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SOME RESULTS ON YAMABE SOLITONS ON NEARLY HYPERBOLIC SASAKIAN MANIFOLDS

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Abstract. We classify almost Yamabe on nearly hyperbolic Sasakian manifolds whose potential vector field is torse-forming admitting semi-symmetric metric connection and quarter symmetric non-metric connection. Certain results of such solitons on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds with respect to such connection are obtained. Finally, a non-trivial example is given to validate some of our results.
Keywords: Sasakian manifolds, CR-sub-manifolds, Yamabe solitons.

1. Introduction

Much progress has been done in recent years in the study of soliton solutions of the Ricci flow, the mean curvature flow and the Yamabe flow. Soliton solutions correspond to self-similar solutions of the corresponding flow. The Yamabe flow,

$$\frac{\partial}{\partial t}g(t) = -R(t)g(t), \tag{1.1}$$

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where $R(t)$ is the scalar curvature of the metric $g(t)$, was introduced by Hamilton [14], as an approach to solve the Yamabe problem. In dimension $n(=2)$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n>2$ the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric while the Ricci flow does not in general.

A Yamabe soliton on a Riemannian manifold (M,g) of dimension n is a special solution of the Yamabe flow. A triplet structure (g,κ,λ) satisfies

$$\frac{1}{2}\mathfrak{L}_{\kappa}g(X,Y)=(\hat{\delta}-\lambda)g(X,Y) \tag{1.2}$$

for all X, Y on M is known as a Yamabe soliton, where $\mathfrak{L}_{\kappa}$ denotes the Lie derivative of the metric g along the vector field κ, $\hat{\delta}$ is the scalar curvature and λ is a constant. The beauty of such =soliton depends on the the flavor of λ. The soliton is said to be expanding, steady or shrinking, according as $\lambda<0$, $\lambda=0$ or $\lambda>0$ respectively. If $\lambda\in C^{\infty}(M)$, then the metric satisfying (1.2) is called almost Yamabe soliton [2]. Thus the almost Yamabe solitons are the generalization of Yamabe solitons. Moreover, if κ is the gradient of some function $\widetilde{\phi}$ on M then it is known as gradient Yamabe soliton. In context of geometry, the Yamabe solitons are special solution of Yamabe flow under some regulation. There are several geometers that light up quite extensively on the beauty of Yamabe flow and Yamabe soliton (see,[9], [11], [12], [16]).

A vector field κ on a Riemannian manifold (M,g) is known as torse-forming vector field [21] if it satisfies

$$\nabla_X\kappa=\psi X+\theta(X)\kappa,\ \ \forall\ X\in\chi(M), \tag{1.3}$$

where $\psi\in C^{\infty}(M)$ and θ is a 1-form. The beauty of such vector field is as follows:

i) It is concircular if the 1-form θ vanishes identically [20],
ii) For concurrent, $\psi=1$ and $\theta=0$ [22],
iii) It is recurrent if $\psi=0$,
iv) For parallel if $\psi=\theta=0$.

In 2017, Chen [8] initiated a new type vector field known as torqued vector field if the vector field κ satisfying (1.2) with $\theta(\kappa)=0$, where ψ is called torqued function with the 1-form θ is the torqued form of κ.

Bejancu introduced the concept of CR-sub-manifolds of Kähler manifold as a generalization of invariant and anti-invariant sub-manifolds [3]. After that, CR-sub-manifolds of Sasakian manifold was studied by Hsu [15] and Kobayashi [17]. Yano and Kon [23] studied contact CR-sub-manifolds. As per this motivation, several geometers studied CR-sub-manifolds of almost contact manifolds (see, [1],[4],[5],[18]). The almost hyperbolic (f,ξ,η,g)-structure was defined and studied by Upadhyay and Dube [19]. CR-sub-manifolds of trans-hyperbolic Sasakian manifold studied by Bhatt and Dube [6]. Apart from that, Golab [13] introduced the idea of semi-symmetric and quarter symmetric connections. Lovejoy Das et al. [10] studied

CR-sub-manifolds of LP-Sasakian manifold with semi-symmetric non-metric connection. CR-sub-manifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by Siddiqi and Rizvi [1].

The sections of this paper are organized as follows. After introduction, Section 2 contains some definitions and basic results. In Section 3, we recall the notion of semi-symmetric metric connection and quarter symmetric non-metric connection on nearly hyperbolic Sasakian manifold. Section 4 is devoted to CR-sub-manifolds of nearly hyperbolic Sasakian manifolds with respect to semi-symmetric metric connection and quarter symmetric non-metric connection. In Section 5, we study Yamabe soliton whose potential vector field is torse-forming vector field on nearly hyperbolic Sasakian manifold with respect to such connection. Section 6 is concerned with the study of Yamabe soliton with a torse-forming vector field on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds. Furthermore, we study almost Yamabe soliton with torse-forming vector field taking κ^t and κ^n as tangential and normal components of such vector field on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds admitting such connection in Section 7.

2. Preliminaries

Let $\mathbb{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \tag{2.1}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y tangent to $\mathbb{M}$ [7]. As per this consequences

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

where I is the identity of the tangent bundle $T\mathbb{M}$, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is Riemannian metric tensor of $\mathbb{M}$. An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\mathbb{M}$ is called hyperbolic Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

for all tangent vectors X, Y and a Riemannian metric g and Riemannian connection ∇ on $\mathbb{M}$. With reference to (2.4), an almost hyperbolic contact metric manifold $\mathbb{M}$ with (ϕ, ξ, η, g)-structure is called a nearly hyperbolic Sasakian manifold if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X. \tag{2.6}$$

Let $\grave{M}$ be a submanifold immersed in $\mathbb{M}$, the Riemannian metric g induced on $\grave{M}$. Let $T\grave{M}$ and $T^{\perp}\grave{M}$ be the Lie algebra of vector fields tangential to $\grave{M}$ and normal to $\grave{M}$ respectively and $\grave{\nabla}$ be the induced Levi-Civita connection on $\grave{M}$, then the Gauss and Weingarten formulae are given respectively by

$$\nabla_X Y = \grave{\nabla}_X Y + h(X, Y), \ \forall\ X, Y \in T\grave{M}, \tag{2.7}$$

$$\nabla_X N = -A_N X + \nabla^{\perp} N, \ \forall\ N \in T^{\perp}\grave{M}, \tag{2.8}$$

where $\nabla_X Y$ and $\{h(X, Y), \nabla^{\perp}_X N\}$ belong to $T\grave{M}$ and $T^{\perp}\grave{M}$, respectively. The second fundamental form h and Weingarten map A_N associated with N as

