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HORIZONTAL LIFT METRIC ON THE TANGENT BUNDLE OF A WEYL MANIFOLD

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Abstract. Let $(M, [g])$ be a Weyl manifold and TM its tangent bundle equipped with the horizontal lift of the base metric. The purpose of this paper is to study the tangent bundle TM endowed with a Weyl structure, and obtain the ide under which conditions such bundle is an Einstein-Weyl or a gradient Weyl-Ricci soliton.
Keywords: Riemannian metric, Weyl structure, tangent bundle

1. Introduction

Weyl geometry is, in a sense, midway between Riemannian geometry and affine geometry. A Weyl manifold is a conformal manifold equipped with an affine connection preserving the conformal structure, called a Weyl connection. It is said to be Einstein-Weyl if and only if the symmetric part of Ricci tensor is proportional to a Riemannian metric in the conformal class (see [5],[6] and [9]). As a generalization, in [4], the authors introduced a new notion, namely gradient Weyl-Ricci soliton, involving Hessian of a smooth function.

There exists a wide range of interesting studies on the geometry of tangent bundles with special types of metrics (Sasaki, Cheeger-Gromoll,...) or more generally g-natural metrics (see [1],[2] and [7]). A pseudo-Riemannian metric on the tangent bundle is defined by the horizontal lift of the base metric (see [8] and [10]).

Tangent bundle of a Weyl manifold is a very recent topic. In [3], Bejan and Gul constructed a Weyl structure on the tangent bundle and find conditions under which

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the tangent bundle is an Einstein-Weyl manifold. In [4], Bejan *et al.* obtained some conditions such that the Weyl structure on the tangent bundle is a gradient Weyl-Ricci soliton. In both studies, the tangent bundle is considered with the Sasaki metric.

In this paper, we introduce a Weyl structure on the tangent bundle of a Weyl manifold and prove that the tangent bundle cannot be an Einstein-Weyl manifold or a gradient Weyl-Ricci soliton unless the base manifold is locally flat. Here, the tangent bundle is endowed with horizontal lift metric.

Unless otherwise stated, throughout the paper, the Einstein summation convention is used and all geometric objects are considered as smooth.

2. Weyl manifolds

We recall the basic information about Weyl geometry from [3]. Let M be an $m-$dimensional manifold endowed with a conformal class of (pseudo) Riemannian metrics $[g]$. A torsion-free connection D is said to be a Weyl connection if it preserves the conformal class $[g]$. For a metric $g \in [g]$, there exists a 1-form ω determined by D as $Dg = -2\omega \otimes g$. If ∇ is the Levi-Civita connection of g, then D is expressed as follows:

$$D_X Y = \nabla_X Y + \omega(Y)X + \omega(X)Y - g(X,Y)\xi, \ \forall X, Y \in \Gamma(TM), \tag{2.1}$$

where ξ is the dual vector field of ω with respect to g. Conversely, if ω is given and if we use the equation (2.1) to define D, then D is a Weyl connection. Note that we have $g(\xi,\xi) = \|\xi\|^2 = \omega(\xi)$ and the relation (2.1) is invariant under the Weyl transformation $e \to e^{2f}g$, $\omega' = \omega - df$. The pair (g,ω) is called a Weyl structure on M.

Denote by $R_g = [\nabla,\nabla] - \nabla_{[,]}$ and $R_{[g]} = [D,D] - D_{[,]}$ the curvature tensors of the Levi-Civita connection ∇ and the Weyl connection D, respectively. Then the relation between them is given by

$$\begin{aligned} R_{[g]}(X,Y)Z \ = \ & R_g(X,Y)Z + d\omega(X,Y)Z - ((\nabla_Y\omega)(Z))X + ((\nabla_X\omega)(Z))Y \\ & +\omega(Y)\omega(Z)X - g(Y,Z)\nabla_X\xi - g(Y,Z)\omega(\xi)X \\ & +g(Y,Z)\omega(X)\xi - \omega(X)\omega(Z)Y + g(X,Z)\nabla_Y\xi \\ & +g(X,Z)\omega(\xi)Y - g(X,Z)\omega(Y)\xi, \forall X,Y,Z \in \Gamma(TM). \end{aligned} \tag{2.2}$$

From (2.2), the relation between the Ricci tensor field $Ric_{[g]}$ of the Weyl connection D and the Ricci tensor field Ric_g of the Levi-Civita connection ∇ is given by

$$\begin{aligned} Ric_{[g]}(X,Y) \ = \ & Ric_g(X,Y) + d\omega(X,Y) + (\delta\omega - (m-2)\|\xi\|^2)g(X,Y) \\ & -(m-2)(\nabla_X\omega)Y + (m-2)\omega(X)\omega(Y), \ \forall X,Y \in \Gamma(TM), \end{aligned}$$

where the co-differential $\delta\omega$ of ω is defined by $\delta\omega = -tr_g\{(U,V) \to (\nabla_U\omega)V\}$.

