_{i,j,h,l}\int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G)\left[\widetilde{F}_l^i(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^i(\widetilde{X}^x) - \widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^i(\widetilde{X}^{x'})\right] \\ &\times \left[\widetilde{F}_h^j(\widetilde{X}^{x+ye_k}) - \widetilde{F}_h^j(\widetilde{X}^x) - \widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_h^j(\widetilde{X}^{x'})\right] dY_s^l dY_s^h\end{aligned}$$

And note that

$$\frac{\partial^2 f}{\partial G_i G_j}(G) = p(p-1)|G|^{p-2}$$

Then for $\overline{I}_1$, we have

$$\sum_{i,j,h,l}\left|\frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2\right| \le d\,|p|\,|p-1|\,|G|^{p-2}|G|^2 \tag{4.34}$$

So

$$|\overline{I}_1| \le d\,|p|\,|p-1| \int_u^{u'} |G_s|^p M_s^2 dN_s dN_s \tag{4.35}$$

$\int_u^{u'} M_s dN_s$ is always a local martingale, so

$$|\overline{I}_1| \le d\,|p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{4.36}$$

For $\overline{I}_2$, we have

$$\begin{aligned}&\sum_{i,j,h,l}\frac{1}{y}\left|\frac{\partial^2 f}{\partial G_i G_j}(G)G_s\widetilde{F}_l^i(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^i(\widetilde{X}^x) - \widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^i(\widetilde{X}^{x'})\right| \\ &\le d\,n\,k_1|p|\,|p-1|\,|G|^{p-2}|G_s|^2\end{aligned} \tag{4.37}$$

Therefore we get

$$|\overline{I}_2| \le d\,n\,k_1|p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{4.38}$$

For $\overline{I}_3$, we have

$$|\overline{I}_3| \le d\,n\,k_1|p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{4.39}$$

For $\overline{I}_4$, we have

$$\overline{I}_4 \le d\,n\,k_1^2|p|\,|p-1| \int_u^{u'} |G_s|^p ds \tag{4.40}$$

Then we have

$$\overline{I} = \frac{1}{2}\left[\overline{I}_1 + \overline{I}_2 + \overline{I}_3 + \overline{I}_4\right] \tag{4.41}$$

Such that

$$\overline{I} \leq \frac{1}{2}\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2)\,d\,|p|\,|p-1| \int_u^{u'} |G_s|^p ds \tag{4.42}$$

From these two inequalities (4.30) and (4.42), we get

$$|G|^p \leq d\,|p|\,(\frac{1}{2}\,|p-1|\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2) + \varphi_t + n\,k_1) \int_u^{u'} |G_s|^p ds \tag{4.43}$$

Therefore

$$\mathbb{E}|G|^p \leq C_{10}^p \int_u^{u'} \mathbb{E}|G_s|^p ds \tag{4.44}$$

By Grönwall's inequality we have

$$\mathbb{E}|G|^p \leq C_{11}^p \tag{4.45}$$

where C_{11}^p is $\exp\left(C_{10}^p(u'-u)\right)$.

The proof is completed.

REFERENCES

1. Fatima Benziadi: *The homeomorphic property of the stochastic flow generated by the one-default model in one-dimensional case*, Facta Universitatis'Series Mathematics and Informatics, (in editing).
2. Jeanblanc M. and Song S: *Random times with given survival probability and their F-martingale decomposition formula*, Stochastic Processes and their Applications 121(6) 1389-1410 (2010-2011).
3. Fatima Benziadi and Abdeldjabbar Kandouci: *Homeomorphic Property of the Stochastic Flow of a Natural Equation in Multi-dimensional Case*,Communications on Stochastic Analysis Journal, Vol. 11, No. 4 (2017) 457-478.
4. N.Ikeda and S.Watanabe: *Stochastic differential equations and diffusion process*, 24 (1989),0-444-873783.
5. Fatima Benziadi and Abdeldjabbar Kandouci: *The continuity of the solution of the natural equation in the one-dimensional case*, Communications on Stochastic Analysis Journal Vol. 10, No. 2 (2016) 239-255.
6. H-Kunita: *Stochastic Differential Equations Based on Levy Processes and Stochastic Flows of Diffeomorphisms*,Department of Mathematical Sciences,Nanzan University, Seirei-cho, Seto, 489-0863, Japan, (2004).
7. P-A-Meyer: *flow of stochastic differential equation*,strasbourg,(1881).
8. A. Y. Veretennikov: *On strong solutions and explicit formulas for solutions of stochastic integral equations*, Math. USSR Sborn 39(3) (1981), 387-403.

9. Z.Brzezniak and K.D.Elworthy: *stochastic flows of diffeomorphisms*,(2015).

10. O.V. Aryasova and A.Y. Pilipenko: *On differentiability of stochastic flow for a multidimensional SDE with discontinuous drift*, 19 (2014), 1-17.

11. Philip E. Protter: *Stochatic integration and Differential equations*,(2004), 0172-4568.

12. H-Kunita, R.M-Dudley and F-Ledrappier: *On the stochatic flow*, Berlin Heidelberg New York Tokyo, 1097 (1984).

13. T. Fujiwara and H. Kunita: *Stochastic differential equations of jump type and Levy flows in diffeomorphisms group*, J. Math. Kyoto Univ. 25 (1985), 71-106.

14. P-A-Meyer: *Flow of stochastic differential equation* ,Strasbourg 15 (1881), 103-117.

15. Andrey Pilipenko: *Generalized differentiability with respect to the initial data of a flow generated by a stochastic equation with reflection*,75 (2007),147-160.

16. Andrey Pilipenko: *Differentiability of stochastic reflecting flow with respect to starting point*, 7,1(2013), 17-37.

17. Aryasova and Pilipenko: *On differentiability of stochastic flow for a multidimensional SDE with discontinuous drift*, 19, 45(2014), 1-17.

18. Krzysztof Burdzy: *Differentiability of stochastic flow of reflected Brownian motions*, 14, 75 (2009), 2182-2240.

19. Attanasio-S: *Stochastic flows of diffeomorphisms for one-dimensional SDE with discontinuous drift*, 15 (2010), 213-226.

20. X-Zhang: *The differentiability of stochastic flow for SDEs without global Lipschitz coefficients*,(2010).

21. E-Fedrizzi and F-Flandoli: *Holder Flow and Differentiability for SDEs with Nonregular Drift*, (2012).

22. Qian Lin: *Differentiability of stochastic differential equations driven by the G-Brownian motion*,(2013).

23. S. E-A-Mohammed, T-Nilssen and F-Proske: *Sobolev differentiable stochastic flows for SDEs with singular coefficients applications to the transport equation*, 43, 3 (2015), 1535-1576.

24. Fatima Benziadi and Abdeldjabbar Kandouci:*The application of Kolmogorov's theorem in the one- default model*, Journal of Mathematical Sciences and Applications E-Notes, 4(2016), No. 2, 71-78.