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.9}$$

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we can write

$$X = PX + QX, \ \ PX \in \Gamma(D), \ QX \in \Gamma(D^{\perp}), \tag{2.10}$$

$$\phi N = BN + CN, \ \ BN \in \Gamma(D^{\perp}), \ CN \in \Gamma(\mu). \tag{2.11}$$

3. Semi-symmetric Metric Connection and Quarter symmetric non-metric connection

Firstly, we define a semi-symmetric metric connection [13]:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{3.1}$$

such that

$$(\widetilde{\nabla}_X g)(Y, Z) = 0. \tag{3.2}$$

With the help of (2.6) and (3.1), we get

$$(\widetilde{\nabla}_X \phi)Y + \phi(\widetilde{\nabla}_X Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) - g(X, \phi Y)\xi. \tag{3.3}$$

On interchanging X and Y, equation (3.3) reduces to

$$(\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_Y X) = (\nabla_Y \phi)X + \phi(\nabla_Y X) - g(Y, \phi X)\xi, \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$\begin{aligned} &(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_X Y - \nabla_X Y) + \phi(\widetilde{\nabla}_Y X - \nabla_Y X) \\ &= (\nabla_X \phi)Y + (\nabla)_Y \phi)X - g(X, \phi Y)\xi - g(Y, \phi X)\xi. \end{aligned} \tag{3.5}$$

Keeping in mind (2.1), (2.3), (2.6) and (3.1) above equation turn up

$$\begin{aligned} &(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X \\ &= 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X. \end{aligned} \tag{3.6}$$

Also from (2.5) and (3.1), we get

$$\widetilde{\nabla}_X \xi = \phi X - X - \eta(X). \tag{3.7}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure (ϕ, ξ, η, g) is called nearly hyperbolic Sasakian manifold with semi-symmetric metric connection if it bearing (3.5) and (3.6). With the help of (2.7), (2.8) and (3.1) the Gauss and Weingarten formulae on nearly hyperbolic Sasakian manifold with semi-symmetric metric connection as follows

$$\widetilde{\nabla}_X Y = \grave{\nabla}_X Y + h(X, Y), \ \ \forall \ X, Y \in T\grave{M}, \tag{3.8}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp} N, \ \ \forall \ N \in T^{\perp}\grave{M}, \tag{3.9}$$

Also we recall the notion of a quarter symmetric non-metric connection [13] by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X, \tag{3.10}$$

such that

$$(\widehat{\nabla}_X g)(Y, Z) = \eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y). \tag{3.11}$$

From (2.6) and (3.9), we have

$$\begin{aligned} (3.12) \qquad & (\widehat{\nabla}_X \phi)Y + (\widehat{\nabla}_Y \phi)X \\ & = 2g(X, Y)\xi - \eta(X)Y - 2\eta(Y)X - 2\eta(X)\phi Y - 2\eta(X)\eta(Y)\xi. \end{aligned}$$

An almost hyperbolic contact manifold is called nearly hyperbolic Sasakian [7] manifold with quarter symmetric non-metric connection if it satisfies (3.11). Therefore from (2.5) and (3.9), we obtain

$$\widehat{\nabla}_X \xi = 2\phi X. \tag{3.13}$$

Therefore Gauss and Weingarten formulae on nearly hyperbolic Sasakian manifold bearing quarter symmetric non-metric connection are given respectively by

$$\widehat{\nabla}_X Y = \grave{\nabla}_X Y + h(X, Y), \ \ \forall \ X, Y \in T\grave{M}, \tag{3.14}$$

$$\widehat{\nabla}_X N = -A_N X + \nabla^{\perp} N, \ \ \forall \ N \in T^{\perp}\grave{M}, \tag{3.15}$$

4. CR-sub-manifolds of a Nearly hyperbolic Sasakian Manifold

Definition 4.1. [4] An m-dimensional Riemannian submanifold (M, g) of an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that

i) D is invariant under ϕ, i.e., $\phi D \subset D$.

ii) The orthogonal complement distribution $D^{\perp} : x \to D_x^{\perp} \subset T_x M$ of the distribution D on M is totally real, i.e., $\phi D^{\perp} \subset T^{\perp} M$.

If dim $D^{\perp}$=0 (resp., dim D=0), then the CR-submanifold is known as an invariant (resp., anti-invariant) submanifold.

Definition 4.2. [4] If the distribution D $(resp., D^{\perp})$ is horizontal (resp., vertical), then the pair $(D, D^{\perp})$ is called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ (resp., $\xi \in \Gamma(D^{\perp})$). The CR-submanifold is also called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ $(resp., \xi \in \Gamma(D^{\perp}))$.

The orthogonal complement $\phi D^{\perp} \in T^{\perp} M$ is given by

$$TM = D \oplus D^{\perp}, \;\; T^{\perp} M = \phi D^{\perp} \oplus \mu, \tag{4.1}$$

where $\phi\mu$=μ.

Let $\grave{M}$ be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with semi-symmetric metric connection $\widetilde{\nabla}$. The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are given, respectively,

$$\widetilde{\nabla}_X Y = \grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y), \tag{4.2}$$

$$\widetilde{\nabla}_X N = -\widetilde{A}_N X + \widetilde{\nabla}_X^{\perp} N \tag{4.3}$$

for any $X, Y \in \Gamma(T\mathbb{M})$, where $\widetilde{\nabla}_X Y$, $\widetilde{A}_N X \in \Gamma(TM)$. Here $\grave{\widetilde{\nabla}}$, $\widetilde{h}$ and $\widetilde{A}_N$ are called the induced connection on M, the second fundamental form and the Weingarten mapping with respect to $\widetilde{\nabla}$, respectively. In view of (3.7), (3.9) and (4.2), we get

$$\grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y) = \grave{\nabla}_X Y + h(X, Y) + \eta(Y)X - g(X, Y)\xi. \tag{4.4}$$

Using (2.10) and (2.11) in the equation (4.4) and comparing the tangential and normal components on both sides, we obtain

$$P\grave{\widetilde{\nabla}}_X Y = P\grave{\nabla}_X Y + \eta(Y)PX - \alpha g(X, Y)P\xi, \tag{4.5}$$

$$\widetilde{h}(X, Y) = h(X, Y) + \eta(Y)\phi QX, \tag{4.6}$$

$$Q\grave{\widetilde{\nabla}}_X Y = Q\grave{\nabla}_X Y - g(X, Y)Q\xi, \tag{4.7}$$

for any $X, Y \in (T\mathbb{M})$.