The symmetric part $Ric^{sym}_{[g]}$ of $Ric_{[g]}$ is given by following formula:

$$
\begin{aligned}
Ric^{sym}_{[g]}(X,Y) &= Ric_g(X,Y) + (\delta\omega - (m-2)\,\|\xi\|^2)g(X,Y) \\
&\quad -\frac{1}{2}(m-2)[(\nabla_X\omega)Y + (\nabla_Y\omega)X] \\
&\quad +(m-2)\omega(X)\omega(Y), \forall X,Y \in \Gamma(TM).
\end{aligned} \tag{2.3}
$$

3. Tangent bundle

Let M be an $m-$dimensional manifold. Its tangent bundle is denoted by TM and $\pi: TM \to M$ is natural projection mapping. Recall that TM is a $2m-$dimensional differentiable manifold. Let (U, x^j) be a coordinate neighborhood of M, where (x^j) is a system of local coordinates defined in the neighborhood U. Let (u^j) be the system of cartesian coordinates in each tangent space of M with respect to the natural frame $\{\frac{\partial}{\partial x^j}\}$. Then, in $\pi^{-1}(U)$, we can introduce the local coordinates $(\pi^{-1}(U), x^j, u^j)$, which are called the induced coordinates. From now on, we denote the induced coordinates by $(x^J) = (x^j, x^{\bar{j}}) = (x^j, u^j)$, $j = 1, ..., m$, $\bar{j} = m+1, ..., 2m$. We also denote the natural frame in $\pi^{-1}(U)$ by $(\frac{\partial}{\partial x^J}) = (\frac{\partial}{\partial x^j}, \frac{\partial}{\partial u^j})$.

If $X = X^i\frac{\partial}{\partial x^i}$ is the local expression of a vector field X in U, then the vertical lift X^V and the horizontal lift X^H of X are given, with respect to the induced coordinates, by

$$
X^V = X^i\frac{\partial}{\partial u^i}, \quad X^H = X^i\frac{\partial}{\partial x^i} - X^j\Gamma^i_{jk}u^k\frac{\partial}{\partial u^i},
$$

where Γ^i_{jk} are the coefficients of a torsion-free affine connection ∇.

If f is a function on M, then the vertical lift f^V of f is defined by $f^V = f \circ \pi$. The horizontal lift f^H of f is $f^H = 0$.

Let ω be a 1-form on M. Then the horizontal lift ω^H of ω is given by the relations $\omega^H(X^H) = 0$, $\omega^H(X^V) = (\omega(X))^V$. The vertical lift ω^V of ω is given by the relations $\omega^V(X^V) = 0$, $\omega^V(X^H) = (\omega(X))^V$.

From [10], the horizontal lift metric G on the tangent bundle TM over the Riemannian manifold (M, g) is defined by the equations

$$
\begin{aligned}
G(X^H, Y^H) &= G(X^V, Y^V) = 0, \\
G(X^V, Y^H) &= G(X^H, Y^V) = g(X,Y), \forall X,Y \in \Gamma(TM).
\end{aligned} \tag{3.1}
$$