Let $\grave{M}$ be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with quarter symmetric metric connection $\widehat{\nabla}$. Then Gauss and Weingarten formulas with respect to $\widehat{\nabla}$ as follows,

$$\widehat{\nabla}_X Y = \grave{\widehat{\nabla}}_X Y + \widehat{h}(X, Y), \tag{4.8}$$

$$\widehat{\nabla}_X N = -\widehat{A}_N X + \widehat{\nabla}^{\perp}_X N \tag{4.9}$$

for any $X, Y \in \Gamma(T\mathbb{M})$, where $\widehat{\nabla}_X Y$, $\widehat{A}_N X \in \Gamma(T\mathbb{M})$. Here $\grave{\widehat{\nabla}}$, $\widehat{h}$ and $\widehat{A}_N$ are called the induced connection on $\mathbb{M}$, the second fundamental form and the Weingarten mapping with respect to $\widehat{\nabla}$, respectively. In view of (3.9), (3.13) and (4.8), we get

$$\widehat{\nabla}_X Y + \widetilde{h}(X, Y) = \grave{\nabla}_X Y + h(X, Y) + \eta(Y)\phi X. \tag{4.10}$$

Using (2.10) and (2.11) in (4.10) and comparing the tangential and normal components on both sides, we obtain

$$P\widehat{\nabla}_X Y = P\grave{\nabla}_X Y + \eta(Y)P\phi X, \tag{4.11}$$

$$\widetilde{h}(X, Y) = h(X, Y), \tag{4.12}$$

$$Q\widehat{\nabla}_X Y = Q\grave{\nabla}_X Y + \eta(Y)Q\phi X, \tag{4.13}$$

for any $X, Y \in (T\mathbb{M})$.

In this sequel we state the following result.

Theorem 4.1. *Let $\grave{M}$ be a CR-Submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi-symmetric metric connection $\widetilde{\nabla}$ then we have*

i) If $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then induced connection $\grave{\widetilde{\nabla}}$ is also a semi-symmetric metric connection.

ii) If $\grave{M}$ ξ-vertical $\Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$ then induced connection $\grave{\widetilde{\nabla}}$ is also a semi-symmetric non-metric connection.

iii) The Gauss formula with respect to semi-symmetric metric connection is of the form

$$\widetilde{\nabla}_X Y = \grave{\widetilde{\nabla}}_X Y + h(X, Y) + \eta(Y)\phi Q X, \tag{4.14}$$

iv) The weingarten formula with respect to semi-symmetric metric connection is of the form

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)X \tag{4.15}$$

Proof. With the help of (4.2) and (4.6) we get (iii). Also, from (2.8) and (3.1) we yield (iv). With reference to (4.5), if $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then result (i) is verifying. On the other hand, with the help of (4.7) if $\grave{M}$ is ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$, we obtain our desired result(ii). This tells us that the proof is completed. $\square$

Theorem 4.2. *Let $\grave{M}$ be a CR-Submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to quarter symmetric non-metric connection $\widehat{\nabla}$ then we have*

i) If $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widehat{\nabla}$ then induced connection $\grave{\widehat{\nabla}}$ is also a quarter symmetric non metric connection.
ii) If $\grave{M}$ ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widehat{\nabla}$ then induced connection $\grave{\widehat{\nabla}}$ is also a quarter symmetric non-metric connection.
iii) The Gauss formula with respect to quarter symmetric non-metric connection is of the form

$$\widehat{\nabla}_X Y = \grave{\nabla}_X Y + h(X, Y), \tag{4.16}$$

iv) The weingarten formula with respect to quarter symmetric non-metric connection is of the form

$$\widehat{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)\phi X \tag{4.17}$$

Proof. With the help of (4.8) and (4.12) we get (iii). Also, from (2.8) and (3.9) we yield (iv). With reference to (4.11), if $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then result (i) is verifying. On the other hand, with the help of (4.13) if $\grave{M}$ is ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$, we obtain our desired result(ii). We completed the proof. □

5. Yamabe solitons with potential vector field is torse-forming

As per this consequence of our analysis in this section we have the following geometric characterization of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ admitting semi-symmetric metric connection $\widetilde{\nabla}$ and quarter symmetric non-metric connection $\widehat{\nabla}$. Thus, in view of my thought,we can state the following result.

Theorem 5.1. *A Yamabe soliton (g, κ, λ) on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi symmetric metric connection $\widetilde{\nabla}$ is invariant if and only if*

$$2\eta(\kappa)g(X, Y) = \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}.$$

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a semi symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widetilde{\mathfrak{L}}_{\kappa} g)(X, Y) = (\widetilde{\hat{\delta}} - \lambda)g(X, Y). \tag{5.1}$$

From the definition of Lie derivative, equations (2.3) and (3.1), we obtain

$$(\widetilde{\mathfrak{L}}_{\kappa} g)(X, Y) = g(\widetilde{\nabla}_X \kappa, Y) + g(X, \widetilde{\nabla}_Y \kappa) \tag{5.2}$$

$$= g(\nabla_X \kappa, Y) + g(X, \nabla_Y \kappa) + 2\eta(\kappa)g(X, Y) - \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}$$
$$= (\mathfrak{L}_{\kappa} g)(X, Y) + 2\eta(\kappa)g(X, Y) - \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.1) and (5.2), we get

$$\frac{1}{2}(\mathfrak{L}_\kappa g)(X,Y) + \eta(\kappa)g(X,Y) - \frac{1}{2}\{g(X,\kappa)\eta(Y) + g(Y,\kappa)\eta(X)\} = (\widetilde{\hat{\delta}} - \lambda)g(X,Y). \tag{5.3}$$

This indicate that proof is completed. □

Theorem 5.2. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi-symmetric metric connection. If κ is a torse-forming vector field, then the soliton (g, κ, λ) is expanding, steady and shrinking according as $\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ <> =0, unless $\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ is constant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a semi-symmetric metric connection$\widetilde{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widetilde{\mathfrak{L}}_\kappa g)(X,Y) = (\widetilde{\hat{\delta}} - \lambda)g(X,Y). \tag{5.4}$$

From the definition of Lie derivative, equations (1.3) and (3.1), we obtain

$$\begin{aligned}(\widetilde{\mathfrak{L}}_\kappa g)(X,Y) &= g(\widetilde{\nabla}_X\kappa, Y) + g(X, \widetilde{\nabla}_Y\kappa)\\ &= 2\psi g(X,Y) + \{\theta(X)g(\kappa,Y) + \theta(Y)g(\kappa,X)\}\\ &\quad +2\eta(\kappa)g(X,Y) - \{\eta(X)g(\kappa,Y) + \eta(Y)g(\kappa,X)\}\end{aligned} \tag{5.5}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.4) and (5.5), we get

$$\begin{aligned}(\psi - \widetilde{\hat{\delta}} + \lambda)g(X,Y) &= \frac{1}{2}\{\eta(Y)g(\kappa,X) + \eta(X)g(\kappa,Y)\}\\ &\quad -\frac{1}{2}\{\theta(X)g(\kappa,Y) + \theta(Y)g(\kappa,X)\} - \eta(\kappa)g(X,Y)\end{aligned} \tag{5.6}$$

On contracting (5.6), we have

$$\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}. \tag{5.7}$$

This leads to the Theorem 5.2 □

In this sequel, we write the following corollaries.

Corollary 5.1. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with respect to a semi-symmetric*

metric connection $\widetilde{\nabla}$. *Then following relations hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} = C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> =0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = C$	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} = C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> =0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$

Theorem 5.3. *A Yamabe soliton* (g, κ, λ) *on an* n*-dimensional nearly hyperbolic Sasakian manifold* $\mathbb{M}^n(\phi, \xi, \eta, g)$ *with respect to quarter symmetric metric connection* $\widehat{\nabla}$ *always invariant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a quarter symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widehat{\mathfrak{L}}_\kappa g)(X, Y) = (\widehat{\hat{\delta}} - \lambda)g(X, Y). \tag{5.8}$$

From the definition of Lie derivative, equations (2.3) and (3.9), we obtain

$$\begin{aligned}(\widehat{\mathfrak{L}}_\kappa g)(X, Y) &= g(\widehat{\nabla}_X \kappa, Y) + g(X, \widehat{\nabla}_Y \kappa)\\ &= g(\nabla_X \kappa, Y) + g(X, \nabla_Y \kappa) + \eta(\kappa)g(\phi X, Y) + \eta(\kappa)g(X, \phi Y)\\ &= (\mathfrak{L}_\kappa g)(X, Y),\end{aligned} \tag{5.9}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.8) and (5.9), we get

$$\frac{1}{2}(\mathfrak{L}_\kappa g)(X, Y) = (\widehat{\hat{\delta}} - \lambda)g(X, Y). \tag{5.10}$$

Proof is completed. □

Theorem 5.4. *Let* (g, κ, λ) *be a Yamabe soliton on an* n*-dimensional nearly hyperbolic Sasakian manifold* $\mathbb{M}^n(\phi, \xi, \eta, g)$ *with respect to quarter symmetric metric connection* $\widehat{\nabla}$. *If* κ *is a torse-forming vector field, then the soliton* (g, κ, λ) *is expanding, steady and shrinking according as* $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> =0$, *unless* $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\}$ *is constant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a quarter symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widehat{\mathfrak{L}}_\kappa g)(X, Y) = (\widehat{\hat{\delta}} - \lambda)g(X, Y). \tag{5.11}$$

From the definition of Lie derivative, equations (1.3) and (3.9), we obtain

$$
\begin{aligned}
(\widehat{\mathfrak{L}}_{\kappa} g)(X,Y) &= g(\widehat{\nabla}_X \kappa, Y) + g(X, \widehat{\nabla}_Y \kappa) \\
&= 2\psi g(X,Y) + \theta(X) g(\kappa, Y) + \theta(Y) g(\kappa, X)
\end{aligned}
\tag{5.12}
$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.11) and (5.12), we get

$$
(\psi - \widehat{\hat{\delta}} + \lambda) g(X,Y) = -\frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(\kappa, X)\}
\tag{5.13}
$$

Taking contraction (5.13), we have

$$
\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\}.
\tag{5.14}
$$

This leads to the Theorem 5.4. $\square$

In this sequel, we write the following corollaries.

Corollary 5.2. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to quarter symmetric metric connection $\widehat{\nabla}$. Then following relations hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} = C$	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <>= 0$
concircular	$\widehat{\hat{\delta}} - \psi = C$	$\widehat{\hat{\delta}} - \psi <>= 0$
concurrent	$\widehat{\hat{\delta}} - 1 = C$	$\widehat{\hat{\delta}} - 1 <>= 0$
recurrent	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} = C$	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} <>= 0$
parallel	$\widehat{\hat{\delta}} = C$	$\widehat{\hat{\delta}} <>= 0$
torqued	$\widehat{\hat{\delta}} - \psi = C$	$\widehat{\hat{\delta}} - \psi <>= 0$

6. Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold

In this section, we study Yamabe soliton whose potential vector field is a torse-forming on CR-sub-manifolds of nearly hyperbolic Sasakian manifold with respect to the induced connection $\grave{\tilde{\nabla}}$ and $\grave{\hat{\nabla}}$. We state the following theorem as:

Theorem 6.1. *Let $\grave{M}$ be a CR-submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting semi-symmetric metric connection $\widetilde{\nabla}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\widetilde{\nabla}$. If (g, κ, λ) be a*

Yamabe soliton on M and κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady and shrinking according as $\widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> =0$, unless $\widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ is constant.

Proof. If $\grave{M}$ is ξ-horizontal for all $X, Y \in \Gamma(D)$ and D is parallel with respect to $\grave{\nabla}$, then in view of (4.5), we have

$$\grave{\grave{\nabla}}_X Y = \grave{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{6.1}$$

With the help of Theorem 5.2 and (3.1), we conclude that the induced connection $\grave{\nabla}$ is also semi-symmetric metric connection. This leads to the proof of the Theorem 6.1 □

In this sequel, we write the following corollaries.

Corollary 6.1. *Let $\grave{M}$ be a CR-submanifold nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a semi-symmetric metric connection $\grave{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\grave{\nabla}}$. If (g, κ, λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then the following results hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}=C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> =0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}=C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> =0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$

Theorem 6.2. *Let $\grave{M}$ be a CR-submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting quarter symmetric non- metric connection $\widehat{\nabla}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\widehat{\nabla}$. If (g, κ, λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady and shrinking according as $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> =0$, unless $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\}$ is constant.*

Proof. If $\grave{M}$ is ξ-horizontal for all $X, Y \in \Gamma(D)$ and D is parallel with respect to $\grave{\hat{\nabla}}$, then in view of (4.11), we have

$$\grave{\hat{\nabla}}_X Y = \grave{\nabla}_X Y + \eta(Y)\phi X, \tag{6.2}$$

With the help of Theorem 5.5 and (3.9), we conclude that the induced connection $\grave{\hat{\nabla}}$ is also quarter symmetric non-metric connection. This leads to the statement of the Theorem 6.2. □

In this sequel, we write the following corollaries.