For the Levi-Civita connection $\bar{\nabla}$ of the metric G, we have

$$
\begin{aligned}
\overline{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H + (R_g(u,X)Y)^V, \\
\overline{\nabla}_{X^H}Y^V &= (\nabla_X Y)^V, \\
\overline{\nabla}_{X^V}Y^H &= 0, \\
\overline{\nabla}_{X^V}Y^V &= 0, \ \forall X,Y \in \Gamma(TM),
\end{aligned} \tag{3.2}
$$

where R_g is the curvature tensor field of the metric g. Non-zero components of the curvature tensor $\bar{R}_G$ and the Ricci tensor $\overline{Ric}_G$ are given by

$$\begin{aligned}\bar{R}_G(X^H,Y^H)Z^H &= (R_g(X,Y)Z)^H + ((\nabla_u R_g)(X,Y)Z)^V,\\ \bar{R}_G(X^H,Y^H)Z^V &= \bar{R}_G(X^H,Y^V)Z^H = (R_g(X,Y)Z)^V,\\ \overline{Ric}_G(X^H,Y^H) &= 2Ric_g(X,Y), \forall X,Y \in \Gamma(TM),\end{aligned}$$

where Ric_g is the Ricci tensor field of the metric g (see [8] and [10]).

4. A Weyl structure on tangent bundle

In this section, we construct a Weyl structure on (TM, G) using the vertical lift of a 1-form on M. Firstly, we write the following proposition from the definition of the metric G in (3.1).

Proposition 4.1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift metric G. Any conformal change $g \to e^{2f}g$ on M corresponds the change of the metric $G \to (e^{2f})^V G$ on TM.*

Now we can express the proposition below.

Proposition 4.2. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift metric G. If the pair (g, ω) is a Weyl structure on M, then the pair (G, ω^V) is a Weyl structure on TM and its Weyl connection is given by*

$$\begin{aligned}\overline{D}_{X^H}Y^H &= (D_XY - g(X,Y)\xi)^H + (R_g(u,X)Y)^V,\\ \overline{D}_{X^H}Y^V &= (\nabla_XY + \omega(X)Y)^V - g(X,Y)\xi^H,\\ \overline{D}_{X^V}Y^H &= \omega(Y)X^V - g(X,Y)\xi^H,\\ \overline{D}_{X^V}Y^V &= 0,\end{aligned} \tag{4.1}$$

where D is the Weyl connection on M, R_g is the curvature tensor field of g and ξ is the dual vector field of ω with respect to g.

Proof. Using the relations (3.2) in (2.1) give the result. □

Lemma 4.1. *Let M be an $m-$dimensional manifold $(m > 2)$ endowed with the Weyl structure (g, ω) and TM its tangent bundle endowed with the Weyl structure (G, ω^V), where G is the horizontal lift metric. The symmetric part $\overline{Ric}^{sym}_{[G]}$ of the Ricci tensor field of the Weyl structure (G, ω^V) satisfies the following relations*

$$\begin{aligned}\overline{Ric}^{sym}_{[G]}(X^H,Y^H) &= 2Ric_g(X,Y) - (m-1)[(\nabla_X\omega)Y + (\nabla_Y\omega)X]\\ &\quad +2(m-1)\omega(X)\omega(Y),\end{aligned} \tag{4.2}$$

$$\overline{Ric}^{sym}_{[G]}(X^V,Y^H) = \delta\omega g(X,Y), \tag{4.3}$$

$$\overline{Ric}^{sym}_{[G]}(X^V,Y^V) = 0, \tag{4.4}$$

where ∇ is the Levi-Civita connection on M and Ric_g is the Ricci tensor field of g.

Proof. We use the formula (2.3). Since TM is a $2m-$dimensional manifold, we have

$$\begin{aligned}
\overline{Ric}^{sym}_{[G]}(X^H,Y^H) &= \overline{Ric}_G(X^H,Y^H) \\
&\quad +(\delta(\omega^V)-2(m-1)G(\xi^H,\xi^H))G(X^H,Y^H) \\
&\quad -(m-1)[(\overline{\nabla}_{X^H}\omega^V)Y^H+(\overline{\nabla}_{Y^H}\omega^V)X^H] \\
&\quad +2(m-1)\omega^V(X^H)\omega^V(Y^H) \\
&= 2(Ric_g(X,Y))^V \\
&\quad -(m-1)[((\nabla_X\omega)Y)^V+((\nabla_Y\omega)X)^V] \\
&\quad +2(m-1)[\omega(X)\omega(Y)]^V \\
&= 2Ric_g(X,Y)-(m-1)[(\nabla_X\omega)Y+(\nabla_Y\omega)X] \\
&\quad +2(m-1)\omega(X)\omega(Y).
\end{aligned}$$

By the same way, we obtain (4.3) and (4.4). □

Now we give the main results.