Corollary 6.2. *Let $\grave{M}$ be a CR-submanifold nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting induced quarter symmetric non-metric connection $\grave{\hat{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\tilde{\hat{\nabla}}}$. If (g, κ, λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then the following results hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} = constant$	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> =0$
concircular	$\widehat{\hat{\delta}} - \psi = constant$	$\widehat{\hat{\delta}} - \psi <> =0$
concurrent	$\widehat{\hat{\delta}} - 1 = constant$	$\widehat{\hat{\delta}} - 1 <> =0$
recurrent	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} = constant$	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} <> =0$
parallel	$\widehat{\hat{\delta}} = constant$	$\widehat{\hat{\delta}} <> =0$
torqued	$\widehat{\hat{\delta}} - \psi = constant$	$\widehat{\hat{\delta}} - \psi <> =0$

7. Almost Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold

In this section, we classify almost Yamabe solitons whose potential field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold with respect to a semi-symmetric metric connection and quarter symmetric non-metric connection. At this stage, we denote κ^t and κ^n as tangential and normal component of such vector field. For almost Yamabe soliton we have the following.

Theorem 7.1. *An almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with a semi-symmetric metric connection of type $\tilde{\nabla}$ satisfies*

$$\begin{aligned}(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) &= g(A_{\kappa^n} X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \\ &+ \frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}\end{aligned} \tag{7.1}$$

for any vector fields X, Y on M.

Proof. In view of (1.3), (3.1), (4.14) and (4.15), we have

$$\begin{aligned}\psi X + \theta(P)\kappa = \widetilde{\nabla}_X \kappa = \widetilde{\nabla}_X(\kappa^t + \kappa^n) = \grave{\nabla}_X \kappa^t + h(X, \kappa^t) + \eta(\kappa^t)\phi Q X \\ -A_{\kappa^n} X + \nabla^{\perp}_X \kappa^n + \eta(\kappa^n) X - g(X, \kappa^n)\xi.\end{aligned} \tag{7.2}$$

On comparing tangential and normal component of (7.2), we obtain

$$\grave{\nabla}_X \kappa^t = \psi X + \theta(P)\kappa + A_{\kappa^n} X - \eta(\kappa^n) X + g(X, \kappa^n)\xi \tag{7.3}$$

and

$$h(X, \kappa^t) = -\nabla^{\perp}_X \kappa^n - \eta(\kappa^n)\phi Q X. \tag{7.4}$$

From the definition of Lie derivative and (7.3), we have

$$\begin{aligned}\mathfrak{L}_{\kappa^t} g(X, Y) = 2\psi g(X, Y) + 2g(A_{\kappa^n} X, Y) - 2\eta(\kappa^n) g(X, Y) + \{\theta(X) g(\kappa, Y) \\ +\theta(Y) g(X, \kappa)\} + \{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}.\end{aligned} \tag{7.5}$$

Using (7.5) in (1.2), we yield

$$\begin{aligned}(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(X, \kappa)\} \\ +\frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}.\end{aligned} \tag{7.6}$$

This proves our assertion. □

Corollary 7.1. *If an almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with semi-symmetric metric connection is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) n = \theta(\kappa). \tag{7.7}$$

Corollary 7.2. *Let (g, κ^t, λ) be an almost Yamabe soliton on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, and ξ-horizontal (resp.ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ satisfies*

$$\begin{aligned}(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) g(X, Y) = g(A_{\kappa^n} X, Y) + \frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(X, \kappa)\} \\ +\frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}\end{aligned} \tag{7.8}$$

for any vector fields X, Y on M.

Corollary 7.3. *If an almost Yamabe soliton (g, κ^t, λ) on CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $(n > 1)$ and ξ-horizontal (resp. ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) n = \theta(\kappa) \tag{7.9}$$

Theorem 7.2. *An almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with quarter symmetric non-metric connection $\widehat{\nabla}$ satisfies*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A_{\kappa^n}X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.10}$$

for any vector fields X, Y on M.

Proof. In view of (1.3), (3.9), (4.16) and (4.17), we have

$$\begin{aligned}\psi X + \theta(P)\kappa = \widehat{\nabla}_X\kappa = \widehat{\nabla}_X(\kappa^t + \kappa^n) &= \grave{\nabla}_X\kappa^t + \widehat{h}(X, \kappa^t) - \widehat{A}_{\kappa^n}X + \widehat{\nabla}^{\perp}_X\kappa^n \\ &= \grave{\nabla}_X\kappa^t + h(X, \kappa^t) - \widehat{A}_{\kappa^n}X + \nabla^{\perp}_X\kappa^n + \eta(\kappa^n)\phi X.\end{aligned} \tag{7.11}$$

On comparing tangential and normal component of (7.11), we obtain

$$\grave{\nabla}_X\kappa^t = \psi X + \theta(X)\kappa + A_{\kappa^n}X - \eta(\kappa^n)\phi X, \tag{7.12}$$

and

$$h(X, \kappa^t) = -\nabla^{\perp}_X\kappa^n. \tag{7.13}$$

From the definition of Lie derivative and (7.12), we have

$$\mathfrak{L}_{\kappa^t}g(X, Y) = 2\psi g(X, Y) + 2g(A_{\kappa^n}X, Y) + \{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\}. \tag{7.14}$$

Using (7.14) in (1.2), we yield

$$(\hat{\delta} - \lambda - \psi)g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.15}$$

This proves our assertion. □

Corollary 7.4. *If an almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with quarter symmetric non-metric connection is minimal, then*

$$(\hat{\delta} - \lambda - \psi)n = \theta(\kappa). \tag{7.16}$$

Corollary 7.5. *Let (g, κ^t, λ) be an almost Yamabe soliton on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, and ξ-horizontal (resp.ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\widehat{\nabla}}$ satisfies*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.17}$$

for any vector fields X, Y on M.

Corollary 7.6. *If an almost Yamabe soliton* (g, κ^t, λ) *on* CR*-submanifold* $\grave{M}$ *of nearly hyperbolic Sasakian manifold* $\mathbb{M}^n(\phi, \xi, \eta, g)$, $(n > 1)$ *and* ξ*-horizontal (resp.* ξ*-vertical),* $X, Y \in \Gamma(D)$, D *is parallel with induced connection* $\grave{\hat{\nabla}}$ *is minimal, then*

$$(\hat{\delta} - \lambda - \psi)n = \theta(\kappa) \tag{7.18}$$

8. Example

Example 8.1. Let us consider on $\mathbb{R}^{2n+1}$ the following hyperbolic Sasakian structure (ϕ, ξ, η, g) given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=n}^{n} y^i dx_i\right), \quad \xi = \frac{\partial}{\partial z},$$

$$g = -\eta \otimes \eta - \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi \circ \left(coshx_i \frac{\partial}{\partial x_i} + sinhy_i \frac{\partial}{\partial y_i} + z\frac{\partial}{\partial z}\right)$$

$$= \sum_{i=1}^{n}\left(sinhy_i \frac{\partial}{\partial x_i} + coshx_i \frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{n} sinhy, y^i \frac{\partial}{\partial z},$$

where $\{x^i, y^i, z\}$,$i = 1, ...n$ are the denoting the Cartesian coordinates.

The equation $t(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, 0, x_5)$ define a CR-sub-manifolds in $\mathbb{R}^5$ with its hyperbolic Sasakian structure (ϕ, ξ, η, g). For this fact we take the orthogonal basis

$$E_1 = coshx_5 \frac{\partial}{\partial x_1} + sinhx_5 \frac{\partial}{\partial x_2}, \quad E_2 = sinhx_5 \frac{\partial}{\partial x_1} + coshx_5 \frac{\partial}{\partial x_2}$$

$$E_3 = coshx_5 \frac{\partial}{\partial x_3} + sinhx_5 \frac{\partial}{\partial x_4}, \quad E_4 = sinhx_5 \frac{\partial}{\partial x_3} + coshx_5 \frac{\partial}{\partial x_4}, \quad E_5 = \frac{\partial}{\partial x_5} = \xi,$$

and define $D = span\{E_1, E_2\}$ and $D^\perp = span\{E_3\}$. In this case it is clear that $TM = D \oplus D^\perp \oplus \langle\xi\rangle$.

Example 8.2. Let us consider the 5-dimensioanl manifold $M = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinated in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = coshx_5 \frac{\partial}{\partial x_1} + sinhx_5 \frac{\partial}{\partial x_2}, \quad e_2 = sinhx_5 \frac{\partial}{\partial x_1} + coshx_5 \frac{\partial}{\partial x_2}$$

$$e_3 = coshx_5 \frac{\partial}{\partial x_3} + sinhx_5 \frac{\partial}{\partial x_4}, \quad e_4 = sinhx_5 \frac{\partial}{\partial x_3} + coshx_5 \frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial x_5} = \xi,$$

which are linearly independent at each point of M and hence form a basis tangent space T_pM.
Let g be the Riemannian metric on M define by

$$g(e_i, e_i) = -1, for 1 \leq i \leq 4 \quad and \quad g(e_5, e_5) = -1, \tag{8.1}$$

$$g(e_i, e_j) = 0, \quad for \quad 1 \neq j \quad and \quad 1 \leq i \leq 5 \quad 1 \leq j \leq 5. \tag{8.2}$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for all $X \in (M)$ and let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = -e_4, \quad \phi(e_4) = -e_3, \quad \phi(e_5) = 0.$$

Thus $e_5 = \xi$, the structure (ϕ, ξ, η, g) define an almost hyperbolic contact metric structure on M. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = -e_2, [e_2, e_5] = -e_1, [e_3, e_5] = e_4, [e_4, e_5] = -e_3,$$

The Levi-Civita connection ∇ of the Riemannian metric g is given by,

$$2g(\nabla_X Y, Z) \tag{8.3}$$

$$= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

which is known as Koszul's formula. After using koszul's formula, we find

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -e_5, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = -e_2,$$

$$\nabla_{e} e_1 = -e_5, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = -e_1,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_4 = -e_5, \quad \nabla_{e_3} e_5 = -e_4,$$

$$\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = -e_5, \quad \nabla_{e_4} e_3 = -e_5, \quad \nabla_{e_4} e_4 = 0, \quad \nabla_{e_4} e_5 = -e_3,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0,$$

By using the definition of semi-symmetric metric connection (3.1) and from above expressions we find

$$\widetilde{\nabla}_{e_1} e_1 = -e_5, \quad \widetilde{\nabla}_{e_1} e_2 = -e_5, \quad \widetilde{\nabla}_{e_1} e_3 = 0, \quad \widetilde{\nabla}_{e_1} e_4 = 0, \quad \widetilde{\nabla}_{e_1} e_5 = -e_1 - e_2,$$

$$\widetilde{\nabla}_{e_2} e_1 = -e_5, \quad \widetilde{\nabla}_{e_2} e_2 = -e_5, \quad \widetilde{\nabla}_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = -e_1 - e_2,$$

$$\widetilde{\nabla}_{e_3} e_1 = 0, \quad \widetilde{\nabla}_{e_3} e_2 = 0, \quad \widetilde{\nabla}_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = -e_5, \quad \nabla_{e_3} e_5 = e_3 - e_4,$$

$$\widetilde{\nabla}_{e_4} e_1 = 0, \quad \widetilde{\nabla}_{e_4} e_2 = -e_5, \quad \widetilde{\nabla}_{e_4} e_3 = -e_5, \quad \widetilde{\nabla}_{e_4} e_4 = -e_5, \quad \widetilde{\nabla}_{e_4} e_5 = -e_3 - e_4,$$

$$\widetilde{\nabla}_{e_5} e_1 = 0, \quad \widetilde{\nabla}_{e_5} e_2 = 0, \quad \widetilde{\nabla}_{e_5} e_3 = 0, \quad \widetilde{\nabla}_{e_5} e_4 = 0, \quad \widetilde{\nabla}_{e_5} e_5 = 0,$$