Theorem 4.1. *Let M be an $m-$dimensional manifold $(m>2)$ and TM be its tangent bundle such that M and TM are endowed with the Weyl structures (g,ω) and (G,ω^V), respectively. If the following conditions are satisfied, then TM is an Einstein-Weyl manifold:*

(i) (M,g) is flat.

$$(ii)(\nabla_X\omega)Y+(\nabla_Y\omega)X=2\omega(X)\omega(Y),\forall X,Y\in\Gamma(TM).$$

Proof. It is known that TM is an Einstein-Weyl manifold if there exists a function $\bar{\alpha}$ such that $\overline{Ric}^{sym}_{[G]}=\overline{\alpha}G(\tilde{X},\tilde{Y})$ for all vector fields $\tilde{X},\tilde{Y}$ on TM.

Assume that $(\nabla_X\omega)Y+(\nabla_Y\omega)X=2\omega(X)\omega(Y)$, then (2.3) becomes

$$Ric^{sym}_{[g]}(X,Y)=Ric_g(X,Y)+(\delta\omega-(m-2)\left\|\xi\right\|^2)g(X,Y), \tag{4.5}$$

$\forall X,Y\in\Gamma(TM)$. If we suppose M is flat, i.e. $R_g=0$, then the formulas (4.2), (4.3) and (4.4) reduce to

$$\begin{aligned}
Ric^{sym}_{[G]}(X^H,Y^H) &= 0, \\
Ric^{sym}_{[G]}(X^V,Y^H) &= \delta\omega g(X,Y), \\
Ric^{sym}_{[G]}(X^V,Y^V) &= 0,\forall X,Y\in\Gamma(TM).
\end{aligned}$$

These equations show that if $\bar{\alpha}=(\delta\omega)^V$, then TM is an Einstein-Weyl manifold. This completes the proof. □

Theorem 4.2. *Let M be an $m-$dimensional manifold $(m>2)$ and TM be its tangent bundle such that M and TM are endowed with the Weyl structures (g,ω) and (G,ω^V), respectively. If the following conditions are satisfied, then the triple (G,ω^V,f^V) is a gradient Weyl-Ricci soliton:*

(i) (M, g) *is flat.*

(ii)

$$(4.6)\quad (\nabla_X\omega)Y + (\nabla_Y\omega)X - 2\omega(X)\omega(Y) = Hess_g f(X,Y), \forall X, Y \in \Gamma(TM),$$

where $Hess_g f$ *denotes the Hessian of the function* f *on* M *with respect to the metric* g.

Proof. For (G, ω^V, f^V) to be a gradient Weyl Ricci soliton, it should satisfy

$$\overline{Ric}^{sym}_{[G]} + Hess_G f^V = \overline{\alpha} G, \tag{4.7}$$

where $\bar{\alpha}$ is a function on TM (see [4]).

For the Hessian of the function f^V with respect to G, we get the following relations by direct computations:

$$\begin{aligned} Hess_G f^V(X^H, Y^H) &= (Hess_g f(X,Y))^V, \\ Hess_G f^V(X^H, Y^V) &= 0, \\ Hess_G f^V(X^V, Y^H) &= 0, \\ Hess_G f^V(X^V, Y^V) &= 0, \forall X, Y \in \Gamma(TM). \end{aligned}$$

Suppose that (4.6) holds, then from (2.3) we have

$$\begin{aligned} Ric^{sym}_{[g]}(X,Y) &= Ric_g(X,Y) + (\delta\omega - (m-2)\|\xi\|^2)g(X,Y) \\ &\quad - \frac{(m-2)}{2(m-1)} Hess_g f(X,Y), \forall X, Y \in \Gamma(TM). \end{aligned} \tag{4.8}$$

If (M, g) flat, then the formulas (4.2), (4.3) and (4.4) turn into

$$\begin{aligned} \overline{Ric}^{sym}_{[G]}(X^H, Y^H) &= -Hess_g f(X,Y) \\ \overline{Ric}^{sym}_{[G]}(X^V, Y^H) &= \delta\omega g(X,Y), \\ \overline{Ric}^{sym}_{[G]}(X^V, Y^V) &= 0, \forall X, Y \in \Gamma(TM). \end{aligned}$$

So, for $\bar{\alpha} = (\delta\omega)^V$, TM is a gradient-Weyl Ricci soliton. This completes the proof. □

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