Therefore, the non-vanishing components of the Riemannian curvatures, the Ricci curvatures and the Scalar curvature with respect to the semi-symmetric metric connection as follows:

$$\widetilde{R}(e_1,e_2)e_1 = 0, \widetilde{R}(e_1,e_2)e_2 = 0, \widetilde{R}(e_1,e_3)e_1 = -e_3 - e_4, \widetilde{R}(e_1,e_3)e_3 = e_1 + e_2,$$

$$\widetilde{R}(e_1,e_2)e_1 = e_2, \widetilde{R}(e_1,e_2)e_2 = -e_1, \widetilde{R}(e_1,e_3)e_1 = 0, \widetilde{R}(e_1,e_3)e_3 = 0,$$

$$\widetilde{R}(e_1,e_4)e_1 = -e_3 - e_4, \widetilde{R}(e_1,e_4)e_4 = e_1 + e_2, \widetilde{R}(e_1,e_5)e_1 = -e_5,$$

$$\widetilde{R}(e_1,e_5)e_5 = -e_1 - e_2, \widetilde{R}(e_2,e_3)e_2 = -e_3 - e_4, \widetilde{R}(e_2,e_3)e_3 = -e_1 - e_2,$$

$$\widetilde{R}(e_2,e_4)e_3 = 0, \widetilde{R}(e_3,e_4)e_4 = 0, \widetilde{R}(e_3,e_5)e_3 = -e_5,$$

$$\widetilde{R}(e_3,e_5)e_5 = -e_3 - e_4, \widetilde{R}(e_4,e_5)e_4 = -e_5, \widetilde{R}(e_4,e_5)e_5 = -e_3 - e_4,$$

From these Riemannian curvatures tensors, we calculate

$$\widetilde{S}(e_1,e_1) = \quad \widetilde{S}(e_2,e_2) = \quad \widetilde{S}(e_3,e_3) = \widetilde{R}(e_4,e_4) = \widetilde{S}(e_5,e_5) = -4$$

Since $\{e_1, e_2, e_3, e_4, e_5\}$ form a basis of a 5-dimensional almost hyperbolic contact metric structure. Thus any vector field $X, Y, Z \in \chi(M^5)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3 + d_1e_4 + t_1e_5,$$

$$Y = a_2e_1 + b_2e_2 + c_2e_3 + d_2e_4 + t_2e_5,$$

$$Z = a_3e_1 + b_3e_2 + c_3e_3 + d_3e_4 + t_3e_5,$$

where $a_i, b_i, c_i, d_i, t_i \in \mathrm{Re}^+,\ i = 1, 2, 3, 4, 5$ such that

$$\left\{\frac{(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_3)}{t_1} + t_1\left(\frac{b_2}{b_1} - \frac{a_2}{a_1} - \frac{c_2}{c_1} - 1\right)\right\} \neq 0.$$

If we consider the 1-form θ by $\theta(X)$=-$g(X, e_5)$, for any $X \in \chi(M)$ and considering $\psi \in C^\infty(M)$ as

$$\psi = \left\{\frac{(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_3)}{t_1} + t_1\left(\frac{b_2}{b_1} - \frac{a_2}{a_1} - \frac{c_2}{c_1} - 1\right)\right\}.$$

So the relation

$$\nabla_X Y = \psi X + \theta(X)Y, \tag{8.4}$$

holds. As per this consequences Y is a torse-forming vector field. Thus from (9.3), we get

$$\begin{aligned}(\mathfrak{L}_Y g)(X,Z) &= g(\nabla_X Y, Z) + g(X, \nabla_Z Y) \\ &= 2\psi g(X,Z) + \theta(X)g(Y,Z) + \theta(Z)g(Y,X).\end{aligned} \tag{8.5}$$

Also, we calculate

$$\begin{cases} g(X,Z) = a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3 \\ g(Y,Z) = a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3 - t_2t_3 \\ g(Y,X) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 - t_1t_2 \end{cases}. \tag{8.6}$$

Also

$$\begin{cases} \theta(X) = t_1 \\ \theta(Y) = t_2 \\ \theta(Z) = t_3 \end{cases}. \tag{8.7}$$

With the help of above equation (9.2) can be reduced

$$\begin{aligned}\tfrac{1}{2}(\mathfrak{L}_Y g)(X,Z) = &\left\{\tfrac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1} + t_1\left(\tfrac{b_2}{b_1} - \tfrac{a_2}{a_1} - \tfrac{c_2}{c_1} - 1\right)\right\} \\ &\times\{a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3 \\ &-\tfrac{1}{2}\, t_1(a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3 - t_2t_3) \\ &+t_3(a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3)\}\end{aligned} \tag{8.8}$$

Also,

$$(\widetilde{\hat{\delta}} - \lambda)g(X,Z) = (-16-\lambda)\{a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3\} \tag{8.9}$$

We consider that $a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3 \neq 0$ and $5t_1(a_2a_3+b_2b_3+c_2c_3+d_2d_3-t_2t_3)+5t_3(a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3)+2t_2(a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3)=0$.

we get (g, Y, λ) is a Yamabe soliton, i.e., $\frac{1}{2}\mathfrak{L}_Y g(X,Z)=(\widetilde{\hat{\delta}} - \lambda)g(X,Z)$ holds, unless

$$\begin{aligned}\lambda &= -16 - \left\{\tfrac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1} + t_1\left(\tfrac{b_2}{b_1} - \tfrac{a_2}{a_1} - \tfrac{c_2}{c_1} - 1\right)\right\} - \tfrac{1}{5}t_2 \\ &= \widetilde{r} - \psi - \tfrac{1}{5}\theta(Y) \\ &= constant\end{aligned}$$

So the existence of Yamabe soliton (g, Y, λ) on a 5-dimensional hyperbolic Sasakian manifold with semi symmetric metric connection $\widetilde{\nabla}$ with potential vector field Y as torse-forming thus the Theorem 5.2 is verified.

Example 8.3. In Example 8.2, we consider the hyperbolic Sasakian manifold $M\ (\phi, \eta, \xi, g)$ with quarter symmetric non-metric connection. Using the equation (3.9), we obtain:

$$\begin{array}{lllll} \widehat{\nabla}_{e_1}e_1 = 0, & \widehat{\nabla}_{e_1}e_2 = -e_5, & \widehat{\nabla}_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \widehat{\nabla}_{e_1}e_5 = -e_2, \\ \widehat{\nabla}_{e}e_1 = -e_5, & \widehat{\nabla}_{e_2}e_2 = 0, & \widehat{\nabla}_{e_2}e_3 = 0, & \widehat{\nabla}_{e_2}e_4 = 0, & \widehat{\nabla}_{e_2}e_5 = -e_1, \\ \widehat{\nabla}_{e_3}e_1 = 0, & \widehat{\nabla}_{e_3}e_2 = 0, & \widehat{\nabla}_{e_3}e_3 = 0, & \widehat{\nabla}_{e_3}e_4 = -e_5, & \widehat{\nabla}_{e_3}e_5 = -e_4, \\ \widehat{\nabla}_{e_4}e_1 = 0, & \widehat{\nabla}_{e_4}e_2 = -e_5, & \widehat{\nabla}_{e_4}e_3 = -e_5, & \widehat{\nabla}_{e_4}e_4 = 0, & \widehat{\nabla}_{e_4}e_5 = -e_3, \\ \widehat{\nabla}_{e_5}e_1 = 0, & \widehat{\nabla}_{e_5}e_2 = 0, & \widehat{\nabla}_{e_5}e_3 = 0, & \widehat{\nabla}_{e_5}e_4 = 0, & \widehat{\nabla}_{e_5}e_5 = 0, \end{array}$$

Therefore, the non-vanishing components of the Riemannian curvatures, the Ricci curvatures and the Scalar curvature with respect to the quarter-symmetric non-metric connection are as follows:

$$\widehat{R}(e_1, e_2)e_1 = e_2, \quad \widehat{R}(e_1, e_2)e_2 = -e_1, \quad \widehat{R}(e_1, e_3)e_1 = 0, \quad \widehat{R}(e_1, e_3)e_3 = 0,$$

$$\widehat{R}(e_1, e_4)e_1 = 0, \quad \widehat{R}(e_1, e_4)e_4 = 0, \quad \widehat{R}(e_1, e_5)e_1 = -e_5, \quad \widehat{R}(e_1, e_5)e_5 = -e_1,$$

$$\widehat{R}(e_2, e_3)e_2 = 0, \quad \widehat{R}(e_2, e_3)e_3 = 0, \quad \widehat{R}(e_2, e_4)e_3 = 0, \quad \widehat{R}(e_3, e_4)e_4 = 0,$$

$$\widehat{R}(e_2, e_5)e_2 = -e_5, \quad \widehat{R}(e_2, e_5)e_5 = -e_2, \quad \widehat{R}(e_3, e_4)e_3 = e_4, \quad \widehat{R}(e_3, e_4)e_4 = -e_3,$$

$$\widehat{R}(e_3, e_5)e_3 = -e_5, \quad \widehat{R}(e_3, e_5)e_5 = -e_3, \quad \widehat{R}(e_4, e_5)e_4 = -e_5, \quad \widehat{R}(e_4, e_5)e_5 = -e_4,$$

From these Riemannian curvatures tensors components with quarter semi-symmetric non-metric connection we calculate:

$$\widehat{S}(e_1, e_1) = \quad \widehat{S}(e_2, e_2) = \quad \widehat{S}(e_3, e_3) = \widehat{R}(e_4, e_4) = 0, \widehat{S}(e_5, e_5) = -4$$

$$\widehat{r} = -4.$$

Therefore, the constructed metric of the hyperbolic Sasakian manifold with quarter-symmetric non-metric connection is Yamabe solion. It is shown that the scalar curvature with respect to the quarter-symmetric non-metric connection $\widehat{r} = -4$ and $\lambda = -4 < 0$ i.e is admitting shrinking Yamabe soliton.

REFERENCES

1. M. Ahmad, M. D. Siddiqi, and S. Rizvi: *CR-sub-manifolds of a nearly hyperbolic Sasakian manifold admitting semi-symmetric semi-metric connection.* International J. Math. Sci. and Engg. Appls. **6** (2012), 145–155.
2. E. Barbosa and E. Ribeiro: *On conformal solution of the Yamabe flow.* Arch. Math. **101** (2013), 79–89.
3. A. Bejancu: *CR-sub-manifolds of a Kähler manifold* I, Proc. Amer. Math. Soc. **69** (1978), 135–142.
4. A. Bejancu: *Geometry of CR-sub-manifolds.* D. Reidel Publ. Co., 1986.
5. A. Bejancu and N. Papaghuic: *CR-sub-manifolds of Kenmotsu manifold.* Rend. Mat. **7** (1984), 607–622.
6. L. Bhatt and K. K. Dube: *CR-sub-manifolds of trans-hyperbolic Sasakian manifold.* Acta Ciencia Indica **31** (2003), 91–96.
7. D. E. Blair: *Contact manifolds in Riemannian geometry.* Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.

8. B. Y. Chen: *Classification of torqued vector fields and its applications to Ricci solitons.* Kragujevac J. Math. **41** (2017), 239–250.

9. B. Y. Chen and S. Deshmukh: *Yamabe and quasi-Yamabe solitons on Euclidean sub-manifolds.* Mediterr. J. Math. **15** (2018), 1–9.

10. L. S. K. Das and M. Ahmad: *CR-sub-manifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric non-metric connection.* Algebras Group Geometry **31** (2014), 313–326.

11. P. Daskalopoulos and N. Sesum: *The classification of locally conformally flat Yamabe solitons.* Adv. Math. **240** (2013), 346–369

12. S. Deshmukh and B.Y. Chen: *A note on Yamabe solitons.* Balkan J. Geom. and Appl. **23** (2018), 37–43.

13. S. Golab: *On semi-symmetric and quarter symmetric linear connections.* Tensor, N.S., Japan, (1975), 249–254.

14. R. S. Hamilton: *The Ricci flow on surfaces, in: Mathematics and General Relativity.* in: Contemp. Math., **71**, (1986), 237–262.

15. C. J. Hsu: *On CR-sub-manifolds of Sasakian manifolds* I, Math. Research Center Reports, Symposium Summer, (1983), 117–140.

16. S. Y. Hsu: *A note on compact gradient Yamabe solitons.* J. Math. Anal. Appl. **388** (2012), 725–726.

17. M. Kobayashi: *CR-sub-manifolds of a Sasakian manifold.* Tensor (N.S.), Japan, (1981), 297–307.

18. M. D. Siddiqi, M. Ahmed and J. P. Ojha: *CR-sub-manifolds of nearly-trans hyperbolic sasakian manifolds admitting semi-symmetric-non-metric connection.* African J. Diaspora bf 17(10) (2014), 93–105.

19. M. D. Upadhyay and K. K. Dubey: *Almost contact hyperbolic (f, ξ, η, g)-structure.* Acta. Math. Acad. Scient. Hung. Tomus, **28** (1976), 1–4.

20. K. Yano: *On semi-symmetric metric connection.* Rev. Roum. Math. Pureset Appl. **15** (1970), 1579–1586.

21. K. Yano: *On torse-forming direction in a Riemannian space.* Proc. Imp. Acad. Tokyo **20** (1944), 340–345.

22. K. Yano and B. Y. Chen: *On concurrent vector fields of immersed manifolds.* Kodai Math. Sem. Rep. bf 23 (1971), 343–350.

23. K. Yano and M. Kom: *Contact CR-sub-manifolds.* Kodai Math. J. (1982), 238–252.